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Research Article

Solutions for nonhomogeneous degenerate quasilinear anisotropic problems

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ABSTRACT. In this article, we consider a class of nonlinear elliptic problems, where anisotropic leading differential operator incorporates the unbounded coefficients and the nonlinear term is a convection term. We prove the solvability of degenerate Dirichlet problem with convection, i.e. the existence of at least one bounded weak solution via the theory of pseudomonotone operators, Nemytskii-type operator and a priori estimate in the degenerate anisotropic Sobolev spaces.

Keywords: Degenerate anisotropic $\vec{p}$-Laplacian, degenerate anisotropic Sobolev spaces, unbounded coefficient, bounded solution, truncation, pseudomonotone operator.

2020 Mathematics Subject Classification: 35P30, 35J20, 47J30, 35J70, 35J92.

1. INTRODUCTION

Anisotropic partial differential equations have various applications in the mathematical modelling of physical and mechanical processes. In particular, they are used in models for the dynamics of fluids in anisotropic media when the conductivities of the media are distinct in different directions, or in biology as a model for the propagation of epidemic diseases in nonhomogeneous clusters. The interest in anisotropic problems has deeply increased recently, because many difficulties arise in passing from the isotropic setting to the anisotropic one. For example some fundamental tools available for the isotropic problem (such as the strong maximum principle) cannot be extended to the anisotropic problem (see [1–5, 7–9, 13, 19–21, 23–28] and the references therein).

One of the most interesting problem in a bounded domain $\Omega \subset \mathbb{R}^N$ is the isotropic case of the degenerate quasilinear Dirichlet elliptic equations with convection

$$-div(a(x)|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u). \tag{1.1}$$

Motreanu and Tornatore [17] developed a sub-supersolution approach to prove the existence of nontrivial, nonnegative and bounded solutions for (1.1). In the anisotropic setting, they analyzed the problem (see [19]),

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(G_i(u)|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = F(x, u, \nabla u), \tag{1.2}$$

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where the coefficients in the principal part are unbounded from above, and obtained the existence of solutions in a weak sense for degenerate anisotropic quasilinear Dirichlet problem (1.2).

In the present work, we extend the results above to a more general case. We consider the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = f(x,u,\nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $N \geq 3$ with a Lipschitz boundary $\partial\Omega$, p_i are given real numbers ($1 < p_i < \infty,\ i = 1, 2, \ldots, N$) and, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$ is a Carathéodory function. The function f depends on the solution and its gradient (usually called convection term) satisfies hypotheses (H_1) and (H_2) (see Section 2). Notice that the problem (1.3) includes the differential operator which is anisotropic with measurable coefficients $\nu_i(x,t)$ ($i = 1, 2, \ldots, N$) that can be written in the form $\nu_i(x,t) = a_i(x)g_i(|t|)$ with functions a_i and g_i that will be defined in Section 2.

The novelty of the paper is the new extension of problems (1.1) and (1.2) to a degenerate one in the anisotropic setting. The extended problem (1.3) is degenerate because the weight functions are decomposed in two parts. The first part, $a_i(x)$ can approach zero or be unbounded, the second part $g_i(t)$ can be unbounded from above. Thus, we need to consider the degenerate anisotropic Sobolev space $W_0^{1,\vec{p}}(\vec{a},\Omega)$ (see Section 2) as a suitable function space. By using the theory of pseudomonotone operators, as well as Nemytskii-type operator, and considering an appropriate truncation and a priori estimate in the anisotropic Sobolev spaces, we prove the existence of at least a bounded weak solution for (1.3) as well as the existence of a uniform bound for the solution set in the anisotropic setting. Our existence result for problem (1.3) is formulated as follows:

Theorem 1.1. *Assume that the weight functions $\nu_i : \Omega \times \mathbb{R} \to \mathbb{R}$ have the structure in (2.4) with positive functions $a_i \in L^1_{loc}(\Omega)$ and continuous functions $g_i : [0,+\infty) \to [\alpha_i,+\infty)$ with $\alpha_i > 0$ for $i = 1, 2, \ldots, N$ satisfying the condition (H_1). Assume also that the Caratheodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions (H_2) and (H_3). Then, problem (1.3) possesses at least a bounded weak solution $u \in W_0^{1,\vec{p}}(\vec{a},\Omega) \cap L^\infty(\Omega)$ in the sense of Definition 3.2.*

The rest of the paper is organized as follows. In Section 2, we state the main hypotheses and the structure of the problem (1.3) and we review some facts about the degenerate anisotropic Sobolev spaces which will be used in the sequel. In Section 3, we study the estimate of the solution set of problem (1.3) in $W_0^{1,\vec{p}}(\vec{a},\Omega)$. In Section 4, we prove the solvability of the auxiliary problem (4.26) obtained, which is used as an appropriate truncation, via the theory of pseudomonotone operators and we prove that the problem (1.3) possesses at least a bounded weak solution $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ in the sense of Definition 3.2.

