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On Convergence in Quaternion-Valued g-Metric Space

Kuaterniyon Değerli g-Metrik Uzayda Yakınsaklık Üzerine

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Abstract

This study presents and investigates the notion of convergence for double sequences in the quaternion-valued g-metric space, as well as a review of certain fundamental features. Moreover, statistical convergence in this context is examined and defined in detail. The final section, focusing on the relationship between the statistical convergence of quaternion-valued g-metric spaces and strong summability, delves into this connection and discusses its implications.

Keywords: Generalized metric spaces, quaternion space, statistical convergence, strong summability.

Öz

Bu makalede, kuaterniyon değerli g-metrik uzayda çift dizilerin yakınsama kavramı tanıtılıp incelenmekte, bazı temel özellikler de ele alınmaktadır. Ayrıca, bu bağlamda istatistiksel yakınsama ayrıntılı olarak incelenip tanımlanmaktadır. Son bölümde ise, kuaterniyon değerli g-metrik uzayların istatistiksel yakınsaması ile güçlü toplanabilirlik arasındaki ilişkiye odaklanılmakta ve bu bağlantının sonuçları tartışılmaktadır.

Anahtar Kelimeler: Genelleştirilmiş metrik uzaylar, kuaterniyon uzayı, istatistiksel yakınsaklık, güçlü toplanabilirlik.

1. Introduction

Fast (1951) conducted the first study on statistical convergence. Three individuals, Tripathy (2003), Mursaleen and Edely (2003), and Moricz (2003), each separately pioneered this field of study on statistical convergence in double sequences.

A distance function or metric expands on the concept of physical distance in mathematical analysis. Large and complex datasets provide a number of issues, hence Khamsi (2015) suggested a number of methods to expand this idea. A 2-metric, as proposed by Gähler (1966), is a more general version of the standard metric, although further studies have not found a connection between these functions. Ha et al. (1988), for example, showed that a 2-metric does not always show continuity with regard to its variables. These results led to Bapure Dhage's (1992) doctoral research on a new class of generalized metric spaces known as D-metric spaces. Determining these spaces' topological characteristics was the goal of Dhage (1992), and it proved essential for later studies in this area. However, studies by Mustafa and Sims (2003) and Naidu et al. (2005) have pointed out inaccuracies in many foundational claims regarding the basic topological characteristics of D-metric spaces, thereby undermining the validity of numerous results obtained in this area.

Mustafa and Sims (2006) pioneered the concept of G-metric space, whereas Choi et al. (2018) generalized this concept to degree l.

Let's review some basic symbols used in quaternion spaces. The quaternion space, denoted as $\mathbf{H}$, consists of four-dimensional real algebra with unity. The zero element of $\mathbf{H}$ is represented as $0_{\mathbf{H}}$, and the multiplicative identity is denoted as $1_{\mathbf{H}}$. Within $\mathbf{H}$, there are three specific imaginary units referred to as i, j, k. These units are defined by the following relationships:

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$i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

For each quaternion $\rho = y_0 + y_1 i + y_2 j + y_3 k$; where y_0, y_1, y_2 and y_3 are real numbers, the elements 1, i, j, k are considered as a basis for the real vector space **H**. Given $\rho = y_0 + y_1 i + y_2 j + y_3 k \in \mathbf{H}$, we recall that:

(i) $\overline{\rho} = y_0 - y_1 i - y_2 j - y_3 k$ is the conjugate quaternion of ρ,
(ii) $|\rho| = \sqrt{\rho\overline{\rho}} = \sqrt{y_0^2 + y_1^2 + y_2^2 + y_3^2} \in \mathbb{R}$

(iii) $\mathrm{Re}(\rho) = \frac{1}{2}(\rho + \overline{p}) = y_0 \in \mathbb{R}$

(iv) $\mathrm{Im}(\rho) = \frac{1}{2}(\rho - \overline{\rho}) = y_1 i + y_2 j + y_3 k$ is the imaginary part of ρ.

When $\rho = \mathrm{Re}(\rho)$, the element $\rho \in \mathbf{H}$ is termed as real. It is evident that ρ is real only if and only if $\rho = \overline{\rho}$. If $\overline{\rho} = -\rho$ or $\rho = \mathrm{Im}(\rho)$, ρ is considered imaginary.

The idea of a complex metric space was introduced by Azam et al. (2011) in the following way.

Definition 1.1. Assume that X is a nonempty set and that $d_C: X \times X \to \mathbf{C}$ is a mapping satisfying the following criteria:

(i) $0 \prec d_C(\mathbb{t}_1, \mathbb{t}_2)$, for all $\mathbb{t}_1, \mathbb{t}_2 \in X$ and $d_c(\mathbb{t}_1, \mathbb{t}_2) = 0$ if and only if $\mathbb{t}_1 = \mathbb{t}_2$,

(ii) $d_C(\mathbb{t}_1, \mathbb{t}_2) = d_C(\mathbb{t}_2, \mathbb{t}_1)$ for all $\mathbb{t}_1, \mathbb{t}_2 \in X$,

(iii) $d_C(\mathbb{t}_1, \mathbb{t}_2) \preceq d_C(\mathbb{t}_1, \mathbb{t}_3) + d_C(\mathbb{t}_3, \mathbb{t}_2)$ for all $\mathbb{t}_1, \mathbb{t}_2, \mathbb{t}_3 \in X$.

As a result, we refer to the pair (X, d_c) as a complex metric space.

Ahmed et al. (2014) broadened the preceding definition to encompass Clifford analysis in the subsequent manner:

Definition 1.2. Assume that X is a nonempty set and that $d_H: X \times X \to \mathbf{H}$ is a mapping satisfying the following criteria:

(i) $0 \prec d_H(\mathbb{t}_1, \mathbb{t}_2)$, for all $\mathbb{t}_1, \mathbb{t}_2 \in X$ and $d_H(\mathbb{t}_1, \mathbb{t}_2) = 0$ if and only if $\mathbb{t}_1 = \mathbb{t}_2$,

(ii) $d_H(\mathbb{t}_1, \mathbb{t}_2) = d_H(\mathbb{t}_2, \mathbb{t}_1)$ for all $\mathbb{t}_1, \mathbb{t}_2 \in X$,

(iii) $d_H(\mathbb{t}_1, \mathbb{t}_2) \preceq d_H(\mathbb{t}_1, \mathbb{t}_3) + d_H(\mathbb{t}_3, \mathbb{t}_2)$ for all $\mathbb{t}_1, \mathbb{t}_2, \mathbb{t}_3 \in X$.

Therefore, the pair (X, d_c) is termed a quaternion-valued metric space.

Ahmed et al. (2014) defined a partial order $\preceq$ on the space **H** (set of all quaternions).

