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Research Article

On discrete orthogonal U-Bernoulli Korobov-type polynomials

Dedicated to Professor Paolo Emilio Ricci, on occasion of his 80th birthday, with respect and friendship.

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ABSTRACT. The primary objective of this paper is to introduce and examine the new class of discrete orthogonal polynomials called U–Bernoulli Korobov-type polynomials. Furthermore, we derive essential recurrence relations and explicit representations for this polynomial class. Most of the results are proven through the utilization of generating function methods. Lastly, we place particular emphasis on investigating the orthogonality relation associated with these polynomials.

Keywords: Bernoulli polynomials, U–Bernoulli Korobov, discrete orthogonal polynomials.

2020 Mathematics Subject Classification: 11B68, 11B83, 11B39, 05A19.

1. INTRODUCTION

The study of discrete Appell polynomials is significant in mathematics due to their special properties and wide range of applications. Analogous to continuous Appell polynomials, they feature a discrete shift operator as their primary differential operator. Moreover, they are closely related to orthogonal polynomials, such as Hermite and Chebyshev polynomials, which are vital in areas like approximation theory and quantum mechanics. Together, these polynomials contribute to the development of special functions that are applied across diverse fields, including mathematics, physics, engineering, and statistics (see, [7, 10, 17]).

In this context, let $f : \mathbb{Z} \to \mathbb{R}$ be any function of the natural numbers, and consider the discrete operator $\Delta f(x) = f(x+1) - f(x)$. This operator plays a crucial role in the definition and analysis of discrete Appell polynomials, further highlighting their importance in both theoretical and applied mathematics.

A discrete Appell sequence $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials such that (see, [6]):

$$\Delta p_k(x) = p_k(x+1) - p_k(x) = kp_{k-1}(x), \quad k \geq 1.$$

It is known that a Taylor series expansion can define Appell sequences (see, [1]):

$$A(z)e^{xz} = \sum_{n=0}^{\infty} P_n(x)\frac{z^n}{n!}, \tag{1.1}$$

where $A(z)$ is a function analytic at $z = 0$ with $A(0) \neq 0$; similarly, discrete Appell sequences can be defined by a Taylor generating expansion

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$$A(z)(1+z)^x = \sum_{n=0}^{\infty} p_n(x)\frac{z^n}{n!}, \tag{1.2}$$

where $A(z)$ is a function analytic at $z=0$ with $A(0)\neq 0$.

There is a substantial body of mathematical literature devoted to studying the families of Appell sequences. Typical examples include the trivial case $\{x^k\}_{k=0}^{\infty}$, whose generating function is given by (1.1) with $A(z)=1$, and the Bernoulli polynomials, which were used by Euler in 1740 to sum $\sum_{n=1}^{\infty} 1/n^{2k}$. Their generating function is (1.1) with $A(z)=\frac{z}{e^z-1}$.

In the case of discrete Appell sequences, the trivial case, obtained from (1.2) with $A(t)=1$, is the family $\{x^{\underline{k}}\}_{k=0}^{\infty}$, where

$$x^{\underline{k}} = x(x-1)\cdots(x-k+1) = \prod_{j=0}^{k-1}(x-j)$$

is the falling factorial (various notations have been used for these polynomials; here we follow [11] and [8, p. 47]). The discrete counterpart to the Bernoulli polynomials is the so-called Bernoulli polynomials of the second kind (see [3]), denoted by $b_k(x)$, which were independently introduced by Jordan [9] and Rey Pastor [16] in 1929. These polynomials, also known as Rey Pastor polynomials (see, [2]), are now defined by a generating function as in (1.1) via

$$\frac{z}{\log(1+z)}(1+z)^x = \sum_{k=0}^{\infty} b_k(x)\frac{z^k}{k!}.$$

We consider the discrete orthonormal polynomials $\{p(x)\}_{n\geq 0}$ corresponding to a positive measure with respect to a discrete weight $\omega(k)$ and satisfies the conditions

$$\sum_{k=0}^{\infty} p_n(k)p_m(k)\omega(k) = \delta_{m,n}, \tag{1.3}$$

where $\delta_{m,n}$ is the Kronecker delta (cf., [14, pp. 586]).

