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SOME CONVERGENCE TYPES OF FUNCTION SEQUENCES AND THEIR RELATIONS

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Abstract. In this paper, we investigate various types of convergence for sequences of functions and examine the relationships among these types. Our findings contribute to a deeper understanding of the structural properties of function sequences and their convergence behaviors.

1. Introduction

The study of convergence for sequences of functions is a fundamental topic in mathematical analysis, with significant implications in various branches of mathematics and applied sciences. Convergence types such as pointwise convergence, uniform convergence, and $\alpha-$convergence provide different lenses through which we can understand the behavior of function sequences under various conditions. These convergence concepts are crucial for solving differential equations, analyzing function spaces, and understanding the limits of functions in mathematical modeling.

In this paper, we aim to systematically investigate the different types of convergence for sequences of functions and elucidate the relationships between them. We begin by defining the basic concepts and notations used throughout the paper.

2. Preliminaries

Let X be a non-empty subset of $\mathbb{R}$. The set of real-valued functions defined on X is denoted by $F(X,\mathbb{R})$:

$$F := F(X,\mathbb{R}) = \{h \mid h : X \to \mathbb{R}\}.$$

The family of function sequences defined over X is denoted by $FS(X)$. For simplicity, if the domain of functions is known, $FS(X)$ is abbreviated to FS.

$$FS = \{(h_n) : \forall n \in \mathbb{N},\ h_n \in F\}.$$

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Definition 2.1. ([10]) Let $(h_n) \in FS$ be given. If there exists a function $C \in F$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq C(t)$ holds, then the function sequence (h_n) is said to be pointwise bounded on X.

The family of pointwise bounded function sequences is denoted by $PBFS(X)$. For simplicity, if the domain of functions is known, $PBFS(X)$ is abbreviated to $PBFS$.

$$PBFS = \{(h_n) \in FS : \exists C \in F : \forall n \in \mathbb{N},\ |h_n| \leq C\}.$$

Example 2.2. On the interval $X = (0, 1)$, the sequence of functions (h_n) defined by $h_n(t) = n/(nt + 1)$ for each $n \in \mathbb{N}$ is pointwise bounded by $C(t) = 1/t$.

Proposition 2.3. The necessary and sufficient condition for a function sequence $(h_n) \in FS$ to be pointwise bounded is that $\sup_{n\in\mathbb{N}} |h_n(t)| < \infty$ for all $t \in X$.

Proof. Let $(h_n) \in PBFS$ and let $t \in X$ be given. In this case, there exists a function $C \in F$ such that for every $n \in \mathbb{N}$, $|h_n(t)| \leq C(t)$. Taking the supremum, we have $\sup_{n\in\mathbb{N}} |h_n(t)| < C(t) < \infty$. On the other hand, suppose $\sup_{n\in\mathbb{N}} |h_n(t)| < \infty$ for every $t \in X$. Then, for each $t \in X$, define $C(t) := \sup_{n\in\mathbb{N}} |h_n(t)|$. Obviously, $C \in F$, and for every $n \in \mathbb{N}$, $|h_n| \leq C$ holds. Thus, (h_n) is pointwise bounded. □

Definition 2.4. ([10]) Let $(h_n) \in FS$ be given. If there exists a number $M > 0$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq M$ holds, then the function sequence (h_n) is said to be uniformly bounded on X.

The family of function sequences that are uniformly bounded on X is denoted by $UBFS(X)$. For simplicity, if the domain of functions is known, $UBFS(X)$ is abbreviated to $UBFS$.

$$UBFS = \{(h_n) \in FS : \exists M > 0 : \forall n \in \mathbb{N},\ |h_n| \leq M.\}$$

Example 2.5. On $X = [0, 1]$, the sequence of functions (h_n) defined by $h_n(t) = t/n$ for each $n \in \mathbb{N}$ is uniformly bounded with $K = 1$.

Proposition 2.6. A function sequence $(h_n) \in FS$ is uniformly bounded if and only if $\sup_{n\in\mathbb{N}} \sup_{t\in X} |h_n(t)| < \infty$.

Proof. Let (h_n) be a sequence on set X that is uniformly bounded. In other words, there exists $K > 0$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq K$. Then, taking the supremum over n and t, we have $\sup_{n\in\mathbb{N}} \sup_{t\in X} |h_n(t)| < K < \infty$, satisfying the desired condition.