Let $\rho_1, \rho_2 \in \mathbf{H}$, then $\rho_1 \preceq \rho_2$ if and only if $\mathrm{Re}(\rho_1) \leq \mathrm{Re}(\rho_2)$ and $\mathrm{Im}_s(\rho_1) \leq \mathrm{Im}_s(\rho_2), \rho_1, \rho_2 \in \mathbf{H}, s = i, j, k$ where $\mathrm{Im}\, m_i = b, \mathrm{Im}\, m_j = c, \mathrm{Im}\, m_k = d$. It was noted that $\rho_1 \preceq \rho_2$, if any of the following conditions hold:

(i) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_{s_1}(\rho_1) = \mathrm{Im}_{s_1}(\rho_2)$ where $s_1 = j, k, \mathrm{Im}_i(\rho_1) < \mathrm{Im}_i(\rho_2)$;

(ii) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_{s_2}(\rho_1) = \mathrm{Im}_{s_2}(\rho_2)$ where $s_2 = j, k, \mathrm{Im}_j(\rho_1) < \mathrm{Im}_j(\rho_2)$;

(iii) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_{s_3}(\rho_1) = \mathrm{Im}_{s_3}(\rho_2)$ where $s_3 = j, k, \mathrm{Im}_k(\rho_1) < \mathrm{Im}_k(\rho_2)$;

(iv) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_{s_1}(\rho_1) = \mathrm{Im}_{s_1}(\rho_2)$,
$\mathrm{Im}\, m_i(\rho_1) = \mathrm{Im}_i(\rho_2)$;

(v) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_{s_2}(\rho_1) = \mathrm{Im}_{s_2}(\rho_2)$,
$\mathrm{Im}\, m_j(\rho_1) = \mathrm{Im}\, m_j(\rho_2)$;

(vi) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_{s_3}(\rho_1) = \mathrm{Im}_{s_3}(\rho_2)$,
$\mathrm{Im}\, m_k(\rho_1) = \mathrm{Im}\, m_k(\rho_2)$;

(vii) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_s(\rho_1) < \mathrm{Im}_s(\rho_2)$;

(viii) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2), \mathrm{Im}_s(\rho_1) = \mathrm{Im}_s(\rho_2)$;

(ix) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2), \mathrm{Im}_{s_1}(\rho_1) = \mathrm{Im}_{s_1}(\rho_2)$,
$\mathrm{Im}_i(\rho_1) < \mathrm{Im}_i(\rho_2)$;

(x) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2), \mathrm{Im}_{s_2}(\rho_1) = \mathrm{Im}_{s_2}(\rho_2)$,
$\mathrm{Im}\, m_j(\rho_1) < \mathrm{Im}\, m_j(\rho_2)$;

(xi) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2), \mathrm{Im}_{s_3}(\rho_1) = \mathrm{Im}_{s_3}(\rho_2)$,
$\mathrm{I}\, m_k(\rho_1) < \mathrm{Im}\, m_k(\rho_2)$;

(xii) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2), \mathrm{Im}_{s_1}(\rho_1) < \mathrm{Im}_{s_1}(\rho_2)$,
$Im_i(\rho_1) = \mathrm{Im}\, m_i(\rho_2)$;

(xiii) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2), \mathrm{Im}_{s_2}(\rho_1) < \mathrm{Im}_{s_2}(\rho_2)$,
$\mathrm{Im}\, m_j(\rho_1) = \mathrm{Im}\, m_j(\rho_2)$;

(xiv) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_{s_3}(\rho_1) = \mathrm{Im}_{s_3}(\rho_2)$,
$Im_k(\rho_1) = \mathrm{Im}(\rho_k)$;

(xv) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2), \mathrm{Im}_s(\rho_1) < \mathrm{Im}_s(\rho_2)$;

(xvi) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2), \mathrm{Im}_s(\rho_1) = \mathrm{Im}_s(m_s)$.

Specifically, we denote $\rho_1 \precnsim \rho_2$ if $\rho_1 \neq \rho_2$ and any one of conditions (i) to (xvi) is satisfied, and $\rho_1 \prec \rho_2$ if only condition (xv) is satisfied.

Remark 1.1. It is important to emphasize that $\rho_1 \preceq \rho_2 \Rightarrow |\rho_1| \leq |\rho_2|$.

Inspired by the research of Ahmed et al. (2014), Adewale et al. (2019) proposed the following definition.

Definition 1.3. Let $G^Q: X \times X \times X \to \mathbf{H}$ be a function that meets the following conditions, and let X be a nonempty set, **H** a collection of quaternions:

(i) $G^Q(\alpha, \beta, \wp) = 0$ if and only if $\alpha = \beta = \wp$,

(ii) $0 \prec G^Q(\alpha,\alpha,\beta), \forall \alpha,\beta \in X$, with $\alpha \neq \beta$,

(iii) $G^Q(\alpha,\alpha,\beta) \preceq G^Q(\alpha,\beta,\wp), \forall \alpha,\beta,\wp \in X$, with $\wp \neq \beta$,

(iv) $G^Q(\alpha,\beta,\wp) = G^Q(\beta,\wp,\alpha) = G^Q(\alpha,\wp,\beta) = \cdots$ (symmetry),

(v) A real number $m \geq 1$ exists such that

$G^Q(\alpha,\beta,\wp) \preceq m[G^Q(\alpha,y,y) + G^Q(y,\beta,\wp)],$ ·

$\forall y,\alpha,\beta,\wp \in X$

The G^Q-metric space is therefore represented as (X, G^Q), and the function G^Q is referred to as a quaternion G-metric. When each Cauchy sequence in a G^Q-metric space converges under G^Q, the space is said to be complete. The following extends the concept of G-metric space to degree l.

Definition 1.4. Assume that X is a non-empty set. If a function $g: X^{l+1} \to \mathbb{R}^+$ satisfies the following requirements, it is defined as a g-metric space with order l on:

(i) $g(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,...,\mathbb{t}_l) = 0$ iff $\mathbb{t}_0 = \mathbb{t}_1 = \cdots \mathbb{t}_l$,

(ii) $g(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,....,\mathbb{t}_l) = g(\mathbb{t}_{\sigma(0)},\mathbb{t}_{\sigma(1)},\mathbb{t}_{\sigma(2)},....,\mathbb{t}_{\sigma(l)})$ for permutation σ on $\{0,1,2,...l\}$,

(iii) $g(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,...,\mathbb{t}_l) \leq g(\mathbb{v}_0,\mathbb{v}_1,\mathbb{v}_2,...,\mathbb{v}_l)$ for each $(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,\ldots,\mathbb{t}_l),(\mathbb{v}_0,\mathbb{v}_1,\mathbb{v}_2,\ldots,\mathbb{t}_l) \in X^{l+1}$ with $\{\mathbb{t}_j : i = 0,1,...l\} \subseteq \{\mathbb{v}_j : i = 0,1,...l\}$,

(iv) For each $\mathbb{t}_0,\mathbb{t}_1,\ldots,\mathbb{t}_\eta,\mathbb{v}_0,\mathbb{v}_1,\ldots\mathbb{v}_\mathfrak{z},u \in X$ with $\eta + \mathfrak{z} + 1 = l$

$$g(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,...,\mathbb{t}_\eta,\mathbb{v}_0,\mathbb{v}_1,\mathbb{v}_2,...,\mathbb{v}_\mathfrak{z}) \leq g(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,...,\mathbb{t}_\eta,u,u,...,u) + g(\mathbb{v}_0,\mathbb{v}_1,\mathbb{v}_2,...,\mathbb{v}_\eta,u,u,...,u).$$

A g-metric space of degree is designated for the pair (X, g). When $l = 1, 2$ it corresponds to a metric space and a G-metric space, respectively.

