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Research Article

C-symmetric Toeplitz operators on Hardy spaces

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ABSTRACT. We characterize all the Toeplitz operators that are complex symmetric with respect to a class of conjugations induced by a permutation. Our results provide an affirmative answer to a conjecture from a paper of Chattopadhyay et al. (2023) [1].

Keywords: Toeplitz operators, conjugations, complex symmetric, Hardy spaces.

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1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The Hardy space H^2 of $\mathbb{D}$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 dm < \infty,$$

where m is the normalized Lebesgue measure on $\mathbb{T}$, i.e., $dm := d\theta/2\pi$. If $f \in H^2$, its radial limit

$$f^*(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})$$

exists m-a.e. on $\mathbb{T}$ and the mapping $f \mapsto f^*$ is an isometry of H^2 onto a closed subspace of $L^2(\mathbb{T}, dm)$. The extension of f to $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$, also denoted by f, is defined such that $f|_{\mathbb{T}} = f^*$. It is known that H^2 is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle_{H^2} := \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, dm \quad \text{for all } f, g \in H^2.$$

The standard orthonormal basis for H^2 is $\{1, z, z^2, \dots\}$. Given $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_\phi : H^2 \to H^2$ is defined by

$$T_\phi f = P(\phi f) \quad \text{for every } f \in H^2,$$

where P is the orthogonal projection from $L^2(\mathbb{T}, dm)$ onto H^2. A more detailed introduction of Toeplitz operators is available in [2, 18]. These operators have also been studied extensively in the literature, for example in [3, 4, 10, 11].

Let $\mathcal{H}$ be a separable complex Hilbert space. A mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be a conjugation if it satisfies the following conditions:

(i) anti-linear (or conjugate-linear), i.e., $A(ax + by) = \overline{a}Ax + \overline{b}Ay$ for all $x, y \in \mathcal{H}$ and $a, b \in \mathbb{C}$,

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(ii) involutive, i.e., $A^2 = I$, the identity operator, and
(iii) isometric, i.e., $\|Ax\| = \|x\|$ for each $x \in \mathcal{H}$.

The adjoint A^* of a bounded and anti-linear operator A is defined to satisfy the property that

$$\langle Ax, y\rangle = \overline{\langle x, A^*y\rangle} = \langle A^*y, x\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

In view of (ii) and (iii), we also have $A^* = A$, i.e., A is self-adjoint.

A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be complex symmetric if there exists a conjugation $C : \mathcal{H} \to \mathcal{H}$ such that $CTC = T^*$ (or equivalently, $CT^* = TC$). In this case, we say T is C-symmetric (or complex symmetric with respect to C). The study of complex symmetric operators was initiated by Garcia et al. in [5, 6, 7, 8]. These operators play a significant role in control theory, signal processing and non-Hermitian quantum mechanics. Examples of complex symmetric maps include normal operators, Hankel operators, Volterra operators and truncated Toeplitz operators.

Investigation of the complex symmetry of Toeplitz operators on Hilbert spaces of analytic functions was motivated by [9], in which the question of characterizing complex symmetric Toeplitz operators on H^2 was posed. Ko and Lee provided a necessary and sufficient condition for T_ϕ to be complex symmetric with respect to a special class of conjugations on H^2 [12, Theorem 2.4]. This result, together with [13, Theorem 2.11(a)] and [16, Theorem 3.6], was generalized by the authors in [17, Theorem 3.4]. Complex symmetric Toeplitz operators on Bergman and Dirichlet spaces have been studied in [14] and [15], respectively.

Let p be a fixed positive integer. In [1], Chattopadhyay et al. introduced the conjugation $C_\sigma : H^2 \to H^2$ defined as

$$C_\sigma\left(\sum_{l=0}^{\infty}\sum_{r=0}^{p-1} a_{pl+r}z^{pl+r}\right) = \sum_{l=0}^{\infty}\sum_{r=0}^{p-1} \overline{\sigma(a_{pl+r})}z^{pl+r},$$

where

(i) $\sum_{l=0}^{\infty}\sum_{r=0}^{p-1} a_{pl+r}z^{pl+r} \in H^2$,
(ii) σ is a permutation on the set $\{a_{pl}, a_{pl+1}, \ldots, a_{pl+p-1}\}$ for $l = 0, 1, \ldots,$ and
(iii) the order of σ is 2.