The moments μ_n of the discrete weight $\omega(k)$ in (1.3) are given by

$$\mu_n = \sum_{k=0}^{\infty} k^n\omega(k),\ n\geq 0.$$

In the special case when the discrete weight has the special form

$$\omega(k) = c(k)z^k, \quad z>0,$$

which is the case for the Charlier polynomials $C_n(k;z)$ and the Meixner polynomials $M_n(k;\alpha;z)$ (see, [5, 13]), then

$$\mu_n(z) = \mu_n = \sum_{k=0}^{\infty} k^n c(k) z^k.$$

Considering the aforementioned context, the main objective of this work is to define and study the discrete U-Bernoulli Korobov-type polynomial. We study the algebraic and differential properties associated with this particular family of polynomials. Furthermore, we introduce an orthogonality relation that satisfies these polynomials.

2. Notation and background

Throughout this paper, let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote, respectively, the set of all natural numbers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers.

The Korobov polynomials $K_n(x;\lambda)$ of the first kind are given by the generating function (cf., [12])

$$\frac{\lambda z}{(1+z)^{\lambda}-1}(1+z)^{x}=\sum_{n=0}^{\infty}K_n(x;\lambda)\frac{z^n}{n!}.$$

When $x=0$, $K_n(\lambda)=K_n(0,\lambda)$ are called the Korobov number.

In [4], L. Carlitz considered the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{z}{(1+\lambda z)^{\frac{1}{\lambda}}-1}(1+\lambda z)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathscr{B}_n(x;\lambda)\frac{z^n}{n!}. \tag{2.4}$$

From (2.4), we have $\lim_{\lambda\to 0}\mathscr{B}_n(x;\lambda)=B_n(x), \qquad (n\geq 0)$.

Additionally, for $n\in\mathbb{N}_0$, we defined the new family U–Bernoulli polynomials $M_n(x)$ of degree n in the variable x by the power series expansion at 0 of the following generating function (see, [15]):

$$f(x;z)=\frac{z}{e^{-z}-1}e^{-xz}=\sum_{n=0}^{\infty}M_n(x)\frac{z^n}{n!},\quad |z|<2\pi.$$

We have for the first few U–Bernoulli polynomials $M_n(x)$, that

$$\begin{aligned}
M_0(x)&=-1, & M_3(x)&=x^3-\frac{3}{2}x^2+\frac{1}{2}x,\\
M_1(x)&=x-\frac{1}{2}, & M_4(x)&=-x^4+2x^3-x^2+\frac{1}{30},\\
M_2(x)&=-x^2+x-\frac{1}{6}, & M_5(x)&=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x.
\end{aligned}$$

When $x=0$ in (2), the U-Bernoulli numbers are defined by the generating function

$$f(z)=\frac{z}{e^{-z}-1}=\sum_{n=0}^{\infty}M_n\frac{z^n}{n!},\quad |z|<2\pi.$$

Some of these numbers are

$$M_0=-1;\quad M_1=-\frac{1}{2};\quad M_2=-\frac{1}{6};\quad M_3=0;\quad M_4=\frac{1}{30};\quad M_5=0.$$

3. U–Bernoulli Korobov-type discrete polynomials

In this section, we introduce the U-Bernoulli Korobov-type discrete polynomials and derive several key results for these polynomials.

Definition 3.1. *The new family of U–Bernoulli Korobov-type discrete polynomials $\mathscr{P}_n(x)$ of degree n in $x\in\mathbb{N}$ are defined by the generating function*

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^{x}=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!},\quad |z|<2\pi. \tag{3.5}$$

The first six U–Bernoulli Korobov-type discrete polynomials $\mathscr{P}_n(x)$, are

$$\mathscr{P}_0(x) = -1, \qquad \mathscr{P}_3(x) = -x^3 + \frac{3}{2}x^2 - x,$$
$$\mathscr{P}_1(x) = -x - \frac{1}{2}, \qquad \mathscr{P}_4(x) = -x^4 + 4x^3 - 4x^2 + 3x + \frac{1}{30},$$
$$\mathscr{P}_2(x) = -x^2 - \frac{1}{6}, \qquad \mathscr{P}_5(x) = -x^5 + \frac{15}{2}x^4 - \frac{65}{3}x^3 + \frac{55}{2}x^2 - \frac{33}{6}x.$$