Recall that a subset T of the set of natural numbers, $\mathbb{N}$ possesses a "natural density" $\delta(T)$ if it fulfills the subsequent conditions:

$$\delta(T) = \lim_{\mathbb{v}\to\infty} \frac{1}{\mathbb{v}} \left| \{\eta \leq \mathbb{v} : \eta \in T\} \right|.$$

The sequence $\mathbb{t} = (\mathbb{t}_\eta)$ considered statistically convergent to number L if, for every $\varrho > 0$

$$\lim_{\mathbb{v}\to\infty} \frac{1}{\mathbb{v}} \left| \{\eta \leq \mathbb{v} : |\mathbb{t}_\eta - L| \geq \varrho\} \right| = 0,$$

and x is termed a statistically Cauchy sequence if, for all $\varrho > 0$ there exists a number $Q = Q(\varepsilon)$ so that

$$\lim_{\mathbb{v}\to\infty} \frac{1}{\mathbb{v}} \left| \{\eta \leq \mathbb{v} : |\mathbb{t}_\eta - \mathbb{t}_Q| \geq \varrho\} \right| = 0.$$

Abazari (2021) provided the following definition.

Definition 1.5. Let $p \in \mathbb{N}$ and define

$T(\alpha) = \{(r_0,r_1,r_2,...r_p) \leq \alpha (\alpha \in \mathbb{N}) : (r_0,r_1,r_2,...r_p) \in T\}.$

Then, the p- dimensional asymptotic (or natural) density of the set T denoted by $\delta_{(p)}(T)$ is defined as:

$$\delta_{(p)}(T) = \lim_{\alpha\to\infty} \frac{p!}{\alpha^p} |T(\alpha)|.$$

Now, we recall the definition of quaternion-valued g-metric space in this section, along with some fundamental characteristics (see, Jan and Jalal (2023)).

Definition 1.6. Assume that X is a non-empty set. A function $g_{\mathrm{H}}: X^{p+1} \to \mathbf{H}$ (where $\mathbf{H}$ is the space of quaternions) is called quaternion valued g-metric space with order p on X if it satisfies the following criterias :

(i) $g_{\mathrm{H}}(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,...,\mathbb{t}_p) = 0$ ifandonlyif $\mathbb{t}_0 = \mathbb{t}_1 = \cdots = \mathbb{t}_p$,

(ii) $g_{\mathrm{H}}(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,...,\mathbb{t}_p) = g_{\mathrm{H}}(\mathbb{t}_{\sigma(0)},\mathbb{t}_{\sigma(1)},\mathbb{t}_{\sigma(2)},...,\mathbb{t}_{\sigma(p)})$ for permutation σ on $\{0,1,2,...p\}$,

(iii) $g_{\mathrm{H}}(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,...,\mathbb{t}_p) \preceq g_{\mathrm{H}}(\mathbb{v}_0,\mathbb{v}_1,\mathbb{v}_2,...,\mathbb{v}_p)$

for all $(\mathbb{t}_0,\mathbb{t}_1,\mathbb{t}_2,\ldots,\mathbb{t}_p),(\mathbb{v}_0,\mathbb{v}_1,\mathbb{v}_2,\ldots,\mathbb{v}_p) \in X^{p+1}$ with $\{\mathbb{t}_i : i = 0,1,\ldots p\} \subsetneq \{\mathbb{v}_i : i = 0,1,\ldots p\}$,

(iv) For all $\mathbb{t}_0,\mathbb{t}_1,\ldots,\mathbb{t}_\eta,\mathbb{v}_0,\mathbb{v}_1,\ldots,\mathbb{v}_\delta,v \in X$ with $\eta + \mathfrak{z} + 1 = p$

$$g_{\mathrm{H}}(\mathbb{t}_0,\mathbb{t}_1,...,\mathbb{t}_\eta,\mathbb{v}_0,\mathbb{v}_1,...\mathbb{v}_\mathfrak{z}) \preceq g_{\mathrm{H}}(\mathbb{t}_0,\mathbb{t}_1,...,\mathbb{t}_\eta,v,v,...v) + g_{\mathrm{H}}(\mathbb{v}_0,\mathbb{v}_1,...,\mathbb{v}_\mathfrak{z},v,v...v)$$

We call the pair (X, g_{H}) a quaternion-valued g_{H}-metric space of degree p. When $p = 1$ and $p = 2$, respectively, it corresponds to quaternion-valued metric space and quaternion-valued G-metric space.

The following theorem demonstrates that quaternion-valued g-metrics extend the concepts of quaternion-valued metric and quaternion-valued G-metric.

Theorem 1.1. Assume X is a given, non-empty set. The following claims are true:

(a) d_{H} is a quaternion valued g-metric of order 1 on X iff d_{H} is a quaternion valued metric on X.

(b) G_{H} is a quaternion valued g-metric of order 2 on X iff G_{H} is a G-metric on X with quaternion values.

These criteria are equivalent to those of metric spaces with quaternion values and G-metric spaces with quaternion values. As a result, quaternion-valued metric and quaternion-valued G-metric spaces are equivalent to quaternion-valued g-metrics of order 1 and 2, respectively.

Proposition 1.1. Consider $(X, g_{\mathbf{H}})$ and $(X, \overline{g}_{\mathbf{H}})$ as quaternion g-metric spaces. Quaternion-valued g-metrics on X are the functions indicated by d_H, are.

(1) $d_{\mathbf{H}}(\mathbb{t}_0, \mathbb{t}_1, ..., \mathbb{t}_p) = g_{\mathbf{H}}(\mathbb{t}_0, \mathbb{t}_1, ..., \mathbb{t}_p) + \overline{g}_{\mathbf{H}}(\mathbb{t}_0, \mathbb{t}_1, ..., \mathbb{t}_p)$

(2) $d_{\mathbf{H}}(\mathbb{t}_0, \mathbb{t}_1, ..., \mathbb{t}_p) = \Phi(d_{\mathbf{H}}(\mathbb{t}_0, \mathbb{t}_1, ..., \mathbb{t}_p))$ where Φ is a function on $\mathbf{H}$

(i) Φ is increasing on $\mathbf{H}$;

(ii) $\Phi(0) = 0$;

(iii) $\Phi(\mathbb{t} + \mathbb{v}) \preceq \Phi(\mathbb{t}) + \Phi(\mathbb{v})$ for all $\mathbb{t}, \mathbb{v} \in \mathbf{H}$

Example 1.1. (Discrete quaternion valued g-metric space) Consider a nonempty set X. Define $d_{\mathbf{H}}: X^{p+1} \to \mathbf{H}$ as follows:

$$d_{\mathbf{H}}(\mathbb{t}_0, \mathbb{t}_1, ...\mathbb{t}_p) = \begin{cases} 0, & if\ \mathbb{t}_0 = \mathbb{t}_1 = \cdots = \mathbb{t}_p \\ 1, & otherwise. \end{cases}$$

for all $\mathbb{t}_0, \mathbb{t}_1, ..., \mathbb{t}_p \in X$. This function $d_{\mathbf{H}}$ represents a quaternion-valued g-metric on X.