2. Preliminaries

In this section, first, we state the main hypotheses and the structure of the problem (1.3) in Sec. 2.1. Then we recall some facts about the suitable function space $W_0^{1,\vec{p}}(\vec{a},\Omega)$ which is necessary for studying the problem (1.3) in Sec. 2.2.

2.1. **Structure of the problem.** The structure that we admit for the weights ν_i entering problem (1.3) is of the form

$$\nu_i(x,t) := a_i(x)g_i(|t|) \text{ for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}, \tag{2.4}$$

with positive functions $a_i \in L^1_{loc}(\Omega)$ and positive continuous functions $g_i : [0,+\infty[\to [\alpha_i,+\infty[$, with $\alpha_i > 0$ for $i = 1,2,\dots,N$. Moreover, for the functions a_i we assume the following hypothesis

(H_1) $a_i^{-s_i} \in L^1(\Omega)$ for some $s_i \in \left(\max\{\frac{N}{p_i}, \frac{1}{p_i-1}\}, +\infty\right)$ for $i = 1,2,\dots,N$.

We point out that the problem (1.3) is degenerate because the weight functions are decomposed in two parts, the first part $a_i(x)$ can approach zero or be unbounded, the second part $g_i(t)$ can be unbounded from above. We set $\vec{p} := (p_1,p_2,\dots,p_N)$, $\vec{a} = (a_1,a_2,\dots,a_N)$ and $p_{s_i} := (p_s)_i = \frac{p_i s_i}{s_i+1}$ for $i = 1,2,\dots,N$, where the real numbers s_i are given by hypothesis (H_1) and consider the vector $\vec{p}_s = (p_{s_1},\dots,p_{s_N})$. We say $\vec{q} \le \vec{p}$ iff $q_i \le p_i$ for all $i = 1,2,\dots,N$, notice that the definition of $\vec{p}_s$ implies $\vec{p}_s \le \vec{p}$. Using (H_1), we have

$$p_{s_i} > 1, \quad i = 1,2,\dots,N$$

and we assume

$$\sum_{i=1}^N \frac{1}{p_{s_i}} > 1, \tag{2.5}$$

and we set the critical exponent

$$p_s^* := \frac{N}{\sum_{i=1}^N \frac{1}{p_{s_i}} - 1}. \tag{2.6}$$

We introduce

$$p^+ := \max\{p_1,\dots,p_N\} \text{ and } p^- := \min\{p_1,\dots,p_N\}, \tag{2.7}$$

and assume that

$$p^+ < p_s^*. \tag{2.8}$$

For the nonlinear term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$, we assume the following hypotheses

(H_2) there exist the constants $b_1 \ge 0$, $b_2 \ge 0$, $b_3 \ge 0$ and $q \in (p^+, p_s^*)$ such that

$$|f(x,t,\xi)| \le b_1 + b_2|t|^{q-1} + b_3\left(\sum_{i=1}^N a_i|\xi_i|^{p_i}\right)^{\frac{1}{q'}},$$

(H_3) there exist the constants $c_1 \ge 0$, $c_2 \ge 0$ with $c_1 + c_2\eta^{p^-}N^{p^- -1} < \alpha_i$ for all $i = 1,2,\dots,N$ and a function $\varrho \in L^1(\Omega)$ such that

$$f(x,t,\xi)t \le c_1 \sum_{i=1}^N a_i(x)|\xi_i|^{p_i} + c_2|t|^{p^-} + \varrho(x)$$

for all $x \in \Omega$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, where η is given by (2.11).