Conversely, if $\sup_{n\in\mathbb{N}} \sup_{t\in X} |h_n(t)| = K < \infty$, then for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq K$. This shows that the sequence (h_n) is uniformly bounded on X. □

Remark 2.7. In Definition 2.1, if we choose the function C to be the number M mentioned in Definition 2.4, such that for every $t \in X$, $C(x) = K$, then it can be seen that every uniformly bounded function sequence is pointwise bounded: $PBFS \subset UBFS$. Nevertheless, the reverse of this does not hold. The function sequence given in Example 2.2 is pointwise bounded on X but not uniformly bounded.

What conditions need to be imposed on the set X for $UBFS = PBFS$ to hold? The answer is provided in the proposition below.

Proposition 2.8. If X is a finite set, then $UBFS = PBFS$.

Proof. Let X be a finite set. It's clear that $UBFS \subset PBFS$. Let $(h_n) \in PBFS$ be arbitrary. Then there exists a function $C : X \to \mathbb{R}^+$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq C(t)$. Since X is finite, we can choose $K = \max_{t \in X} C(t)$, and for every $n \in \mathbb{N}$ and every $t \in X$, $|h_n(t)| \leq K$. Thus, $PBFS \subset UBFS$. □

If X is not finite, can Proposition 2.8 still hold true? Is it possible to eliminate the condition that the set X is finite? The answer to these questions is provided in the remark below.

Remark 2.9. If X is infinite, then it has a countable subset $S = \{t_1, t_2, \cdots\}$. Without loss of generality, we can choose (t_n) to be a monotone sequence. This leads to two cases:

(i) $\lim |t_n| = \infty$
(ii) $\lim |t_n| = a < \infty$

In (i)

$$h_n(t) = \begin{cases} \dfrac{n|t|}{n+1}, & x \in S \\ 0, & t \in X \setminus S \end{cases}$$

In (ii)

$$h_n(t) = \begin{cases} \dfrac{n-1}{n(|t|-a)}, & x \in S \\ 0, & t \in X \setminus S \end{cases}$$

is pointwise bounded but not uniformly bounded. Thus, the condition that X must be finite cannot be removed for the equality $UBFS = PBFS$.

3. Some Convergence Types of Function Sequences

3.1. **Pointiwse Convergence.**

Definition 3.1. ([10]) Let $(h_n) \in FS$ and $h \in F$ be given. If for every $t \in X$, $\lim_{n\to\infty} h_n(t) = h(t)$ holds, then the sequence (h_n) is said to converge pointwise to the function h on the set X, and it is denoted by $h_n \xrightarrow{X} h$.

The above definition can also be expressed as follows: The sequence (h_n) is said to converge pointwise to the function h on the set X if for every $\varepsilon > 0$ and every $t \in X$, there exists a natural number n_0 such that for every $n \geq n_0$, $|h_n(t) - h(t)| < \varepsilon$ holds.

If the convergence set is specified, for simplicity, $h_n \xrightarrow{X} h$ is replaced with $h_n \to h$. The family of pointwise convergent function sequences on set X is denoted by $PCFS(X)$. If the domain set is specified, for simplicity, $PCFS(X)$ is replaced with $PCFS$.

$$PCFS = \Big\{(h_n) \in FS : \exists h \in F : \forall t \in X, \lim_{n\to\infty} h_n(t) = h(t)\Big\}$$

Example 3.2. $h_n : [0,1] \to \mathbb{R}$ defined by $h_n(t) = t^n$, the sequence (h_n) of functions is pointwise convergent on $[0,1]$ to the function

$$h(t) = \begin{cases} 1, & t = 1 \\ 0, & t \in [0,1). \end{cases}$$

Proposition 3.3. *$PCFS \subset PBFS$.*

Proof. Let (h_n) be a sequence in $PCFS$ and let $t \in X$. Then, there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $|h_n(t) - h(t)| < 1$. From this, we have

Let $C(t) := \max_{t\in X} \{|h_1(x)|, |h_2(t)|, \ldots, |h_{n_0-1}(t)|, |h(t)| + 1\}$. Performing this operation for every $t \in X$, we define the function C on X such that for every $n \in \mathbb{N}$ and every $t \in X$,

$$|h_n(t)| \leq C(t)$$

Thus, (h_n) is pointwise bounded on X. □

3.2. **Uniform Convergence.**

Definition 3.4. [10] Let $(h_n) \in FS$ and $h \in F$ be given. If for every $\varepsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $t \in X$, $|h_n(t) - h(t)| < \varepsilon$, then the sequence (h_n) is said to converge uniformly to the function h on the set X, and it is denoted by $h_n \overset{X}{\rightrightarrows} h$.