A special case of C_σ is the operator $C_p^{i,j} : H^2 \to H^2$ given by

$$C_p^{i,j}\left(\sum_{l=0}^{\infty}\sum_{r=0}^{p-1} a_{pl+r}z^{pl+r}\right) = \sum_{l=0}^{\infty} \overline{a_{pl+j}}z^{pl+i} + \sum_{l=0}^{\infty} \overline{a_{pl+i}}z^{pl+j} + \sum_{l=0}^{\infty}\sum_{\substack{r=0\\ r\neq i,j}}^{p-1} \overline{a_{pl+r}}z^{pl+r},$$

where i, j are any fixed integers such that $0 \le i < j < p$. They characterized all $C_p^{i,j}$-symmetric Toeplitz operators on H^2 with additional assumptions on i, j and p [1, Theorem 2.2]. In the next section, we provide an affirmative answer to Remark 2.3 therein that these characterizations are valid whenever $0 \le i < j < p$.

2. Main Results

We will characterize all the Toeplitz operators T_ϕ on H^2 that are complex symmetric with respect to $C_p^{i,j}$ in terms of the Fourier coefficients of ϕ.

Theorem 2.1. *Let $\phi(z) = \sum_{n=-\infty}^{\infty} \hat{\phi}(n)z^n \in L^\infty(\mathbb{T})$ and i, j, p be integers such that $0 \le i < j < p$. Then the operator T_ϕ is $C_p^{i,j}$-symmetric if and only if*

$$\hat{\phi}(pl) = \hat{\phi}(-pl)$$

and

$$\hat{\phi}(pl+r)=0$$

for every integer l and $r=1,2,\ldots,p-1$.

Proof. Suppose $\hat{\phi}(pl)=\hat{\phi}(-pl)$ and $\hat{\phi}(pl+r)=0$ for all integers l and $r=1,2,\ldots,p-1$. Let l' be any fixed non-negative integer. Then,

$$\begin{aligned}
T_\phi C_p^{i,j} z^{pl'+i} &= T_\phi z^{pl'+j}\\
&= P\left(\hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)(z^{p(l+l')+j}+\overline{z}^{pl}z^{pl'+j})\right)\\
&= \begin{cases} \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j} & \text{if } l'=0;\\ \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j}+\sum_{l=1}^{l'}\hat{\phi}(pl)z^{p(l'-l)+j} & \text{if } l'\geq 1;\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
C_p^{i,j}T_\phi^* z^{pl'+i} &= C_p^{i,j}T_{\overline{\phi}} z^{pl'+i}\\
&= C_p^{i,j}P\left(\overline{\hat{\phi}(0)}z^{pl'+i}+\sum_{l=1}^{\infty}\overline{\hat{\phi}(pl)}(z^{p(l+l')+i}+\overline{z}^{pl}z^{pl'+i})\right)\\
&= \begin{cases} C_p^{i,j}\left(\overline{\hat{\phi}(0)}z^{pl'+i}+\sum_{l=1}^{\infty}\overline{\hat{\phi}(pl)}z^{p(l+l')+i}\right) & \text{if } l'=0;\\ C_p^{i,j}\left(\overline{\hat{\phi}(0)}z^{pl'+i}+\sum_{l=1}^{\infty}\overline{\hat{\phi}(pl)}z^{p(l+l')+i}+\sum_{l=1}^{l'}\overline{\hat{\phi}(pl)}z^{p(l'-l)+i}\right) & \text{if } l'\geq 1;\end{cases}\\
&= \begin{cases} \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j} & \text{if } l'=0;\\ \hat{\phi}(0)z^{pl'+j}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+j}+\sum_{l=1}^{l'}\hat{\phi}(pl)z^{p(l'-l)+j} & \text{if } l'\geq 1.\end{cases}
\end{aligned}$$

Similarly,

$$T_\phi C_p^{i,j} z^{pl'+j}=C_p^{i,j}T_\phi^* z^{pl'+j}.$$

When $r=0,1,\ldots,p-1$ with $r\neq i,j$, we have

$$\begin{aligned}
T_\phi C_p^{i,j} z^{pl'+r} &= C_p^{i,j}T_\phi^* z^{pl'+r}\\
&= \begin{cases} \hat{\phi}(0)z^{pl'+r}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+r} & \text{if } l'=0;\\ \hat{\phi}(0)z^{pl'+r}+\sum_{l=1}^{\infty}\hat{\phi}(pl)z^{p(l+l')+r}+\sum_{l=1}^{l'}\hat{\phi}(pl)z^{p(l'-l)+r} & \text{if } l'\geq 1.\end{cases}
\end{aligned}$$

Thus, $T_\phi C_p^{i,j}=C_p^{i,j}T_\phi^*$.