For $x = 0$ in (3.5) the U–Bernoulli Korobov-type discrete numbers $\mathscr{P}_n(0)$ are defined by the generating function

$$\frac{z}{e^{-z}-1} = \sum_{n=0}^{\infty} \mathscr{P}_n \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{3.6}$$

Some of these numbers are

$$\mathscr{P}_0 = -1; \quad \mathscr{P}_1 = -\frac{1}{2}; \quad \mathscr{P}_2 = -\frac{1}{6}; \quad \mathscr{P}_3 = 0; \quad \mathscr{P}_4 = \frac{1}{30}; \quad \mathscr{P}_5 = 0.$$

A consequence of (3.5) and (3.6) is the following proposition, which highlights several properties satisfied by this family of polynomials.

Proposition 3.1. *The U-Bernoulli Korobov-type discrete polynomials in the variable x, they satisfy the following relations*

(i) $$\mathscr{P}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} (y)_k \mathscr{P}_{n-k}(x),$$

(ii) $$\mathscr{P}_n(x) = \sum_{k=0}^{n-1} n \binom{n-1}{k} (x)_k + \sum_{k=0}^{n} \binom{n}{k} \mathscr{P}_k(x),$$

(iii) $$\sum_{k=0}^{n} \binom{n}{k} \mathscr{P}_k(x+y)\mathscr{P}_{n-k} = \sum_{k=0}^{n} \binom{n}{k} \mathscr{P}_{n-k}(x)\mathscr{P}_k(y),$$

(iv) $$\mathscr{P}_n(x) = \mathscr{P}_n + \sum_{k=0}^{n-1} \frac{n}{(k+1)} \binom{n-1}{k} (x)_{k+1} \mathscr{P}_{n-1-k},$$

(v) $$\mathscr{P}_n(x) = \sum_{k=0}^{\infty} \binom{x}{k} \frac{n!}{(n-k)!} \mathscr{P}_{n-k},$$

(vi) $$\mathscr{P}_n(x) = \mathscr{P}_n(x+1) - n\mathscr{P}_{n-1}(x).$$

Proof. (see (iii)). Let's consider the following expressions

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^x = \sum_{n=0}^{\infty} \mathscr{P}_n(x) \frac{z^n}{n!} \tag{3.7}$$

and

$$\left(\frac{z}{e^{-z}-1}\right)(1+z)^y = \sum_{n=0}^{\infty} \mathscr{P}_n(y) \frac{z^n}{n!}. \tag{3.8}$$

Of (3.7) and (3.8), we have

$$\left[\frac{z}{e^{-z}-1}\right]^2(1+z)^{x+y} = \left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(y)\frac{z^n}{n!}\right)$$
$$\left(\sum_{n=0}^{\infty}\mathscr{P}_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x+y)\frac{z^n}{n!}\right) = \left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{P}_n(y)\frac{z^n}{n!}\right)$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}\mathscr{P}_k(x+y)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x)\mathscr{P}_k(y)\frac{z^n}{n!}$$
$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}\mathscr{P}_k(x+y) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{P}_{n-k}(x)\mathscr{P}_k(y).$$

Therefore,

$$\sum_{k=0}^{n}\binom{n}{k}\left[\mathscr{P}_k(x+y)\mathscr{P}_{n-k} - \mathscr{P}_{n-k}(x)\,\mathscr{P}_k(y)\right] = 0.$$

□

Theorem 3.1 (Differential expressions). *For $n \in \mathbb{N}$, let $\{\mathscr{P}_n(x)\}_{n\geq 0}$ be the sequences of U–Bernoulli Korobov-type discrete polynomials in the variable $x \in \mathbb{N}$, they satisfy the following relations*