If the convergence set is specified, for simplicity, $h_n \overset{X}{\rightrightarrows} h$ is replaced with $h_n \rightrightarrows h$. The family of uniformly convergent function sequences on set X is denoted by $UCFS(X)$. If the domain set is specified, for simplicity, $UCFS(X)$ is replaced with $UCFS$.

$$UCFS = \{(h_n) \in FS : \forall\varepsilon > 0, \exists n_0 : \forall t \in X, \forall n \geq n_0, |h_n(t) - h(t)| < \varepsilon\}$$

Example 3.5. The sequence of functions (h_n) defined by $h_n(t) = t/n$ for every $n \in \mathbb{N}$ is uniformly convergent to the function $h(t) = 0$ on the interval $[0, 1]$.

Remark 3.6. It is evident that $UCFS$ is a subset of $PCFS$, Nevertheless, the reverse of this does not hold. Even though the function sequence given in Example 3.2 is pointwise convergent, it is not uniformly convergent.

Under what conditions on the set X does $UCFS = PCFS$ hold? The answer is provided in the proposition below.

Proposition 3.7. If X is a finite set, then $UCFS = PCFS$ holds.

Proof. Let X be a finite set. It is obvious that $PCFS \subset UCFS$. Let $h_n \to h$ for functions h and h_n defined on the finite set $X = \{t_1, \cdots, t_k\}$, and let $\varepsilon > 0$ be arbitrary. Then, there exist $n_1, \cdots, n_k \in \mathbb{N}$ such that for every $n \geq n_i$, $|h_n(t_i) - h(t_i)| < \varepsilon$ for $i = \overline{1, k}$. If we choose $N = \max_{1\leq i\leq k} n_i$, then for every $n \geq N$, we have $\sup_{t\in X} |h_n(t) - h(t)| < \varepsilon$. Thus, $UCFS \subset PCFS$ is satisfied. □

If X is not finite, can Proposition 3.7 still hold true? Can the requirement that the set X be finite be removed? The answer to these questions is provided in the remark below.

Remark 3.8. If X is infinite, then it has a countable subset $S = \{t_1, t_2, \cdots\}$. Without loss of generality, we can choose (t_n) to be a monotone sequence. This leads to two cases:

(i) $\lim |t_n| = \infty$
(ii) $\lim |t_n| = a < \infty$

In (i)

$$h_n(t) = \begin{cases} \arctan \dfrac{|t|}{n}, & t \in S \\ 0, & t \in X \setminus S \end{cases}$$

In (ii)

$$h_n(t) = \begin{cases} \left(\frac{|t|}{a}\right)^n, & t \in S \\ 0, & t \in X \setminus S \end{cases}$$

is pointwise convergent but not uniformly convergent. Thus, the condition that X must be finite cannot be removed for the equality $UBFS = PBFS$.

3.3. **Relationships Between Types of Boundedness and Convergence on Certain Sets.** Let a sequence $(h_n) \in FS$ be given. Here, given S as any finite set, some examples of the relationships between types of boundedness and convergence according to the set X can be provided as follows: For each $n \in \mathbb{N}$,

(1) Let $X \in \{[0,1], (0,1), S, \mathbb{N}, \mathbb{R}\}$ and $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = n|t| + n$. Then the sequence of functions (h_n) is not in $PBFS$.
(2) Let $X \in \{[0,1], (0,1), \mathbb{R}\}$ and $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = ((-1)^n nt)/(1+nt^2)$. The sequence of functions (h_n) is in $PBFS$, but it is not in $PCFS$.
(3) Let $h_n : \mathbb{N} \to \mathbb{R}$ be defined as $h_n(t) = (-1)^n t$. The sequence of functions (h_n) is in $PBFS$, but it is not in both $PCFS$ and $UBFS$.
(4) For $X \in \{[0,1], (0,1), S, \mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = (-1)^n \sin t$. The sequence of functions (h_n) is in $UBFS$, but it is not in $PCFS$.
(5) For $X \in \{[0,1], (0,1)\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = t^n$ for each $n \in \mathbb{N}$. The sequence of functions (h_n) is in both $UBFS$ and $PCFS$, but it is not in $UCFS$.
(6) For $X \in \{\mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = \arctan(t/n)$. The sequence of functions (h_n) is in both $UBFS$ and $PCFS$, but it is not in $UCFS$.
(7) For $X \in \{[0,1], (0,1), S, \mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = \sin t/n$. Then the sequence of functions (h_n) is in both $UBFS$ and $UCFS$.
(8) Let $h_n : [0,1] \to \mathbb{R}$ be defined as

$$h_n(t) = \begin{cases} \frac{n-t}{nt}, & t \in (0,1] \\ 0, & t = 0. \end{cases}$$