Conversely, assume T_ϕ is $C_p^{i,j}$-symmetric. Let l,m be any non-negative integers. We first show that $\hat{\phi}(n)=0$ for $|n|=pl+i,\ldots,pl+j$. Since

$$\begin{aligned}
\left\langle P(\overline{\phi}z^{pl+i}),C_p^{i,j}z^m\right\rangle_{H^2} &= \left\langle T_{\overline{\phi}}z^{pl+i},C_p^{i,j}z^m\right\rangle_{H^2}=\left\langle z^m,C_p^{i,j}T_\phi^* z^{pl+i}\right\rangle_{H^2}\\
&= \left\langle z^m,T_\phi C_p^{i,j}z^{pl+i}\right\rangle_{H^2}=\left\langle T_{\overline{\phi}}z^m,C_p^{i,j}z^{pl+i}\right\rangle_{H^2}\\
&= \langle\overline{\phi}z^m,z^{pl+j}\rangle_{H^2}=\left\langle\sum_{n=-\infty}^{\infty}\overline{\hat{\phi}(n)}z^{m-n},z^{pl+j}\right\rangle_{H^2}\\
&= \overline{\phi(m-pl-j)}
\end{aligned}$$

and $\{C_p^{i,j} z^m\}_{m=0}^{\infty}$ is an orthonormal basis for H^2, it follows that

$$\left\|P(\overline{\phi} z^{pl+i})\right\|^2 = \sum_{m=0}^{\infty} \left|\left\langle P(\overline{\phi} z^{pl+i}), C_p^{i,j} z^m \right\rangle_{H^2}\right|^2 = \sum_{m=0}^{\infty} |\hat{\phi}(m-pl-j)|^2 = \sum_{n=-pl-j}^{\infty} |\hat{\phi}(n)|^2.$$

On the other hand,

$$P(\overline{\phi} z^{pl+i}) = P\left(\sum_{n=-\infty}^{\infty} \overline{\hat{\phi}(n)} z^{pl+i-n}\right) = \sum_{n=-\infty}^{pl+i} \overline{\hat{\phi}(n)} z^{pl+i-n}$$

and so,

$$\left\|P(\overline{\phi} z^{pl+i})\right\|^2 = \sum_{n=-\infty}^{pl+i} |\hat{\phi}(n)|^2.$$

Thus,

$$\sum_{n=-pl-j}^{\infty} |\hat{\phi}(n)|^2 = \sum_{n=-\infty}^{pl+i} |\hat{\phi}(n)|^2. \tag{2.1}$$

By considering $P(\overline{\phi} z^{pl+j})$, we obtain

$$\sum_{n=-pl-i}^{\infty} |\hat{\phi}(n)|^2 = \sum_{n=-\infty}^{pl+j} |\hat{\phi}(n)|^2 \tag{2.2}$$

in a similar fashion. From (2.1) and (2.2),

$$\sum_{n=-pl-j}^{-pl-i-1} |\hat{\phi}(n)|^2 = -\sum_{n=pl+i+1}^{pl+j} |\hat{\phi}(n)|^2$$

which implies

$$\hat{\phi}(n) = 0 \quad \text{for } |n| = pl+i+1, \ldots, pl+j. \tag{2.3}$$

Note that

$$\begin{aligned} T_\phi C_p^{i,j} z^{pl+i} = T_\phi z^{pl+j} &= P\left(\sum_{n=-\infty}^{\infty} \hat{\phi}(n) z^{pl+j+n}\right) \\ &= \sum_{n=-pl-j}^{\infty} \hat{\phi}(n) z^{pl+j+n} = \sum_{n=0}^{\infty} \hat{\phi}(n-pl-j) z^n \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} C_p^{i,j} T_\phi^* z^{pl+i} &= C_p^{i,j} P\left(\sum_{n=-\infty}^{\infty} \overline{\hat{\phi}(n)} z^{pl+i-n}\right) \\ &= C_p^{i,j}\left(\sum_{n=-\infty}^{pl+i} \overline{\hat{\phi}(n)} z^{pl+i-n}\right) \\ &= C_p^{i,j}\left(\sum_{n=0}^{\infty} \overline{\hat{\phi}(pl+i-n)} z^n\right). \end{aligned} \tag{2.5}$$