2.2. **Function space.** In this section, we define the degenerate anisotropic Sobolev spaces (see [14, 15, 21, 22, 26, 27] and references therein). Set

$$\vec{p} := (p_1, p_2, \ldots, p_N)$$

with $1 < p_1, p_2, \ldots, p_N < \infty$ and $\sum_{i=1}^N \frac{1}{p_i} > 1$. We introduce p^+, p^- and p^* as in (2.7) and (2.6), respectively. We recall the anisotropic Sobolev space

$$W^{1,\vec{p}}(\Omega) := \left\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ i = 1, 2, \ldots, N \right\}$$

with the norm $\|u\|_{W^{1,\vec{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}$. The space $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm.

We recall the following theorem [12, Theorem 1].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. If*

$$p_i > 1, \textit{ for all } i = 1, 2, \ldots, N, \quad \sum_{i=1}^N \frac{1}{p_i} > 1,$$

then for all $r \in [1, p_\infty]$ where, $p_\infty = \max\{p^*,\ p^+\}$, there is a continuous embedding $W_0^{1,\vec{p}}(\Omega) \subset L^r(\Omega)$. For $r < p_\infty$, the embedding is compact.

The degenerate Banach space with weight $a \in L^1_{loc}(\Omega)$ which satisfies the condition $a^{-s} \in L^1(\Omega)$ for some $s \in (\frac{N}{p_i}, +\infty) \cap [\frac{1}{p_i - 1}, +\infty)$ is

$$L^{p_i}(a, \Omega) := \left\{ u : \Omega \to \mathbb{R} :\ u \text{ is measurable and } \int_\Omega a(x)|u(x)|^{p_i} dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p_i}(a,\Omega)} = \left(\int_\Omega a(x)|u(x)|^{p_i} dx \right)^{\frac{1}{p_i}}.$$

The degenerate weighted Sobolev space is defined by

$$W^{1,p_i}(a, \Omega) := \left\{ u \in L^{p_i}(\Omega)\ :\ \int_\Omega a(x)|u(x)|^{p_i} dx < \infty \right\}$$

and endowed with the norm

$$\|u\|_{W^{1,p_i}(a,\Omega)} = \|u\|_{L^{p_i}(\Omega)} + \||\nabla u|\|_{L^{p_i}(a,\Omega)}.$$

The space $W_0^{1,p_i}(a, \Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p_i}(a,\Omega)}$. Furthermore,

$$\|u\|_{W_0^{1,p}(a,\Omega)} := \left(\int_\Omega a(x)|\nabla u(x)|^p dx \right)^{\frac{1}{p}} \tag{2.9}$$

for all $u \in W_0^{1,p}(a, \Omega)$, is an equivalent norm on $W_0^{1,p}(a, \Omega)$ for which $W_0^{1,p}(a, \Omega)$ becomes a uniformly convex Banach space.

Now, we recall the next Proposition from [16, Proposition 1] which establishes the continuous embedding of degenerate Sobolev space $W^{1,p}(a, \Omega)$ into the classical Sobolev space $W^{1,p_s}(\Omega)$.

Proposition 2.1. *Let be $p > 1$ and $a \in L^1_{loc}(\Omega)$ which satisfies the condition $a^{-s} \in L^1(\Omega)$ for some $s \in (\frac{N}{p}, +\infty) \cap [\frac{1}{p-1}, +\infty)$. Then, there are continuous embeddings*

$$W^{1,p}(a, \Omega) \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow L^p(\Omega),$$

where $p_s = \frac{ps}{s+1}$. In addition, the embedding $W^{1,p_s}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

Let be $\vec{p} = (p_1, p_2, \dots, p_N)$ and $\vec{a} = (a_1, a_2, \dots, a_N)$ such that condition (H_1) holds, the degenerate anisotropic Sobolev space is given by

$$W^{1,\vec{p}}(\vec{a}, \Omega) = \left\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(a_i, \Omega) \text{ for } i = 1, \dots, N \right\}$$

with the norm $\|u\|_{W^{1,\vec{p}}(\vec{a},\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(a_i,\Omega)}$. The anisotropic Sobolev space $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ with the following norm

$$\|u\| = \|u\|_{W_0^{1,\vec{p}}(\vec{a},\Omega)} := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}$$

is a separable and reflexive Banach space [11, 15].

Finally, by Proposition 2.1, we can prove the following proposition.