The sequence of functions (h_n) is in $UCFS$, but it is not in $UBFS$.
(9) Let $h_n : (0,1) \to \mathbb{R}$ be defined as $h_n(t) = (n-t)/(nt)$. The sequence of functions (h_n) is in $UCFS$, but it is not in $UBFS$.
(10) For $X \in \{\mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = t + 1/n$. The sequence of functions (h_n) is in $UCFS$, but it is not in $UBFS$.
(11) For $X \in \{[0,1], (0,1), \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = (nt)/(1+nt^2)$. The sequence of functions (h_n) is in $PCFS$, but it is not in both $UBFS$ and $UCFS$.
(12) Let $h_n : \mathbb{N} \to \mathbb{R}$ be defined as $h_n(t) = t^2/n + t$. The sequence of functions (h_n) is in $PCFS$, but it is not in both $UBFS$ and $UCFS$.

X	$[0,1]$	$(0,1)$	S	$\mathbb{N}$	$\mathbb{R}$
	(1)	(1)	(1)	(1)	(1)
	(2)	(2)	(Prop. 2.8) $\emptyset$	(3)	(2)
	(4)	(4)	(4)	(4)	(4)
	(5)	(5)	(Prop. 3.7) $\emptyset$	(6)	(6)
	(7)	(7)	(7)	(7)	(7)
	(8)	(9)	(Prop. 2.8) $\emptyset$	(10)	(10)
	(11)	(11)	(Prop. 3.7) $\emptyset$	(12)	(11)

TABLE 1. Relations between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$.

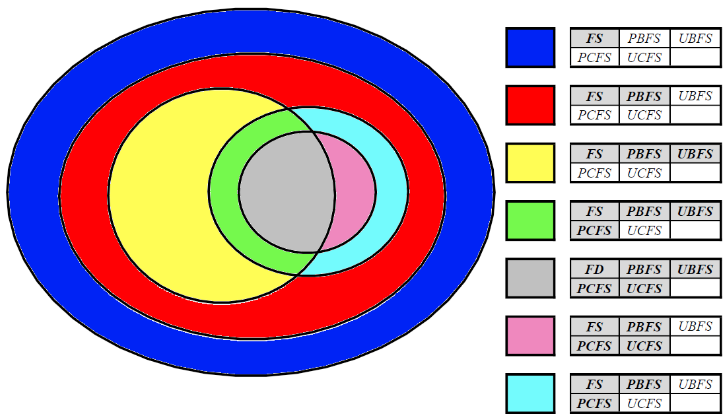

FIGURE 1. Relations between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$ on $[0,1], (0,1), \mathbb{N}$ and $\mathbb{R}$.

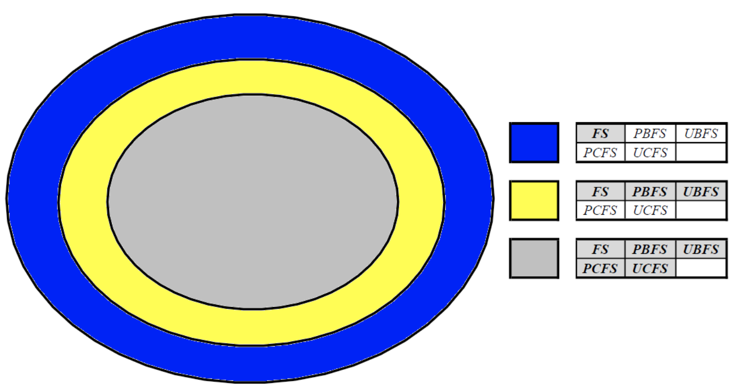

FIGURE 2. Relations Between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$ on Finite Set S.

3.4. $\alpha-$**convergence.**

Definition 3.9. ([4]) Let $(h_n) \in FS$ and $h \in F$ be given. If for every $t \in X$ and for every sequence (t_n) in X such that $t_n \to t$, we have $h_n(t_n) \to h(t)$, then the sequence (h_n) is said to be α-convergent to the function h on the set X, and it is denoted by $h_n \overset{X}{\to}_\alpha h$.