(1)

$$(n-1)\mathscr{P}_n(x) - n\psi(x;z)\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x) = 0,$$

where

$$\psi(x;z) = \left[\frac{x}{(z+1)\log(z+1)} + \frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right],$$

(2)

$$\frac{\partial \mathscr{P}_n(x)}{\partial x} = \sum_{k=0}^{n-1} n\binom{n-1}{k}(-1)^k\frac{k!}{k+1}\mathscr{P}_{n-k-1}(x).$$

Proof. For the proof of (1). Consider the following equations

$$L(x;z) = \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}, \tag{3.9}$$

$$L(x;z) = \frac{z}{e^{-z}-1}(1+z)^x. \tag{3.10}$$

Partially differentiating with respect to z in (3.9) and (3.10) , the result is

$$\frac{\partial L(x;z)}{\partial z} = \sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^{n-1}}{n!}$$

and

$$\frac{\partial L(x;z)}{\partial z} = \frac{(1+z)^x}{e^{-z}-1} + \left[\frac{z(1+z)^x}{e^{-z}-1}\right]\frac{x}{1+z} + \left[\frac{z(1+z)^x}{e^{-z}-1}\right]\frac{e^{-z}}{e^{-z}-1}. \tag{3.11}$$

Partially differentiating with respect to x in (3.10) , we have

$$\frac{\partial L(x;z)}{\partial x} = \frac{z\log(z+1)(1+z)^x}{e^{-z}-1}.$$

Of (3.11), we have

$$0=\frac{\partial L(x;z)}{\partial z}-\frac{(1+z)^x}{e^{-z}-1}-\left[\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}\right]\frac{x}{(1+z)\log(z+1)}$$
$$-\left[\frac{z\log(z+1)(1+z)^x}{e^{-z}-1}\right]\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}$$
$$0=\frac{z\partial L(x;z)}{\partial z}-\left[\frac{zx}{(1+z)\log(z+1)}+\frac{ze^{-z}}{(e^{-z}-1)\log(z+1)}\right]\frac{\partial L(x;z)}{\partial x}-\frac{z(1+z)^x}{e^{-z}-1}$$
$$0=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{nz^n}{n!}-\sum_{n=0}^{\infty}\left[\frac{x}{(1+z)\log(z+1)}+\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right]\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x)\frac{nz^n}{n!}$$
$$-\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}$$
$$0=(n-1)\mathscr{P}_n(x)-\left[\frac{x}{(1+z)\log(z+1)}+\frac{e^{-z}}{(e^{-z}-1)\log(z+1)}\right]n\frac{\partial}{\partial x}\mathscr{P}_{n-1}(x).$$

This completes the proof of (1).

For the proof of (2). Partially differentiating with respect to x in (3.1), we have

$$\left(\frac{z}{e^{-z}-1}\right)\frac{\partial}{\partial x}\left[(1+z)^x\right]=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}$$
$$\left(\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{n+1}\right)=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\mathscr{P}_{n-1-k}(x)(-1)^k\binom{n-1}{k}\frac{k!}{(k+1)}n\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathscr{P}_n(x)\frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ in both sides of the equation, the result is

$$\frac{\partial}{\partial x}\mathscr{P}_n(x)=\sum_{k=0}^{n-1}n\binom{n-1}{k}(-1)^k\frac{k!}{k+1}\mathscr{P}_{n-k-1}(x).$$

□

4. Orthogonality relation of the U-Bernoulli Korobov-type discrete polynomials

In this section, we will present a comprehensive demonstration of the orthogonality relationship associated with the U-Bernoulli Korobov-type discrete polynomials.

Theorem 4.2. *The U–Bernoulli Korobov-type discrete polynomials $\mathscr{P}_n(x)$, fulfill the following orthogonality relation*

$$\int_0^{\infty}\mathscr{P}_m(x)\;\mathscr{P}_n(x)\,d\mu(x)=(-1)^{n-1}\,m!\,n\,\delta_{m,n},$$

where

$$d\mu(x)=\omega(x,\lambda_1,\sigma_1)dx=\frac{(1-e^{\lambda_1})(1-e^{\sigma_1})\,e}{x!}dx,$$

with $x\in\mathbb{N}$, $z,v\in\mathbb{C}$, and $\lambda_1\in\mathrm{Re}(z),\sigma_1\in\mathrm{Re}(v)$.