Theorem 1.2. On a nonempty set X, let $(X, g_{\mathbf{H}})$ represent a quaternion-valued g-metric of order n. The following claims are accurate:

1. $g_{\mathbf{H}}(\underbrace{\mathbb{t}, ..., \mathbb{t}}_{\eta \text{ times}}, y, ..., y) \preceq g_{\mathbf{H}}(\underbrace{\mathbb{t}, ..., \mathbb{t}}_{\eta \text{ times}}, v, ..., v) + g_{\mathbf{H}}(\underbrace{\mathbb{t}, ..., \mathbb{t}}_{\eta \text{ times}}, y, ..., y)$,

2. $g_{\mathbf{H}}(\underbrace{\mathbb{t}, ..., \mathbb{t}}_{\eta \text{ times}}, v, ..., v) \preceq \eta g_{\mathbf{H}}(\mathbb{t}, v, ..., v)$ and

$g_{\mathbf{H}}(\underbrace{\mathbb{t}, ..., \mathbb{t}}_{\eta \text{ times}}, v, ..., v) \preceq (n + 1 - \eta) g_{\mathbf{H}}(v, \mathbb{t}, ..., \mathbb{t})$,

3. $g_{\mathbf{H}}(\mathbb{t}_0, \mathbb{t}_1, ..., \mathbb{t}_n) \preceq \sum_{i=0}^{n} g_{\mathbf{H}}(\mathbb{t}_i, v, ..., v)$,

4. $g_{\mathbf{H}}(\mathbb{t}, v, ..., v) \preceq (1 + (\eta - 1)(n + 1 - \eta)) g_{\mathbf{H}}(\underbrace{\mathbb{t}, ..., \mathbb{t}}_{\eta \text{ times}}, v, ..., v)$.

Definition 1.7. Let $(X, g_{\mathbf{H}})$ be a g-metric space with quaternion values.

(i) Any point $\mathbb{t}_0 \in X$ is said to be the interior of a set $A \subset X$, if there exists $q \in \mathbf{H}: 0 \prec q$ such that

$B_{g_{\mathbf{H}}}(\mathbb{t}_0, q) := \{y \in X \mid g_{\mathbf{H}}(\mathbb{t}_0, y, y, ...y) \prec q\} \subset A$.

(ii) When every point in subset A is an interior point in subset A, then A is said to be open in subset X.

Here, we investigate the notion of convergence for double sequences in the context of g-metric spaces with quaternion values and discuss some of their basic characteristics. This study is significant as it extends the traditional framework of metric spaces by incorporating quaternion values, which are essential in various applications such as signal processing, control theory, and three-dimensional computer graphics. We meticulously examine statistical convergence within this context, providing a detailed definition and analysis. By doing so, we contribute to the mathematical foundation necessary for advanced theoretical research and practical applications in areas where quaternion-valued functions are prevalent. Furthermore, our investigation culminates in the final section, where we explore the intricate relationship between statistical convergence in quaternion-valued g-metric spaces and the concept of strong summability. This exploration not only connects theoretical concepts but also sheds light on their practical implications, offering new insights and tools for researchers and practitioners working with complex multidimensional data structures. The results of this study can potentially lead to new methodologies and algorithms that improve the efficiency and accuracy of computational processes in various scientific and engineering fields.

2. New Concepts

This section defines quaternion valued g-metric space and goes over some of its fundamental properties.

Definition 2.1. Assume that $(X, g_{\mathbf{H}})$ is a quaternion valued g-metric space, $\mathbb{t} \in X$ is a point and $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\} \subseteq X$ is a sequence. $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\} g_{\mathbf{H}}$ -converges to $\mathbb{t}$ denoted by $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\} \xrightarrow{g_{\mathbf{H}}} \mathbb{t}$ if for every $0 \prec q \in \mathbf{H}$ there exists $N \in \mathbb{N}$ such that

$$r_1, r_2, ...r_p \geq N, u_1, u_2, ...u_p \geq N \Rightarrow g_{\mathbf{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec q$$

In such a situation, $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ is called $g_{\mathbf{H}}$-convergent in X and $\mathbb{t}$ is said to be $g_{\mathbf{H}}$-limit of $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$.

Definition 2.2. Let $(X, g_{\mathbf{H}})$ be a quaternion valued g-metric space. Let $\mathbb{t} \in X$ be a point and $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\} \subseteq X$ be a sequence. $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ is called $g_{\mathbf{H}}$-Cauchy if for all $q \in \mathbf{H}, 0 \prec q$, there exists $N \in \mathbb{N}$ such that $r_0, r_1, r_2, ...r_p \geq N, u_0, u_1, u_2, ...u_p \geq N$
$\Rightarrow g_{\mathbf{H}}(\mathbb{t}_{r_0u_0}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec q$.

Definition 2.3. Assume that $(X, g_{\mathbf{H}})$ is a g-metric space with quaternion values. If all $g_{\mathbf{H}}$-Cauchy sequence in $(X, g_{\mathbf{H}})$ is $g_{\mathbf{H}}$-convergent in $(X, g_{\mathbf{H}})$, then $(X, g_{\mathbf{H}})$ is complete.

Proposition 2.1. The following claims are accurate:

(a) In a quaternion valued g-metric space, the limit of a $g_{\mathbf{H}}$ -convergent sequence is unique.

(b) In a quaternion valued g-metric space, every convergent sequence is a $g_{\mathbf{H}}$ Cauchy sequence.

Proof. (a) Assume that $(X, g_{\mathbf{H}})$ is a quaternion valued g-metric space and $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\} \subseteq X$. By Definition 2.1 for $0 \prec q \in \mathbf{H}$, there exists N_1 and N_2 such that

$g_{\mathbb{H}}(\alpha, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec \frac{q}{w+1}$ for all

$r_1, r_2, ..., r_p > N_1,\ u_1, u_2, ...u_p > N_1,$

$g_{\mathbb{H}}(\beta, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec \frac{q}{w+1}$ for all

$r_1, r_2, ..., r_p > N_2,\ u_1, u_2, ...u_p > N_2.$

Set $N = \max\{N_1, N_2\}$. If $m, n \geq N$, then by the condition (iv) of definition 1.6 and Theorem 1.1, it follows that

$$|g_{\mathbb{H}}(\alpha, \beta, \beta, .., \beta)| \leq |g_{\mathbb{H}}(\alpha, \mathbb{t}_{mn}, \mathbb{t}_{mn}, .., \mathbb{t}_{mn})|$$
$$+|g_{\mathbb{H}}(\mathbb{t}_{mn}, \beta, \beta, ..., \beta)| \leq |g_{\mathbb{H}}(\alpha, \mathbb{t}_{mn}, \mathbb{t}_{mn}, .., \mathbb{t}_{mn})|$$
$$+w|g_{\mathbb{H}}(\beta, \mathbb{t}_{mn}, \mathbb{t}_{mn}, .., \mathbb{t}_{mn})| < \frac{q}{w+1} + \frac{wq}{w+1} = q.$$

Since $0 \prec q \in \mathbf{H}$ is arbitrary, $g_{\mathbb{H}}(\alpha, \beta, \beta, .., \beta) = 0$. Hence $\alpha = \beta$.