If the convergence set is specified, for simplicity, $h_n \overset{X}{\to}_\alpha h$ is replaced with $h_n \to_\alpha h$. The family of α-convergent function sequences on set X is denoted by $\alpha CFS\,(X)$. If the domain set is specified, for simplicity, $\alpha CFS\,(X)$ is replaced with αCFS.

$$\alpha CFS = \{(h_n) \in FS : \forall t \in X, \forall (t_n)(t_n \to t) \implies h_n(t_n) \to h(t)\}.$$

In [2], Athanassiadou et al. (2015) have proven the proposition equivalent to the definition of α-convergence.

Proposition 3.10. ([2]) A necessary and sufficient condition for $h_n \overset{X}{\to}_\alpha h$ is that for every given $\varepsilon > 0$ and every $t_0 \in X$, there exist $\delta(\varepsilon, t_0) > 0$ and $n_0(\varepsilon, t_0) \in \mathbb{N}$ such that for every $t \in X$ satisfying $|t - t_0| < \delta$, we have $|h_n(t) - h(t_0)| < \varepsilon$.

From here, the following question can be asked: Is there a relationship between α-convergence and other types of convergence, as there is between uniform and pointwise convergence? The answer to this question is provided in the following examples.

Example 3.11. For every $n \in \mathbb{N}$, let $h_n : [0,1] \to \mathbb{R}$ be defined as $h_n(t) = t^n$. The sequence (h_n) is pointwise convergent to the function

$$h(t) = \begin{cases} 1, & t = 1 \\ 0, & 0 \le t < 1 \end{cases}$$

while it is not α-convergent.

Example 3.12. Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous function, and consider an arbitrary sequence (t_n) converging to t such that $h_n(t_n) \not\to h(t)$. If we choose h_n such that $h_n \equiv h$ for every $n \in \mathbb{N}$, then $h_n \not\to_\alpha h$, although $h_n \rightrightarrows h$.

Example 3.13. For every $n \in \mathbb{N}$, let $h_n : (0,1] \to \mathbb{R}$ be defined as follow,

$$h_n(t) = \begin{cases} 1 - nt, & 0 < t \le 1/n \\ 0, & 1/n < t \le 1. \end{cases}$$

Then, $h_n \not\rightrightarrows 0$, but $h_n \to_\alpha 0$.

One might wonder: Under what conditions can connections between different types of convergence be made? Below are some of the situations where such connections occur.

Proposition 3.14. ([7]) $X \subset \mathbb{R}$ and let $h, h_n : X \to \mathbb{R}$ for every $n \in \mathbb{N}$. In this case,

(a) If $h_n \to_\alpha h$, then $h_n \to h$.
(b) If $h_n \rightrightarrows h$ and h is continuous, then $h_n \to_\alpha h$.
(c) If X is a compact set and $h_n \to_\alpha h$, then $h_n \rightrightarrows h$.
(d) X is compact if and only if $h_n \to_\alpha h$ implies $h_n \rightrightarrows h$.

Which properties need to be imposed on the set X for $\alpha CFS = PCFS = UCFS$? The answer to this question is provided in the proposition below.

Proposition 3.15. If X is a finite set then $\alpha CFS = PCFS = UCFS$.

Proof. Let $X = \{t_1, t_2, \cdots, t_k\}$ be a finite set. According to Proposition 3.7, $UCFS = PCFS$ on X. It suffices to show that $PCFS \subset \alpha CFS$. Let $(t_n) \subset X$ be an arbitrary sequence converging to t_i ($t_i \in X$) with $\lim_{n\to\infty} h_n = h$. Then, there exists an $n_i \in \mathbb{N}$ such that for all $n \geq n_i$, $|h_n(t_i) - h(t_i)| < \varepsilon$. Since X is a finite set, for any convergent sequence defined on this set, after a certain index, every term will be the same. Therefore, there exists $\overline{n}_i \in \mathbb{N}$ such that for all $n \geq \overline{n}_i$, $|t_n - t_i| = 0$. Choosing δ as any positive number and $n_0 = \max\{n_i, \overline{n}_i\}$, for all $n \geq n_0$,

$$|h_n(t_n) - h(t_i)| = |h_n(t_n) - h_n(t_i)| + |h_n(t_i) - h(t_i)| < 0 + \varepsilon = \varepsilon$$

Therefore, $(h_n) \in \alpha CFS$, and $PCFS \subset \alpha CFS$ is established. □

Proposition 3.15 can also be proven using Proposition 3.14 (b) and (d) aimed at showing $\alpha CFS = UCFS$.