Similarly,

$$T_\phi C_p^{i,j} z^{pl+j} = \sum_{n=0}^{\infty} \hat{\phi}(n-pl-i) z^n \tag{2.6}$$

and

$$C_p^{i,j} T_\phi^* z^{pl+j} = C_p^{i,j} \left(\sum_{n=0}^{\infty} \overline{\hat{\phi}(pl+j-n)} z^n \right). \tag{2.7}$$

Suppose $i \neq 0$ and k is any integer such that $0 \leq k \leq i-1$. We claim that $\hat{\phi}(n) = 0$ for $|n| = pl+i-k$. Comparing the coefficients of z^k in the right most expressions of (2.4) and (2.5) as well as those of z^k in (2.6) and (2.7), we have

$$\hat{\phi}(k-pl-j) = \hat{\phi}(pl+i-k) \text{ and } \hat{\phi}(k-pl-i) = \hat{\phi}(pl+j-k), \tag{2.8}$$

respectively. Since $i < j$, we also have

$$pl+i-k \leq pl+j-1-k \leq pl+j. \tag{2.9}$$

When $k = 0$, it follows from (2.8) and the fact $\hat{\phi}(-pl-j) = 0 = \hat{\phi}(pl+j)$ that $\hat{\phi}(n) = 0$ for $|n| = pl+i$. Assume there exists an integer k' such that $0 \leq k' < i-1$ and $\hat{\phi}(n) = 0$ if $|n| = pl+i-k', \ldots, pl+i$. By taking $k = k'+1$ and $k = k'$ in (2.8) and (2.9) respectively, it follows from the induction assumption and (2.3) that $\hat{\phi}(pl+i-k'-1) = 0 = \hat{\phi}(k'+1-pl-i)$. Therefore,

$$\hat{\phi}(n) = 0 \text{ for } |n| = pl+1, \ldots, pl+j \tag{2.10}$$

(if $i = 0$, then (2.10) is also true in light of (2.3)).

It remains to show that $\hat{\phi}(n) = 0$ for $|n| = pl+k$, where k is any integer with $j+1 \leq k \leq p-1$. Since p does not divide $2pl+k$, we have $2pl+i+k \neq pl'+i$ for all $l' \in \mathbb{N}$. Moreover, $2pl+i+k \neq pl'+j$ for every $l' \in \mathbb{N}$. Otherwise, $i-j+k = p(l'-2l)$. This equality is absurd, because $i+1 \leq i-j+k \leq p-1+i-j \leq p-2 < p$. Comparing the coefficients of $z^{2pl+i+k}$ in the right most expressions of (2.4) and (2.5) as well as those of $z^{2pl+i+k}$ in (2.6) and (2.7) gives

$$\hat{\phi}(pl+i-j+k) = \hat{\phi}(-pl-k) \text{ and } \hat{\phi}(pl+k) = \hat{\phi}(-pl-i+j-k), \tag{2.11}$$

respectively. Furthermore,

$$pl+i+1 \leq pl+i-j+k \leq pl+k-1. \tag{2.12}$$

When $k = j+1$, it follows from (2.11) and the fact $\hat{\phi}(pl+i+1) = 0 = \hat{\phi}(-pl-i-1)$ that $\hat{\phi}(n) = 0$ for $|n| = pl+j+1$. Assume there is an integer k' for which $j+1 \leq k' < p-1$ and $\hat{\phi}(n) = 0$ for $|n| = pl+j+1, \ldots, pl+k'$. Taking $k = k'+1$ in (2.11) and (2.12), it follows from the induction assumption, (2.3), (2.11) and (2.12) that $\hat{\phi}(n) = 0$ for $|n| = pl+k'+1$.