Proposition 2.2. *Assume that (H_1), (2.5), (2.6) and (2.8) hold. There are continuous embeddings*

$$W_0^{1,\vec{p}}(\vec{a}, \Omega) \hookrightarrow W_0^{1,\vec{p}_s}(\Omega) \hookrightarrow L^r(\Omega) \tag{2.10}$$

for all $1 \leq r \leq {p_s}^*$ In addition, the embedding $W^{1,\vec{p}}(\vec{a}, \Omega) \hookrightarrow L^r(\Omega)$ is compact for $r < {p_s}^*$. Furthermore $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ is a uniformly convex Banach space.

Proof. In order to prove the first inclusion in (2.10), let $u \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$. Using Hölder's inequality and condition (H_1) (note $p_{s_i} < p_i$), we infer that

$$\begin{aligned}
\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_{s_i}} dx =& \int_\Omega \left(a_i(x)^{\frac{p_{s_i}}{p_i}} |\frac{\partial u}{\partial x_i}|^{p_{s_i}} \right) a_i(x)^{-\frac{p_{s_i}}{p_i}} dx \\
\leq& \left(\int_\Omega a_i(x) |\frac{\partial u}{\partial x_i}|^{p_i} dx \right)^{\frac{p_{s_i}}{p_i}} \left(\int_\Omega a_i(x)^{-\frac{p_{s_i}}{p_i - p_{s_i}}} dx \right)^{\frac{p_i - p_{s_i}}{p_i}} \\
\leq& \|a_i^{-s_i}\|_{L^1(\Omega)}^{\frac{1}{s_i+1}} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}^{p_{s_i}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u\|_{W_0^{1,\vec{p}_s}(\Omega)} =& \sum_{i=1}^N \left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_{s_i}} dx \right)^{\frac{1}{p_{s_i}}} \leq \sum_{i=1}^N \|a_i^{-s_i}\|_{L^1(\Omega)}^{\frac{1}{p_i s_i}} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)} \\
\leq& \Upsilon \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)} = \Upsilon \|u\|
\end{aligned}$$

where $\Upsilon = \max\{\|a_i^{-s_i}\|_{L^1(\Omega)}^{\frac{1}{p_i s_i}} : i = 1, 2, \dots, N\}$. The continuous inclusion $W_0^{1,\vec{p}}(\vec{a}, \Omega) \hookrightarrow W_0^{1,\vec{p}_s}(\Omega)$ is proven.

Also we know by Theorem 2.2 and using (2.8) that the embedding $W_0^{1,\vec{p}_s}(\Omega) \hookrightarrow L^r(\Omega)$ for $1 \leq r < p_s^*$ is compact, then the compactness of the second inclusion in (2.10) follows.

It remains to show that $W_0^{1,\vec{p}}(\vec{a},\Omega)$ is a uniformly convex Banach space. It suffices to have $a_i^{-\frac{1}{p_i-1}} \in L^1(\Omega)$ for all $i = 1, 2, \ldots, N$(see [10, Theorem 1.3]). From hypothesis (H_1), it is known that $a_i^{-s_i} \in L^1(\Omega)$ with $s_i \geq \frac{1}{p_i-1}$, for all $i = 1, 2, \ldots, N$, which results in

$$\begin{aligned}\int_\Omega a_i(x)^{-\frac{1}{p_i-1}}\,dx &= \int_{\{a_i(x)<1\}} a_i(x)^{-\frac{1}{p_i-1}} + \int_{\{a_i(x)\geq 1\}} a_i(x)^{-\frac{1}{p_i-1}}\\ &\leq \int_\Omega a_i(x)^{-s_i} + |\Omega| < \infty,\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Thus completing the proof. □

Taking into account Proposition 2.2, definition of $\vec{p}_s$, (2.7) and (2.8), there exists a positive constant η such that

$$\|u\|_{L^{p^-}(\Omega)} \leq \eta\|u\|, \quad \text{for all } u \in W_0^{1,\vec{p}}(\vec{a},\Omega). \tag{2.11}$$

The degenerate anisotropic $\vec{p}$-Laplacian operator with the weights $a_i \in L^1_{loc}(\Omega)$, $i = 1, 2, \ldots, N$ is defined by the map

$$-\Delta_{\vec{p},\vec{a}}(\cdot) = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x)\left|\frac{\partial(\cdot)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\cdot)}{\partial x_i}\right) : W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^*.$$