Proposition 3.16. Let X be a countable set without accumulation points, and let $(h_n) \in FS$ and $h \in F$ be given. Then, $h_n \to_\alpha h \iff h_n \to h$.

Proof. The necessary condition is clear from Proposition 3.14 (a). On the other hand, suppose $h_n \to h$ for any arbitrary $t \in X$ and $\varepsilon > 0$. Then, there exists an $n_t \in \mathbb{N}$ such that for every $n \geq n_t$, $|h_n(t) - h(t)| < \varepsilon$. Now, choosing $\delta < \inf\{|t_i - t_j| : t_i, t_j \in X, i \neq j\}$ and $\overline{n}_t = n_t$, for every $y \in X$ such that $|y - t| < \delta$ and every $n \geq \overline{n}_t$, we have $|h_n(y) - h(t)| = |h_n(t) - h(t)| < \varepsilon$. □

Proposition 3.17. If $K \subset \mathbb{R}$ is a compact set and $h_n \xrightarrow{K}_\alpha h$, then $(h_n) \in UBFS$.

Proof. Let's assume that $h_n \xrightarrow{K} \alpha h$, and for each $i \in I$, $t_i \in K$ is given. For $\varepsilon = 1$, there exist $\delta_i > 0$ and $n_i \in \mathbb{N}$ such that for all $t \in K$ satisfying $|t - t_i| < \delta_i$ and all $n \geq n_i$, $|h_n(t_n) - h(t_i)| < 1$. Since $h_n \xrightarrow{K}_\alpha h$, h is continuous and attains its maximum value $||h||_\infty = \max_{t\in K} |h(t)|$ on the compact set K. Hence,

$$|h_n(t)| - |h(t_i)| \leq |h_n(t) - h(t_i)| < 1 \implies |h_n(t)| \leq 1 + |h(t_i)| \leq 1 + ||h||_\infty = M_0.$$

Here, $B = \bigcup_{t_i \in K} B(t_i, \delta_i)$ is an open cover of the compact set K, which has a finite subcover. Without loss of generality, let's consider this subcover as $\bigcup_{i=1}^{k} B(t_i, \delta_i)$. By choosing $M_i = \max\{|h_1(t_i)|, |h_2(t_i)|, \cdots, |h_{n_0-1}(t_i)|, M_0\}$ for each i, and then selecting $M = \max_{1\leq i\leq k} M_i$, we ensure that for all $t \in K$ and all $n \in \mathbb{N}$, $|h_n(t)| \leq M$. Therefore, the sequence (h_n) is uniformly bounded on X. □

3.5. **Relationships Between $\alpha-$ Convergence, Boundedness, and Other Types of Convergence On Some Sets.** Let a sequence $(h_n) \in FS$ be given. Here, some examples of the relationships between types of boundedness and convergence according to the structure of the set X, with S denoting any finite set, can be given as follows: For each $n \in \mathbb{N}$

(13) Let $h_n : (0, 1) \to \mathbb{R}$, $h_n(t) = nt^n(1 - t)$. The sequence of functions (h_n) belongs to both PCFS and UBFS. However, the sequence (h_n) does not belong to either αCFS or $UCFS$.

(14) Let $h_n : \mathbb{R} \to \mathbb{R}$, $h_n(t) = (nt)/(1 + n^2t^2)$. The sequence of functions (h_n) belongs to both $PCFS$ and $UBFS$. However, the sequence (h_n) does not belong to either $UCFS$ or αCFS.

(15) For $X \in \{[0,1], (0,1)\}$, let $h_n : X \to \mathbb{R}$,

$$h_n(t) = \begin{cases} 1, & t \leq \frac{1}{2} \\ t/n, & t > \frac{1}{2} \end{cases}$$

The sequence of functions (h_n) belongs to both $UCFS$ and $UBFS$. However, the sequence (h_n) does not belong to αCFS.

(16) Let $h_n : \mathbb{R} \to \mathbb{R}$,

$$h_n(t) = \begin{cases} \dfrac{n + \sin t}{n}, & t > 0 \\ \\ \dfrac{-n \sin t}{n}, & t \leq 0 \end{cases}$$

The sequence of functions (h_n) belongs to both $UCFS$ and $UBFS$. However, the sequence (h_n) does not belong to αCFS.