Hence $\hat{\phi}(pl+r) = 0$ for all integers l and $r = 1, \ldots, p-1$. Comparing the coefficients of z^j in the right most expressions of (2.4) and (2.5), we have

$$\hat{\phi}(-pl) = \hat{\phi}(pl)$$

for all non-negative integers l. The proof of the theorem is now complete. □

Chattopadhyaya et al. introduced another special case of C_σ in [1], namely the operator $C_n : H^2 \to H^2$ defined by

$$C_n \left(\sum_{l=0}^{\infty} \sum_{r=0}^{n-1} a_{nl+r} z^{nl+r} \right) = \sum_{l=0}^{\infty} \sum_{r=0}^{n-1} \overline{a_{nl+n-r-1}} z^{nl+r},$$

where $\sum_{l=0}^{\infty}\sum_{r=0}^{n-1} a_{nl+r}z^{nl+r} \in H^2$ and n is any fixed positive integer. They also obtained the following characterizations for T_ϕ to be C_n-symmetric.

Theorem 2.2. [1, Theorem 3.1] *Let $\phi(z) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k)z^k \in L^\infty(\mathbb{T})$. Then the operator T_ϕ is C_n-symmetric if and only if*

$$\hat{\phi}(nl) = \hat{\phi}(-nl)$$

and

$$\hat{\phi}(nl + r) = 0$$

for every integer l and $r = 1, 2, \ldots, n-1$.

The method adopted in proving Theorem 2.1 furnishes an alternative proof to the necessity part of Theorem 2.2: Assume T_ϕ is C_n-symmetric. Since $C_n z^{nl} = z^{nl+n-1}$, we have

$$\langle P(\overline{\phi} z^{nl}), C_n z^m\rangle_{H^2} = \overline{\hat{\phi}(m - nl - n + 1)}$$

for all non-negative integers l and m. The fact that $\{C_n z^m\}_{m=0}^{\infty}$ is an orthonormal basis for H^2 implies

$$\left\|P(\overline{\phi} z^{nl})\right\|^2 = \sum_{m=0}^{\infty} |\hat{\phi}(m - nl - n + 1)|^2 = \sum_{k=-nl-n+1}^{\infty} |\hat{\phi}(k)|^2.$$

Moreover,

$$\left\|P(\overline{\phi} z^{nl})\right\|^2 = \sum_{k=-\infty}^{nl} |\hat{\phi}(k)|^2.$$

Thus,

$$\sum_{k=-nl-n+1}^{\infty} |\hat{\phi}(k)|^2 = \sum_{k=-\infty}^{nl} |\hat{\phi}(k)|^2. \tag{2.13}$$

Considering $P(\overline{\phi} z^{nl+n-1})$ likewise, we obtain

$$\sum_{k=-nl}^{\infty} |\hat{\phi}(k)|^2 = \sum_{k=-\infty}^{nl+n-1} |\hat{\phi}(k)|^2. \tag{2.14}$$

Now, it follows from (2.13) and (2.14) that

$$\sum_{k=-nl-n+1}^{-nl-1} |\hat{\phi}(k)|^2 = -\sum_{k=nl+1}^{nl+n-1} |\hat{\phi}(k)|^2,$$

i.e., $\hat{\phi}(k) = 0$ for $|k| = nl + 1, \ldots, nl + n - 1$. Furthermore,

$$\begin{aligned} T_\phi C_n z^{nl+n-1} = T_\phi z^{nl} &= P\left(\sum_{k=-\infty}^{\infty} \hat{\phi}(k) z^{nl+k}\right) \\ &= \sum_{k=-nl}^{\infty} \hat{\phi}(k) z^{nl+k} = \sum_{k=0}^{\infty} \hat{\phi}(k - nl) z^k \end{aligned} \tag{2.15}$$

and

$$
\begin{aligned}
C_n T_\phi^* z^{nl+n-1} &= C_n P\left(\sum_{k=-\infty}^{\infty} \overline{\hat{\phi}(k)} z^{nl+n-1-k}\right) \\
&= C_n \left(\sum_{k=-\infty}^{nl+n-1} \overline{\hat{\phi}(k)} z^{nl+n-1-k}\right) \\
&= C_n \left(\sum_{k=0}^{\infty} \overline{\hat{\phi}(nl+n-1-k)} z^{k}\right). \qquad (2.16)
\end{aligned}
$$

Upon comparing the constant terms in the right most expressions of (2.15) and (2.16), we have

$$\hat{\phi}(-nl) = \hat{\phi}(nl).$$

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