(b) Assume that $(X, g_{\mathbb{H}})$ is a quaternion valued g-metric space and $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\} \subseteq X$ is a sequence that

$g_{\mathbb{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec \frac{q}{w+1}$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$.

Following from Theorem 1.2 and quaternion valued g-metric space's monotonicity requirement,

$$g_{\mathbb{H}}(\mathbb{t}_{r_0u_0}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \preceq \sum_{k=0}^{w} g_{\mathbb{H}}(\mathbb{t}_{r_ku_k}, \mathbb{t}, ...\mathbb{t}) \preceq \sum_{k=0}^{w} \frac{q}{w+1} = q.$$

Thus, $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ is $g_{\mathbb{H}}$-Cauchy in $(X, g_{\mathbb{H}})$.

Proposition 2.2. Assume that $(X, g_{\mathbb{H}})$ is a quaternion valued g-metric space and $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ is a sequence in X. Then $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ converges to $\mathbb{t}$ iff

$|g_{\mathbb{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \to 0$ as $\{r_1, r_2, ...r_p\} \to \infty$ and $\{u_1, u_2, ...u_p\} \to \infty$.

Proof. Suppose that $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ converges to $\mathbb{t}$. Assume $q = \frac{\varrho}{2} + i\frac{\varrho}{2} + j\frac{\varrho}{2} + k\frac{\varrho}{2}$ and a real number $\varrho > 0$. Thus, $0 \prec q \in \mathbf{H}$ and there is natural number N such that $g_{\mathbb{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| < |q| = \varrho$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$. Hence $|g_{\mathbb{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \to 0$ as $\{r_1, r_2, ...r_p\} \to \infty$ and $\{u_1, u_2, ...u_p\} \to \infty$.

Conversely, suppose that $|g_{\mathbb{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \to 0$ as $\{r_1, r_2, ...r_p\} \to \infty$ and $\{u_1, u_2, ...u_p\} \to \infty$. Then, given $q \in \mathbf{H}$ with $0 \prec q$, there is a real number $\delta > 0$, such that, for $h \in \mathbf{H}$,

$|h| < \delta \Rightarrow h \prec q$.

For this δ, there is a natural number N such that $|g_{\mathbb{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| < \delta$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$. Implying that $g_{\mathbb{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec q$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$, hence $\{t_{r_1u_1}, t_{r_2u_2}, ..., t_{r_pu_p}\}$ converges to $\mathbb{t}$.

Proposition 2.3. Assume that $(X, g_{\mathbb{H}})$ is a quaternion valued g-metric space and $\{t_{r_1u_1}, t_{r_2u_2}, ..., t_{r_pu_p}\}$ is a sequence in X. Then $\{t_{r_1u_1}, t_{r_2u_2}, ..., t_{r_pu_p}\}$ is a Cauchy sequence if and only if

$|g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \to 0$ as $r_1, u_1, r_2, u_2 ...r_p, u_p \to \infty$.

Proof. Suppose that $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ converges to $\mathbb{t}$. Assume $q = \frac{\varrho}{2} + i\frac{\varrho}{2} + j\frac{\varrho}{2} + k\frac{\varrho}{2}$ and a real number $\varrho > 0$. Thus, $0 \prec q \in \mathbf{H}$ and there is natural number N such that $g_{\mathbb{H}}(\mathbb{t}_{r_0u_0}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec q$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$. Therefore, $|g_{\mathbb{H}}(\mathbb{t}_{r_0u_0}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| < |q| = \varrho$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$. Hence $|g_{\mathbb{H}}(t_{r_0u_0}, t_{r_1u_1}, t_{r_2u_2}, ..., t_{r_pu_p})| \to 0$ as $\{r_1, r_2, ...r_p\} \to \infty$ and $\{u_1, u_2, ...u_p\} \to \infty$.

Conversely, suppose that $|g_{\mathbb{H}}(\mathbb{t}_{r_0u_0}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \to 0$ as $\{r_1, r_2, ...r_p\} \to \infty$ and $\{u_1, u_2, ...u_p\} \to \infty$. Then, given $q \in \mathbf{H}$ with $0 \prec q$, there is a real number $\delta > 0$, such that, for $h \in \mathbf{H}$,

$|h| < \delta \Rightarrow h \prec q$.

For this δ, there is a natural number N such that $|g_{\mathbb{H}}(\mathbb{t}_{r_0u_0}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| < \delta$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$. Implying that $g_{\mathbb{H}}(\mathbb{t}_{r_0u_0}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec q$ for all $\{r_1, r_2, ...r_p\} > N$ and $\{u_1, u_2, ...u_p\} > N$, hence $\{t_{r_1u_1}, t_{r_2u_2}, ..., t_{r_pu_p}\}$ is a Cauchy sequence to $\mathbb{t}$.

Definition 2.4. Assume that $(X, g_{\mathbb{H}})$ is a quaternion valued g-metric space and $0 \prec q \in \mathbf{H}$ is given.

(i) A set $A \subset X$ is said to be $q, g_{\mathbb{H}}$-net of $(X, g_{\mathbb{H}})$ if for $\mathbb{t} \in X$, there exists $a \in A$ such that $\mathbb{t} \in B_{g\mathbb{H}}(a, q)$. A set is referred to as finite $q, g_{\mathbb{H}}$-net of $(X, g_{\mathbb{H}})$ if it is finite.

(ii) A quaternion valued g-metric space $(X, g_{\mathbb{H}})$ is called totally $g\mathbf{H}$-bounded if for all $0 \prec q \in \mathbf{H}$ there exists a finite $q, g_{\mathbb{H}}$-net.

(iii) If a quaternion valued g-metric space $(X, g_{\mathbb{H}})$ is complete and totally $g_{\mathbb{H}}$-bounded, it is referred to as $g_{\mathbb{H}}$-compact.

Definition 2.5. Let $(X, g_{\mathbb{H}_1})$ and $(X, g_{\mathbb{H}_2})$ be quaternion valued g-metric spaces. A mapping $T: X_1 \to X_2$ is said to be $g_{\mathbb{H}}$-continuous at point $\mathbb{t} \in X_1$ provided that for each open ball $B_{g_{\mathbb{H}_2}}(T(x), q)$ there exists an open ball $B_{g_{\mathbb{H}_1}}(x, \delta) \subseteq B_{g_{\mathbb{H}_2}}(T(x), q)$.