Proof. Let's consider the following equality

$$L(x,z)=\left(\frac{z}{e^{-z}-1}\right)(1+z)^{x}=\sum_{n=0}^{\infty}\mathscr{P}_n(x)\frac{z^n}{n!},\qquad |z|<2\pi.$$

Then

$$\begin{aligned}L(x,z)&=\left(\frac{z}{e^{-z}-1}\right)(1+z)^{x}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k}\frac{\mathscr{P}_{n-k}}{(n-k)!}\,z^n\end{aligned}\tag{4.12}$$

$$L(x,z)=\sum_{n=0}^{\infty}L_n(x)z^n,\tag{4.13}$$

where we get

$$L_n(x)=\sum_{k=0}^{n}\binom{x}{n}\frac{\mathscr{P}_{n-k}}{(n-k)!}$$

$$L_n(x)=\sum_{k=0}^{n}\frac{1}{k!}\left(x(x-1)(x-2)\cdots(x-k+1)\right)\frac{\mathscr{P}_{n-k}}{(n-k)!}.\tag{4.14}$$

Note that (4.14) is a polynomial of degree n and $L_n(x)$, which coincides with the so-called U–Bernoulli Korobov-type discrete polynomial.

On the other hand, let

$$\begin{aligned}L(x,v)&=f(v)g(x,v)\\&=\left(\frac{v}{e^{-v}-1}\right)(1+v)^{x}\\&=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{x}{m}\frac{\mathscr{P}_{m-k}}{(m-k)!}\,v^m\end{aligned}\tag{4.15}$$

$$L(x,v)=\sum_{m=0}^{\infty}L_m(x)\,v^m,\tag{4.16}$$

where it is obtained

$$L_m(x)=\sum_{k=0}^{m}\binom{x}{m}\frac{\mathscr{P}_{m-k}}{(m-k)!}.$$

Considering the product of (4.12) and (4.15), we obtain

$$\begin{aligned}L(x,z)L(x,v)&=\left(\frac{z}{e^{-z}-1}\right)(1+z)^{x}\left(\frac{v}{e^{-v}-1}\right)(1+v)^{x}\\&=\left[\frac{zve^{z+v}}{(1-e^{z})(1-e^{v})}\right]\left[(1+z)^{x}(1+v)^{x}\right]\end{aligned}$$

then,

$$L(x,z)L(x,v)=\left[\frac{zve^{z+v}}{(1-e^{z})(1-e^{v})}\right]\left[(1+z)(1+v)\right]^{x}.$$

For any $k=x$, we have

$$(-1)^k L(x,z)L(x,v)=\left[\frac{zve^{z+v}}{(1-e^z)(1-e^v)}\right]\left[(-1)(1+z)(1+v)\right]^k.$$

Then,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k L(x,z)L(x,v)}{k!}&=\frac{zve^{z+v}}{(1-e^z)(1-e^v)}\sum_{k=0}^{\infty}\frac{\left[-(1+z)(1+v)\right]^k}{k!}\\&=\frac{zve^{z+v}}{(1-e^z)(1-e^v)}\,e^{-(1+z)(1+v)}\\&=\left[\frac{zv\,e^{-1}}{(1-e^z)(1-e^v)}\right]e^{-zv}.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{\infty}\frac{(-1)^k L(x,z)L(x,v)}{k!}=\sum_{n=0}^{\infty}\left[\frac{zv\,e^{-1}(-1)^n}{(1-e^z)(1-e^v)}\right]\frac{z^n\,v^n}{n!},\tag{4.17}$$

from (4.13) and (4.16) on (4.17) left side we have

$$\sum_{k=0}^{\infty}\frac{(-1)^k L_m(k\,,z)L_n(k\,,v)}{k!}=\sum_{m\,,n=0}^{\infty}\sum_{k=0}^{\infty}L_m(k),L_n(k)\,\frac{(-1)^k}{k!}\,z^n v^m.\tag{4.18}$$