This means that

$$\langle -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right), v\rangle = \sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx$$

for all $u,\ v \in W_0^{1,\vec{p}}(\vec{a},\Omega)$.
The definition makes sense as can be seen through Hölder's inequality

$$\begin{aligned}&\left|\sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx\right|\\ \leq&\sum_{i=1}^N \int_\Omega \left(a_i(x)^{\frac{p_i-1}{p_i}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\right)\left(a_i(x)^{\frac{1}{p_i}}\left|\frac{\partial v}{\partial x_i}\right|\right)dx\\ \leq&\sum_{i=1}^N \left(\int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)^{\frac{p_i-1}{p_i}}\left(\int_\Omega a_i(x)\left|\frac{\partial v}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}} < \infty.\end{aligned}$$

Remark 2.1. *The ordinary definition of p-Laplacian is recovered when $p_i = p$ and $a_i(x) = 1$ in Ω, for $i = 1, 2, \ldots, N$.*

Before ending this section we recall the definition of pseudomonotone map.

Definition 2.1. *The map $A : X \to X^*$ is called pseudomonotone if for each sequence $\{u_n\} \subset X$ satisfying $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, it holds*

$$\langle A(v), u - v\rangle \leq \liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \text{ for all } v \in X.$$

The main theorem for pseudomonotone operators reads as follows (see, e.g., [6, Theorem 2.99]).

Theorem 2.3. *Let X be a reflexive Banach space. If the mapping $A : X \to X^*$ is pseudomonotone, bounded and coercive, then it is surjective.*

3. Bounded solutions

We start with the estimate of the solution set of problem (1.3) in $W_0^{1,\vec{p}}(\vec{a},\Omega)$. But before that, we recall the definition of a weak solution for problem (1.3).

Definition 3.2. *The function $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ is called a weak solution to problem (1.3) if $f(x,u,\nabla u)v$ and $\nu_i(x,u)|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$ for $i=1,2,\ldots,N$ are integrable on Ω and*

$$\sum_{i=1}^{N}\int_\Omega \nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx = \int_\Omega f(x,u,\nabla u)vdx \tag{3.12}$$

for all $v \in W_0^{1,\vec{p}}(\vec{a},\Omega)$.

Lemma 3.1. *Under assumptions (H_1) and (H_3), the set of solutions to problem (1.3) is bounded in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ with a bound depending on g_i only through its lower bound α_i, for $i=1,2,\ldots,N$.*

Proof. Set $v = u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ in (3.12), we get

$$\sum_{i=1}^{N}\int_\Omega \nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx = \int_\Omega f(x,u,\nabla u)udx.$$

Hypothesis (H_3), $p_i > 1$ for $i=1,2,\ldots,N$ in conjunction with (2.4) and Proposition 2.2 ensures that

$$\begin{aligned}
&\sum_{i=1}^{N}\alpha_i\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}\\
&\leq \sum_{i=1}^{N}\int_\Omega \nu_i(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx = \int_\Omega f(x,u,\nabla u)udx\\
&\leq c_1\sum_{i=1}^{N}\left(\int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right) + c_2\left(\int_\Omega |u|^{p^-}dx\right) + \int_\Omega \varrho(x)dx\\
&\leq c_1\sum_{i=1}^{N}\left(\int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right) + c_2\,\|u\|^{p^-}_{L^{p^-}(\Omega)} + \|\varrho\|_{L^1(\Omega)}.\\
&\leq c_1\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} + c_2\eta^{p^-}\|u\|^{p^-} + \|\varrho\|_{L^1(\Omega)}\\
&\leq c_1\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} + c_2\eta^{p^-}\left(\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}\right)^{p^-} + \|\varrho\|_{L^1(\Omega)}\\
&\leq c_1\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} + c_2\eta^{p^-}N^{p^--1}\left(N+\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)}\right) + \|\varrho\|_{L^1(\Omega)}.
\end{aligned}$$

Thus

$$\sum_{i=1}^{N}(\alpha_i - c_1 - c_2\eta^{p^-}N^{p^--1})\left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} \leq \|\varrho\|_{L^1(\Omega)} + c_2\eta^{p^-}N^{p^-}.$$

But $\alpha_i - c_1 - c_2\eta^{p^-}N^{p^--1} > 0$ for all $i=1,2,\ldots,N$ and $\varrho \in L^1(\Omega)$ hence the proof is complete. ∎

Theorem 3.4. *Assume that conditions (H_1), (H_2) and (H_3) are fulfilled. Then there exists a constant $C > 0$ such that for each weak solution $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ to problem (1.3) it holds the uniform estimate $\|u\|_{L^\infty(\Omega)} \le C$. The constant C depends on g_i, for $i = 1, 2, \dots, n$, only through its lower bound α_i $(i = 1, 2, \dots, N)$.*

Proof. Let $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ be a weak solution to problem (1.3). We can write $u = u^+ - u^-$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. We have to show that u^+ and u^- are both uniformly bounded by a constant independent of u. We only provide the proof for u^+ because in the case of u^- one can argue similarly.