3. Statistical Convergence of Double Sequences in Quaternion Valued g-metric Space

In this section, we introduce statistical convergence for double sequences in quaternion-valued g-metric space and give several key characteristics.

Definition 3.1. Let $(X, g\mathbf{H})$ be a quaternion valued g-metric space, $\mathbb{t} \in X$ be a point, and $\{\mathbb{t}_{\alpha\beta}\} \subseteq X$ be a sequence.

(i) $\{\mathbb{t}_{\alpha\beta}\}$ statistically converges to $\mathbb{t}$ if for each $q \prec \mathbf{H}$ with $0 \prec q$ such that

$$\lim_{\alpha,\beta\to\infty} \frac{p!}{(\alpha\beta)^p} \Big| \{((r_1, r_2, ..., r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ..., r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}) : | g_{\mathrm{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) | \geq | q | \} \Big| = 0.$$

and denoted by $g\mathbf{H}(\eta\mathbb{t}) - \lim_{\alpha,\beta\to\alpha} \{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\} = \mathbb{t}$ or $\{\mathbb{t}_{\alpha\beta}\} \xrightarrow{g\mathrm{H}(\eta\mathbb{t})} \mathbb{t}$.

(ii) $\{\mathbb{t}_{\alpha\beta}\}$ is said to be statistically g_{H}-Cauchy if for every $q \in \mathbf{H}$ with $0 \prec q$, there exists $r_m, u_n \prec \mathbf{H}$ such that

$$\lim_{\alpha,\beta\to\infty} \frac{p!}{(\alpha\beta)^p} \Big| \{((r_1, r_2, ..., r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ..., r_p \leq\leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}) : | g_{\mathrm{H}}(x_{r_mu_n}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) | \geq | q | \} \Big| = 0.$$

$(X, g\mathbf{H})$ is called a complete quaternion valued -metric space.

(iii) $\{\mathbb{t}_{\alpha\beta}\}$ is bounded if there exists a positive number M such that $\left| \{t_{r_1u_1}, t_{r_2u_2}, ..., t_{r_pu_p}\} \right| \leq M$ for all $\{(r_1, r_2, ..., r_p), (u_1, u_2, ..., u_p)\}$.

The set of all bounded sequences will be represented by ℓ_∞.

Theorem 3.1. If a sequence $\{\mathbb{t}_{\alpha\beta}\}$ is statistically convergent in $(X, g\mathbf{H})$ then $g\mathbf{H}(\eta\mathbb{t}) - \lim(\mathbb{t}_{\alpha\beta})$ is unique.

Proof. Suppose that $\{\mathbb{t}_{\alpha\beta}\}$ statistically converges in $(X, g\mathbf{H})$. Let $g\mathbf{H}(\eta\mathbb{t}) - \lim(\mathbb{t}_{\alpha\beta}) = \gamma_1$ and $g\mathbf{H}(\eta\mathbb{t}) - \lim(\mathbb{t}_{\alpha\beta}) = \gamma_2$.

Given $\varepsilon > 0$ and $0 \prec q \in \mathbf{H}$, let

$$q = \frac{\varrho}{4p} + i\frac{\varrho}{4p} + j\frac{\varrho}{4p} + k\frac{\varrho}{4p}.$$

Define the following sets as:

$$K_1(\varrho) = \Big\{((r_1, r_2, ..., r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ..., r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}) : \left| g_{\mathrm{H}}(\gamma_1, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \right| \geq | q | = \frac{\varrho}{2p} \Big\},$$

$$K_2(\varrho) = \Big\{((r_1, r_2, ..., r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ..., r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}) : \left| g_{\mathrm{H}}(\gamma_2, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \right| \geq | q | = \frac{\varrho}{2p} \Big\}.$$

Since $g_{\mathrm{H}}(\eta\mathbb{t}) - \lim(\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) = \gamma_1$, we have $\delta(K_1(\varrho)) = 0$.

Similarly $g_{\mathrm{H}}(\eta\mathbb{t}) - \lim(\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) = \gamma_2$, implies $\delta(K_2(\varrho)) = 0$.

Let $K(\varrho) = K_1(\varrho) \cup K_2(\varrho)$. Then $\delta(K(\varrho)) = 0$ and we have $K^c(\varrho)$ is non-empty and $\delta(K^c(\varrho)) = 1$. Suppose $\{r_1, r_2, ..., r_p\}, \{u_1, u_2, ..., u_p\} \in K^c(\varrho)$, then by Theorem 1.2, we have:

$$\begin{aligned}
&| g_{\mathrm{H}}(\gamma_1, \gamma_2, \gamma_2 ..., \gamma_2) | \leq | g_{\mathrm{H}}(\gamma_1, \mathbb{t}_{mn}, \mathbb{t}_{mn}, ..., \mathbb{t}_{mn}) | \\
&+ | g_{\mathrm{H}}(\mathbb{t}_{mn}, \gamma_2, \gamma_2, ..., \gamma_2) | \leq | g_{\mathrm{H}}(\gamma_1, \mathbb{t}_{mn}, \mathbb{t}_{mn}, ..., \mathbb{t}_{mn}) | \\
&+ p | g_{\mathrm{H}}(\gamma_2, \mathbb{t}_{mn}, \mathbb{t}_{mn}, ..., \mathbb{t}_{mn}) | \leq | g_{\mathrm{H}}(\gamma_1, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) | \\
&+ p | g_{\mathrm{H}}(\gamma_2, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) | \leq p | g_{\mathrm{H}}(\gamma_1, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) | \\
&+ p | g_{\mathrm{H}}(\gamma_2, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) | \leq p\left(\frac{\varrho}{2p} + \frac{\varrho}{2p}\right) = \varrho.
\end{aligned}$$

Since $\varrho > 0$ was arbitrary, we get $g_{\mathrm{H}}(\gamma_1, \gamma_2, \gamma_2 ..., \gamma_2) = 0$, therefore $\gamma_1 = \gamma_2$.

Theorem 3.2. If $g_{\mathrm{H}} - \lim(\mathbb{t}_{\alpha\beta}) = \mathbb{t}$, then $g_{\mathrm{H}}(\eta\mathbb{t}) - \lim(\mathbb{t}_{\alpha\beta}) = \mathbb{t}$ the opposite need not always hold.

Proof. Assume that $g_{\mathrm{H}} - \lim(\mathbb{t}_{\alpha\beta}) = \mathbb{t}$. Thus for all $0 \prec q \in \mathbf{H}$ there exists $N \in \mathbb{N}$ such that $r_1, r_2, ... r_p \geq N, u_1, u_2, ... u_p \geq N \Rightarrow g_{\mathrm{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) \prec q$.