From (4.17) and (4.18) we get

$$\sum_{n=0}^{\infty}\left[\frac{zv\,e^{-1}(-1)^n}{(1-e^z)(1-e^v)}\right]\frac{z^n v^n}{n!}=\sum_{m\,,n=0}^{\infty}\sum_{k=0}^{\infty}L_m(k),L_n(k)\,\frac{(-1)^k}{k!}\,z^n v^m.\tag{4.19}$$

Which gives:

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\frac{zv\,e^{-1}(-1)^n}{(1-e^z)(1-e^v)}\right]\frac{z^n v^n}{n!}&=\sum_{n=0}^{\infty}\left[\frac{e^{-1}(-1)^n}{(1-e^z)(1-e^v)}\right]\frac{z^{n+1}v^{n+1}}{n!}\\&=\sum_{n=0}^{\infty}\left[\frac{e^{-1}(-1)^{n-1}}{(1-e^z)(1-e^v)}\right]\frac{nz^n v^n}{n!}.\end{aligned}$$

Therefore, in (4.19) we have

$$\sum_{n=0}^{\infty}\left[\frac{e^{-1}(-1)^{n-1}}{(1-e^z)(1-e^v)}\right]\frac{nz^n v^n}{n!}=\sum_{m\,,n=0}^{\infty}\sum_{k=0}^{\infty}L_m(k)L_n(k)\,\frac{(-1)^k}{k!}\,z^n v^m,\tag{4.20}$$

by comparing the coefficients in (4.20), one has

$$\sum_{k=0}^{\infty}L_m(x)\,L_n(x)\,\frac{(-1)^k}{k!}=\begin{cases}\dfrac{(-1)^{n-1}\,e^{-1}}{n!}\left[\dfrac{n}{(1-e^z)(1-e^v)}\right]; & \text{if } m=n.\\ 0; & \text{if } m\neq n.\end{cases}\tag{4.21}$$

In (4.21), we note that $\{L_n(x)\}_{n\geq 0}$, is a sequence of orthogonal polynomials with respect to the weight function $\dfrac{(-1)^k}{k!}$, $k=0\,,1\,,2,\ldots$

Remark 4.1. *We define about (4.21) the functional $\mathscr{L}$ as follows:*

$$\mathscr{L}\left[L_m(x)L_n(x)\frac{(-1)^k}{k!}\right]=\frac{(-1)^{n-1}}{n!}\,e^{-1}\,\psi_n\,\delta_{m,n}, \tag{4.22}$$

where

$$\psi_n=\frac{n}{(1-e^z)(1-e^v)}.$$

Therefore (4.22) can be expressed in terms of the Riemann-Stieltjes integral as follows.

$$\int_0^\infty L_m(x)L_n(x)\,\frac{(-1)^k}{k!}\,dx=\frac{(-1)^{n-1}\,e^{-1}}{n!}\,\psi_n\,\delta_{m,n}. \tag{4.23}$$

In (4.13) and (4.16), we have that,

$$L(x,z)=\sum_{n=0}^{\infty}L_n(x)\,z^n \qquad \text{and} \qquad L(x,v)=\sum_{m=0}^{\infty}L_m(x)\,v^m.$$

Therefore,

$$L_n(x)=\frac{P_n(x)}{n!} \qquad \text{and} \qquad L_m(x)=\frac{P_m(x)}{m!}. \tag{4.24}$$

Of (4.24) and (4.23) we obtain

$$\int_0^\infty \frac{P_m(x)}{m!}\,\frac{P_n(x)}{n!}\,d\lambda=\frac{(-1)^{n-1}\,e^{-1}}{n!}\,\psi_n\,\delta_{m,n}, \tag{4.25}$$

where

$$d\,\lambda=\frac{(-1)^k}{k!}\,dx, \quad k=0\,,1\,,2,\ldots$$

$$\int_0^\infty P_m(x)\,P_n(x)\,d\mu(x)=(-1)^{n-1}\,m!\psi_n\,\delta_{m,n}, \tag{4.26}$$

where $d\mu(x)=\dfrac{(-1)^x\,e}{x!}\,dx.$

□

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