Our first goal is to prove that

$$u^+ \in L^r(\Omega) \text{ for all } r \in [1, +\infty). \tag{3.13}$$

To this end we insert in (3.12) the test function $v = u^+ u_h^{kp_j} \in W_0^{1,\vec{p}}(\vec{a},\Omega)$, where $u_h := \min\{u^+, h\}$ with arbitrary constants $h > 0$, $k > 0$ and $1 \le j \le N$, thus obtaining

$$\sum_{i=1}^{N} \int_\Omega \nu_i(x,u) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i}(u^+ u_h^{kp_j})dx = \int_\Omega f(x,u,\nabla u) u^+ u_h^{kp_j} dx. \tag{3.14}$$

By means of (2.4), the left-hand side of (3.14) can be estimated from below as

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \nu_i(x,u) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial u^+ u_h^{kp_j}}{\partial x_i} dx \\ =&\sum_{i=1}^{N} \int_\Omega a_i(x) g_i(|u|) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \left(u_h^{kp_j} \frac{\partial u^+}{\partial x_i} + kp_j u^+ u_h^{kp_j-1} \frac{\partial u_h}{\partial x_i}\right) dx \\ \ge&\sum_{i=1}^{N} \alpha_i (kp_j+1) \int_\Omega a_i(x) u_h^{kp_j} \left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} dx > \sum_{i=1}^{N} \alpha_i \int_\Omega a_i(x) u_h^{kp_j} \left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} dx. \end{aligned} \tag{3.15}$$

On the other hand the right hand side of (3.14) by (H_2) implies

$$\begin{aligned} &\int_\Omega f(x,u,\nabla u) u^+ u_h^{kp_j} dx \\ \le& b_3 \int_\Omega \left(\sum_{i=1}^{N} a_i(x) \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \right)^{\frac{1}{q'}} u^+ u_h^{kp_j} dx + b_2 \int_\Omega |u|^{q-1} u^+ u_h^{kp_j} dx + b_1 \int_\Omega u^+ u_h^{kp_j} dx \\ \le& b_4 \sum_{i=1}^{N} \left(\int_\Omega a_i(x)^{\frac{1}{q'}} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{p_i}{q'}} u^+ u_h^{kp_j} dx \right) + b_2 \int_\Omega (u^+)^q u_h^{kp_j} dx + b_1 \int_\Omega u^+ u_h^{kp_j} dx \end{aligned} \tag{3.16}$$

with a constant $b_4 > 0$. We observe that through Young's inequality, for any $\varepsilon > 0$ and a constant $c(\varepsilon) > 0$ we get

$$\begin{aligned} &b_4 \sum_{i=1}^{N} \left(\int_\Omega a_i(x)^{\frac{1}{q'}} \left|\frac{\partial u}{\partial x_i}\right|^{\frac{p_i}{q'}} u^+ u_h^{kp_j} dx \right) \\ =&\varepsilon \sum_{i=1}^{N} \int_\Omega a_i(x) \left|\frac{\partial u^+}{\partial x_i}\right|^{p_i} u_h^{kp_j} dx + C(\varepsilon) \sum_{i=1}^{N} \int_\Omega (u^+)^q u_h^{kp_j} dx, \end{aligned} \tag{3.17}$$

taking into account of previous relation and since

$$\int_\Omega u^+ u_h^{kp_j} dx \le \int_\Omega (u^+)^q u_h^{kp_j} dx + |\Omega|$$

from (3.15) we obtain

$$\begin{aligned}&\int_\Omega f(x,u,\nabla u)u^+u_h^{kp_j}\,dx\\ \text{(3.18)}\qquad &\leq\varepsilon\sum_{i=1}^N\int_\Omega a_i(x)\left|\frac{\partial u^+}{\partial x_i}\right|^{p_i}u_h^{kp_j}\,dx+b_5\left(\sum_{i=1}^N\int_\Omega(u^+)^q u_h^{kp_j}\,dx+1\right)\end{aligned}$$

with a constant $b_5>0$. Then, we derive by (3.15) and (3.18)

$$\text{(3.19)}\qquad \sum_{i=1}^N(\alpha_i-\varepsilon)\int_\Omega a_i(x)\left|\frac{\partial u^+}{\partial x_i}\right|^{p_i}u_h^{kp_j}\,dx\leq b_6\left(\int_\Omega(u^+)^q u_h^{kp_j}\,dx+1\right),$$

with a constant $b_6>0$ and for every $j=1,2,\dots,N$.