The set

$$A(\varrho) = \{((r_1, r_2, ... r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ... r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}) : | g_{\mathrm{H}}(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) | \geq | q | = \varrho\} \subset \{(1,1), (2,2), (3,3), ...\}^p,$$

where $q = \frac{\varrho}{2} + i\frac{\varrho}{2} + j\frac{\varrho}{2} + k\frac{\varrho}{2}, \delta(A(\varrho)) = 0$. Hence $g_{\mathrm{H}}(\eta\mathbb{t}) - \lim(\mathbb{t}_{\alpha\beta}) = \mathbb{t}$.

The following example demonstrates that the reverse does not have to be true.

Example 3.1. Let $X = \mathbb{R}$ and $G_{\mathrm{H}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbf{H}$ be a quaternion valued G-metric space defined by

$$\begin{aligned}
G_{\mathrm{H}}(p_1, p_2, p_3) &= | z_0^1 - z_0^2 | + | z_0^1 - z_0^3 | \\
&+ i(| z_1^1 - z_1^2 | + | z_1^1 - z_1^3 | + | z_1^2 - z_1^3 |) \\
&+ j(| z_2^1 - z_2^2 | + | z_2^1 - z_2^3 | + | z_2^2 - z_2^3 |) \\
&+ k(| z_3^1 - z_3^2 | + | z_3^1 - z_3^3 | + | z_3^2 - z_3^3 |)
\end{aligned}$$

where $y_r = y_0^r + y_0^r i + y_0^r j + y_0^r k$ for $r = 1, 2, 3$. Let $\mathbb{t}_{mn} = \{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ be a sequence defined as

$$\mathbb{t}_{mn} = \begin{cases} mn, & \textit{if } m, n \textit{ is a square} \\ 1, & \textit{if not.} \end{cases}$$

It is easy to see that $g_{\mathrm{H}}(\eta\mathbb{t}) - \lim(\mathbb{t}_{mn}) = 1$, since $|(m, n) \in \mathbb{N}^p, m \leq \alpha, n \leq \beta\,(\alpha, \beta \in \mathbb{N}) : | g_{\mathrm{H}}(1, \mathbb{t}_{mn}) | > | q | = \varrho | \leq \sqrt{mn}$ for every $\varrho > 0$ and $q = \frac{\varrho}{2} + i\frac{\varrho}{2} + j\frac{\varrho}{2} + k\frac{\varrho}{2}$. But $\{\mathbb{t}_{mn}\}$ is neither convergent nor bounded.

Theorem 3.3. Allow (X, g_H) to be the complete g-metric space with quaternion values. A sequence $\{\mathbb{t}_{\alpha\beta}\}$ of points in (X, g_H) is considered statistically g-convergent iff it is statistically g_H-Cauchy.

Proof. Assume $g_H - \lim(\mathbb{t}_{\alpha\beta}) = \mathbb{t}$. So, we have $\delta(A(\varrho)) = 0$, where

$$A(\varrho) = \{((r_1, r_2, ...r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ...r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}): |g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \geq |q| = \frac{\varrho}{2}\},$$

where $q = \frac{\varrho}{4} + i\frac{\varrho}{4} + j\frac{\varrho}{4} + k\frac{\varrho}{4}$.

This implies that

$$\delta(A^c(\varrho)) = \{((r_1, r_2, ...r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ...r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}): |g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| < |q| = \frac{\varrho}{2}\} = 1.$$

Let $(j_1, j_2, ..., j_\eta), (k_1, k_2, ..., k_\eta) \in A^c(\varrho)$. Then $|g_H(\mathbb{t}, \mathbb{t}_{j_1k_1}, \mathbb{t}_{j_2k_2}, ..., \mathbb{t}_{j_\eta k_\eta})| < |q| = \frac{\varrho}{2}$.

Let

$$B(\varrho) = \{((r_1, r_2, ...r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ...r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}): |g_H(\mathbb{t}_{j_1k_1}, \mathbb{t}_{j_2k_2}, ..., \mathbb{t}_{j_\eta k_\eta}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \geq \varrho\},$$

we need to show that $B(\varrho) \subset A(\varrho)$ Let $\alpha, \beta \in B(\varrho)$. Then $|g_H(\mathbb{t}_{j_1k_1}, \mathbb{t}_{j_2k_2}, ..., \mathbb{t}_{j_\eta k_\eta}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \geq \varrho$ and hence $|g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \geq \frac{\varrho}{2}$.

That is $r_1, r_2, ...r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}) \in A(\varrho)$.

Otherwise if $|g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \leq \varrho$ then

$$\varrho \leq |g_H(\mathbb{t}_{j_1k_1}, \mathbb{t}_{j_2k_2}, ..., \mathbb{t}_{j_\eta k_\eta}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \leq g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}) + (\mathbb{t}, \mathbb{t}_{j_1k_1}, \mathbb{t}_{j_2k_2}, ..., \mathbb{t}_{j_\eta k_\eta}) < \frac{\varrho}{2} + \frac{\varrho}{2} = \varrho$$

which is not possible. Hence $B(\varrho) \subset A(\varrho)$, which implies that $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ is $g_H(\eta\mathbb{t})$-Cauchy.

Conversely, suppose that $\{\mathbb{t}_{\alpha\beta}\}$ is $g_H(\eta\mathbb{t})$-Cauchy but not $g_H(\eta\mathbb{t})$-convergent. So, there exists $(j_1, j_2, ..., j_\eta), (k_1, k_2, ..., k_\eta) \in \mathbb{N}^p \times \mathbb{N}^p$ such that $\delta(G(\varrho)) = 0$ where

$$G(\varrho) = \{((r_1, r_2, ...r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ...r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}): |g_H(\mathbb{t}_{j_1k_1}, \mathbb{t}_{j_2k_2}, ..., \mathbb{t}_{j_\eta k_\eta}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \geq \varrho\}$$

and $\delta(D(\varrho)) = 0$, where

$$D(\varrho) = \{((r_1, r_2, ...r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ...r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}): |g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| < \frac{\varrho}{2}\},$$

that is, $\delta(D^c(\varrho)) = 1$.

Since

$$|g_H(\mathbb{t}_{j_1k_1}, \mathbb{t}_{j_2k_2}, ..., \mathbb{t}_{j_\eta k_\eta}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \leq 2|g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \leq \varrho$$

if $|g_H(\mathbb{t}, \mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p})| \leq \frac{\varrho}{2}$.

Therefore $\delta(G^c(\varrho)) = 0$ that is $\delta(G(\varrho)) = 1$, which leads the contradiction, since $\{\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}\}$ was $g_H(\eta\mathbb{t})$-Cauchy. Hence $\{\mathbb{t}_{\alpha\beta}\}$ is $g_H(\eta\mathbb{t})$-convergent.

4. Strong Summability

The relationship between g_H-statistical convergence and strong summability in quaternion valued g-metric space is established in this section.

Definition 4.1. A sequence $\{\mathbb{t}_{\alpha\beta}\}$ is called strongly p-Cesàro summable $(0 < p < \infty)$ to limit $\mathbb{t}$ in (X, g_H) if

$$\lim_{\alpha,\beta\to\infty} \frac{p!}{(\alpha\beta)^p} \sum_{r_1, r_2, ..., r_p = 1}^{\alpha} \sum_{u_1, u_2, ..., u_p = 1}^{\beta} (g_H(\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}, \mathbb{t})^p) = 0.$$

and we write it as $\mathbb{t}_{\alpha\beta} \to \mathbb{t}[C_1, g_H]_p$. Here, l represents the $[C_1, g_H]_p$-limit of $\{\mathbb{t}_{\alpha\beta}\}$.