(17) Let $h_n : (0,1) \to \mathbb{R}$,

$$h_n(t) = \begin{cases} t\left(t + \frac{3}{4}\right)^n, & 0 < t < \frac{1}{4} \\ \left(\frac{1}{2} - t\right)\left(\frac{5}{4} - t\right)^n, & \frac{1}{4} \leq t < \frac{1}{2} \\ 0, & \frac{1}{2} \leq t \leq 1 - \frac{1}{2n} \\ n + 2n^2(t-1), & 1- < \frac{1}{2n} \leq t < 1 \end{cases}$$

The sequence of functions (h_n) belongs to PCFS, but it does not belong to $UBFS$, $UCFS$, or αCFS.

(18) For $X \in \{[0,1], (0,1), \mathbb{R}\}$ and let $h_n : X \to \mathbb{R}$.

$$h_n(t) = \begin{cases} \dfrac{n - t}{nt}, & 0 < t < \frac{1}{2} \\ 1, & \text{otherwise} \end{cases}$$

The sequence of functions (h_n) belongs to $UBFS$, $UCFS$, and αCFS.

(19) Let $h_n : (0,1) \to \mathbb{R}$,

$$h_n(t) = \begin{cases} 1 - nt, & 0 < t \leq \frac{1}{n} \\ 0, & \frac{1}{n} < t < 1 \end{cases}$$

The sequence of functions (h_n) is in $UBFS$ and αCFS, but it is not in $UCFS$.

(20) Let $h_n : \mathbb{N} \to \mathbb{R}$,

$$h_n(t) = \begin{cases} 0, & t > n \\ 1, & t \leq n \end{cases}$$

The sequence of functions (h_n) is in $UBFS$ and αCFS, but it is not in $UCFS$.

(21) For $X \in \{[0,1], (0,1), S, N, \mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$, $h_n(t) = 1/(n|t| + n)$. The sequence of functions (h_n) is in $UBFS$, $UCFS$, and αCFS.

(22) Let $h_n : (0,1) \to \mathbb{R}$, $h_n(t) = n/(nt + 1)$. The sequence of functions (h_n) is in αCFS, but it is not in $UBFS$ or $UCFS$.

	$[0,1]$	$(0,1)$	S	$\mathbb{N}$	$\mathbb{R}$
■	(1)	(1)	(1)	(1)	(1)
■	(2)	(2)	(Prop. 2.8) $\emptyset$	(3)	(2)
■	(4)	(4)	(4)	(4)	(4)
■	(5)	(13)	(Prop. 3.15) $\emptyset$	(Prop. 3.16) $\emptyset$	(14)
■	(15)	(15)	(Prop. 3.15) $\emptyset$	(Prop. 3.16) $\emptyset$	(16)
■	(11)	(17)	(Prop.3.15) $\emptyset$	(Prop. 3.16) $\emptyset$	(11)
■	(18)	(18)	(Prop. 3.15) $\emptyset$	(Prop. 3.16) $\emptyset$	(18)
■	(Prop. 3.14 (c)) $\emptyset$	(19)	(Prop. 3.15) $\emptyset$	(20)	(6)
□	(21)	(21)	(21)	(21)	(21)
■	(Prop. 3.17) $\emptyset$	(9)	(Prop. 2.8) $\emptyset$	(10)	(10)
■	(Prop. 3.17) $\emptyset$	(22)	(Prop. 3.15) $\emptyset$	(12)	(12)

TABLE 2. Relations Between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$-αCFS.

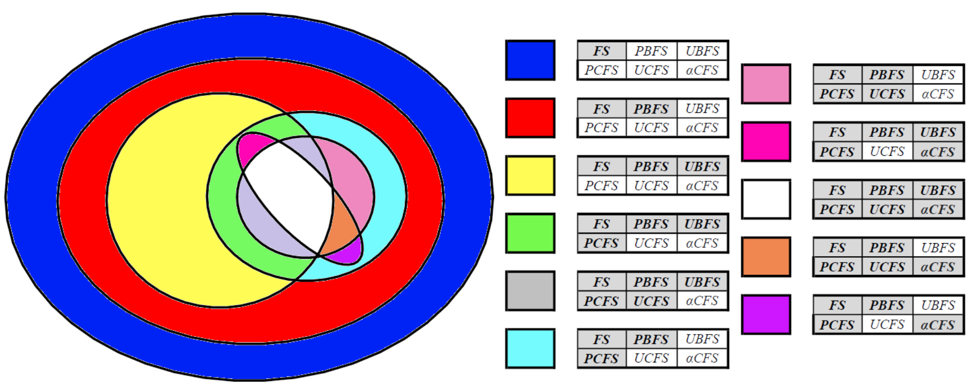

FIGURE 3. Relations Between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$-αCFS on $(0,1)$ and $\mathbb{R}$.