We observe that

$$\left|\frac{\partial}{\partial x_i}(u^+u_h^k)\right|\leq(k+1)u_h^k\left|\frac{\partial u^+}{\partial x_i}\right|.$$

Taking into account this relation, from (3.19) if $\varepsilon>0$ is sufficiently small, we obtain for each $j=1,2,\dots,N$

$$\text{(3.20)}\qquad \left\|\frac{\partial(u_h^ku^+)}{\partial x_j}\right\|_{L^{p_j}(a_j,\Omega)}\leq(k+1)b_6^{\frac{1}{p_j}}\left(\int_\Omega(u^+)^q u_h^{kp_j}\,dx+1\right)^{\frac{1}{p_j}}.$$

By hypothesis (H_2), we can find $r\in(p_j,q)$ satisfying

$$\text{(3.21)}\qquad \frac{(q-p_j)r}{r-p_j}\leq p_s^*,\quad j=1,2,\dots,N.$$

We can estimate the left-hand side of (3.19), using Hölder's inequality getting for any $k>0$

$$\begin{aligned}\int_\Omega u_h^{kp_j}\left(u^+\right)^q dx&=\int_\Omega\left(u_h^ku^+\right)^{p_j}\left(u^+\right)^{q-p_j}dx\\ &\leq\left(\int_\Omega\left(u^+\right)^{\frac{(q-p_j)r}{r-p_j}}dx\right)^{\frac{r-p_j}{r}}\left(\int_\Omega\left(u_h^ku^+\right)^r dx\right)^{\frac{p_j}{r}}\\ &\leq M\|u_h^ku^+\|_{L^r(\Omega)}^{p_j},\end{aligned}$$

where $M>0$ is a constant that does not depend on the solution u of (1.3). The independence of M with respect to the solution u is a consequence of Lemma 3.1 and the continuous embedding $W_0^{1,\vec{p}}(\vec{a},\Omega)\hookrightarrow L^{\frac{(q-p_j)r}{r-p_j}}(\Omega)$ that follows from Proposition 2.2 and (3.21), morover constant M depends on g_i, for $i=1,2,\dots,N$, only for its lower bound α_i.

Inserting the previous inequality into (3.20), we obtain

$$\left\|\frac{\partial(u_h^ku^+)}{\partial x_j}\right\|_{L^{p_j}(a_j,\Omega)}\leq(k+1)b_6^{\frac{1}{p_j}}\left(M\|u_h^ku^+\|_{L^r(\Omega)}^{p_j}+1\right)^{\frac{1}{p_j}}.$$

Summing on j from 1 to N, we have

$$\left\|u_h^ku^+\right\|\leq(k+1)b_7N\left(\|u_h^ku^+\|_{L^r(\Omega)}+1\right)$$

with a constant $b_7>0$.

From the continuous embedding $W_0^{1,\vec{p}}(\vec{a},\Omega)\hookrightarrow L^{p_s^*}(\Omega)$ and using Fatou's lemma, we get

$$\text{(3.22)}\qquad \|u^+\|_{L^{(k+1)p_s^*}(\Omega)}\leq b_8^{\frac{1}{k+1}}(k+1)^{\frac{1}{k+1}}N^{\frac{1}{k+1}}\left(\|u^+\|_{L^{(k+1)r}(\Omega)}+1\right)$$

with a constant $b_8>0$.