Theorem 4.1. (a). If $0 < p < \infty$ and $\mathbb{t}_{\alpha\beta} \to \mathbb{t}[C_1, g_H]_p$, then $\{\mathbb{t}_{\alpha\beta}\}$ is statistically g_H-convergent to $\mathbb{t}$ in (X, g_H).

(b). If g_H-statistically convergent to $\mathbb{t}$ in (X, g_H) and $\{\mathbb{t}_{\alpha\beta}\}$ is bounded, then $\mathbb{t}_{\alpha\beta} \to \mathbb{t}[C_1, g_H]_p$.

Proof. (a) Let

$$K_\varrho(p) = \{((r_1, r_2, ...r_p), (u_1, u_2, ..., u_p)) \in \mathbb{N}^p \times \mathbb{N}^p, r_1, r_2, ...r_p \leq \alpha, u_1, u_2, ..., u_p \leq \beta(\alpha, \beta \in \mathbb{N}): |g_H(\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p}, \mathbb{t})|^p \geq \varrho\}.$$

Now since $\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pu_p} \to \mathbb{t}[\mathrm{C}_1, \mathrm{g}_H]_p$, then

$$0 \leftarrow \frac{p!}{(\alpha\beta)^p} \sum_{r_1, r_2, ..., r_p = 1}^{\alpha} \sum_{r_1, r_2, ..., r_p = 1}^{\beta} (g_H(\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pup}, \mathbb{t})^p)$$

$$= \frac{p!}{(\alpha\beta)^p} \Bigg(\sum_{r_1, r_2, ..., r_p = 1, r_i \notin K_\varrho(p)}^{\alpha} \sum_{u_1, u_2, ..., u_p = 1, u_i \notin K_\varrho(p)}^{\beta} (g_H(\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pup}, \mathbb{t})^p)$$

$$+ \sum_{r_1, r_2, ..., r_p = 1, r_i \in K_\varrho(p)}^{\alpha} \sum_{u_1, u_2, ..., u_p = 1, u_i \in K_\varrho(p)}^{\beta} (g_H(\mathbb{t}_{r_1u_1}, \mathbb{t}_{r_2u_2}, ..., \mathbb{t}_{r_pup}, \mathbb{t})^p) \Bigg)$$

$$\geq \frac{p!}{(\alpha\beta)^p} |K_\varrho(p)| \varrho^p, as\ \alpha, \beta \to \beta.$$

That is, $\lim_{\alpha,\beta\to\infty}\frac{p!}{(\alpha\beta)^p}|K_\varrho(p)|=0$ and $\delta(K_\varrho(p))=0$.

(b) Suppose that $\{\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p}\}$ is bounded and statistically g_{H}-convergent to $\mathbb{t}$ in (X,g_{H}). So, for $\varrho>0$, we have $\delta(K_\varrho(p))=0$. Since $\{\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p}\}\in\ell_\infty$, there exists $M>0$ such that $|g_{\mathrm{H}}(\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p},\mathbb{t})|^p\leq M$. We have

$$\frac{p!}{(\alpha\beta)^p}\sum_{r_1,r_2,...,r_p=1}^{\alpha}\sum_{u_1,u_2,...,u_p=1}^{\beta}\left(g_{\mathrm{H}}(\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p},\mathbb{s})^p\right)$$
$$=\frac{p!}{(\alpha\beta)^p}\sum_{r_1,r_2,...,r_p=1,r_i\notin K_\varrho(p)}^{\alpha}\sum_{u_1,u_2,...,u_p=1,u_i\notin K_\varrho(p)}^{\beta}\left(g_{\mathrm{H}}(\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p},\mathbb{s})^p\right)$$
$$+\frac{p!}{(\alpha\beta)^p}\sum_{r_1,r_2,...,r_p=1,r_i\in K_\varrho(p)}^{\alpha}\sum_{u_1,u_2,...,u_p=1,u_i\in K_\varrho(p)}^{\beta}\left(g_{\mathrm{H}}(\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p},\mathbb{s})^p\right)$$
$$=U_1(\varrho)+U_2(\varrho),$$

where

$$U_1(\varrho)=\frac{p!}{(\alpha\beta)^p}\sum_{\substack{r_1,r_2,...,r_p=1,r_i\notin K_\varrho(p)\\u_1,u_2,...,u_p=1,u_i\notin K_\varrho(p)}}^{\alpha,\beta}\left(g_{\mathrm{H}}(\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p},\mathbb{s})^p\right)$$

and

$$U_2(\varrho)=\frac{p!}{(\alpha\beta)^p}\sum_{\substack{r_1,r_2,...,r_p=1,r_i\notin K_\varrho(p)\\u_1,u_2,...,u_p=1,u_i\notin K_\varrho(p)}}^{\alpha,\beta}\left(g_{\mathrm{H}}(\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p},\mathbb{s})^p\right).$$

Now if $\{r_1,r_2,...r_p\},\{u_1,u_2,...,u_p\}\notin K_\in(\varrho)$ then $U_1(\varrho)<\varrho^p$. For $\{r_1,r_2,...r_p\},\{u_1,u_2,...,u_p\}\in K_\in(\varrho)$,

we deduce that

$$U_2(\varrho)\leq\sup|g_{\mathrm{H}}(\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p},\mathbb{t})|\left(\frac{p!|K_\varrho(p)|}{(\alpha\beta)^p}\right)$$
$$\leq M\frac{p!|K_\varrho(p)|}{(\alpha\beta)^p}\to 0$$

as $\alpha,\beta\to\infty$, since $\delta(K(\varrho))=0$. Hence $\mathbb{t}_{r_1u_1},\mathbb{t}_{r_2u_2},...,\mathbb{t}_{r_pu_p}\to\mathbb{t}[\mathrm{C}_1,\mathrm{g}_{\mathrm{H}}]_p$.

5. Conclusion and Suggestions

In this manuscript, we have introduced and explored the concept of convergence for double sequences within the quaternion-valued g-metric space, examining several foundational properties. Furthermore, we have conducted a detailed analysis of statistical convergence in this context, aiming to provide a comprehensive understanding of sequence behavior in this specialized metric environment.

The final part of our investigation has focused on the relationship between statistical convergence in quaternion-valued g-metric spaces and the concept of strong summability. This exploration has not only deepened our understanding of how sequences behave under statistical measures but has also shed light on their summation properties within this unique framework. These findings underscore the intricate connections between statistical convergence and summability, offering valuable insights into the nature of convergence in quaternion-valued g-metric spaces.

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Author contribution: Author Kolancı, Author Gürdal and Author Kişi: Planned and designed the study, gathered and analyzed data about the study, and wrote the article by analyzing the study.

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