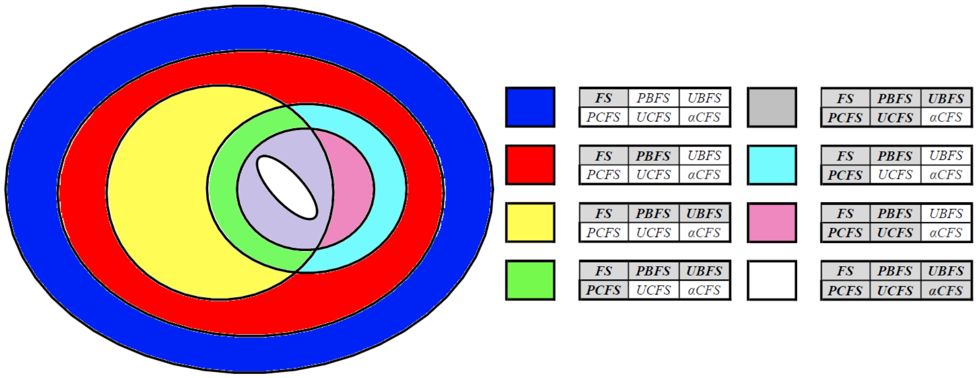

FIGURE 4. Relations Between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$-αCFS on $[0,1]$.

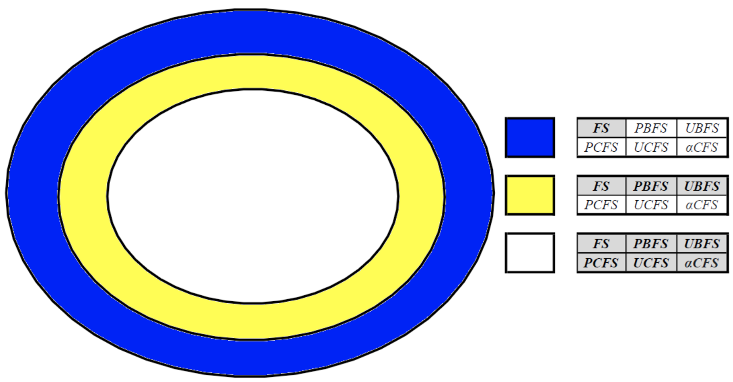

FIGURE 5. Relations Between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$-αCFS on Finite Set.

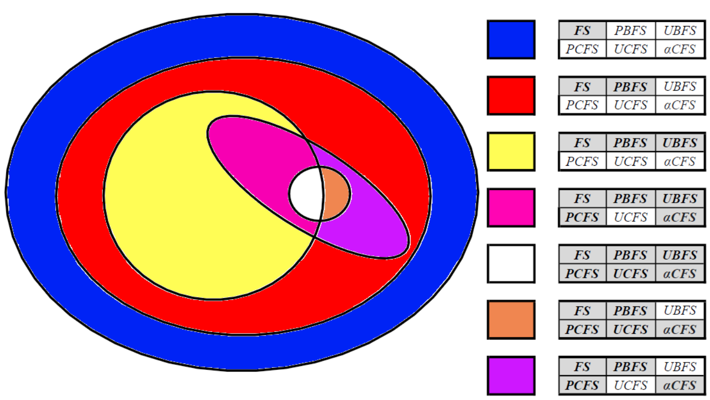

FIGURE 6. Relations Between FS-$PBFS$-$UBFS$-$PCFS$-$UCFS$-αCFS on $\mathbb{N}$.

4. Conclusion

In conclusion we systematically studied various convergence types of function sequences, including pointwise, uniform, and $\alpha-$convergence, along with their relationships. By examining the conditions under which these convergence types are equivalent or not, we provide examples into the structural properties of function sequences. Notably, our results highlight the critical role of the underlying set's structure (e.g., finite, countable, or compact) in determining these relationships.

The theoretical results presented here contribute to an extended understanding of convergence behaviors for further research in function analysis and its applications. Future work could extend these results to explore additional convergence concepts or their implications in applied sciences.

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The Declaration of Research and Publication Ethics
The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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