Without loss of generality, we may suppose that $\|u^+\|_{L^{(k+1)r}(\Omega)} > 1$ except for finitely many k (otherwise conclusion readily follows), moreover, since the sequence $(k+1)^{\frac{1}{\sqrt{k+1}}}$ is bounded, (3.22) gives rise to a constant $b > 0$ such that accordingly, (3.22) amounts to saying that

$$\|u^+\|_{L^{(k+1)p_s^*}(\Omega)} \leq b^{\frac{1}{\sqrt{k+1}}} \|u^+\|_{L^{(k+1)r}(\Omega)} \tag{3.23}$$

with a constant $b > 0$ independent of k and of the solution u, and for which the dependence on g_i, for $i = 1, 2, \ldots, N$, reduces to the dependence on α_i. At this point, we can implement the Moser iteration with $(k_n+1)r = (k_{n-1}+1)p_s^*$ posing $(k_1+1)r = p_s^*$ if $\|u^+\|_{L^{(k+1)r}(\Omega)} > 1$ for all k and $(k_1+1)r = (k_0+1)p_s^*$ if $\|u^+\|_{L^{(k_0+1)r}(\Omega)} \leq 1$ and $\|u^+\|_{L^{(k+1)r}(\Omega)} > 1$ for all $k > k_0$. Then (3.23) renders

$$\|u^+\|_{L^{(k_n+1)p_s^*}(\Omega)} \leq b^{\sum_{1\leq i\leq n} \frac{1}{\sqrt{k_i+1}}} \|u^+\|_{L^{p_s^*}(\Omega)}, \text{ for all } n \geq 1. \tag{3.24}$$

Letting $n \to \infty$ in (3.24) since the series converges and $k_n \to +\infty$ as $n \to \infty$ the uniform boundedness of the solution set of (1.3) is achieved then there exists a positive constant C such that $\|u\|_{L^\infty(\Omega)} \leq C$. A careful reading of the proof shows that the dependence of the uniform bound C on g_i, for $i = 1, 2, \ldots, N$, arises just through the lower bound α_i of g_i for $i = 1, 2, \ldots, N$. This completes the proof. □

4. TRUNCATED WEIGHT AND ASSOCIATED OPERATOR

For any number $R > 0$ we consider the following truncation of the weights $\nu_i(x,u)$, for $i = 1, 2, \ldots, N$ in problem (1.3):

$$\nu_{iR}(x,t) = a_i(x) g_{iR}(|t|), \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where

$$g_{iR}(t) = \begin{cases} g_i(t) & \text{if} \quad t \in [0,R], \\ g(R) & \text{if} \quad t > R. \end{cases} \tag{4.25}$$

Corresponding to the truncation in (4.25), we state the auxiliary problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\nu_{iR}(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = f(x,u,\nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.26}$$

Our approach to study problem (4.26) is based on the theory of pseudomonotone operators. In this respect, we introduce the mapping, corresponding to an $R > 0$

$$A_R : W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^*$$

as

$$A_R = \mathcal{A} - \mathcal{N}, \tag{4.27}$$

with the degenerate anisotropic operator associated to the truncated weights $\nu_{iR}(x,t)$ ($i = 1, 2, \ldots, N$), $\mathcal{A} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^\star$ defined by

$$\langle \mathcal{A}(u), v\rangle = \sum_{i=1}^{N} \int_\Omega \nu_{iR}(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx \text{ for all } u, v \in W_0^{1,\vec{p}}(\vec{a},\Omega) \tag{4.28}$$

and a Nemytskii-type operator $\mathcal{N} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to L^{\overline{p}'}(\Omega)$ defined by

$$\langle \mathcal{N}(u), v\rangle = \int_\Omega f(x,u(x),\nabla u(x))v(x)dx \text{ for all } u, v \in W_0^{1,\vec{p}}(\vec{a},\Omega). \tag{4.29}$$

Remark 4.2. *One has that $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ is a (weak) solution to problem (4.26) if and only if it solves the equation $A_R(u) = 0$ with A_R given in (4.27).*

The next propositions focus on the properties of the operators $\mathcal{A}$ and $\mathcal{N}$.

Proposition 4.3. *Given $R > 0$, let $\mathcal{A} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^*$ be as in (4.28). Then, $\mathcal{A}$ is well defined, bounded and continuous, moreover it has the S_+-property, that is, any sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\vec{a},\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ and*

$$\limsup_{n\to+\infty} \langle \mathcal{A}(u_n), u_n - u\rangle \le 0 \tag{4.30}$$

satisfies $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$.

Proof. By (2.4), (4.25), the continuity of g_i, for $i = 1, 2, \ldots, N$ and Hölder's inequality, we get

$$\begin{aligned}
|\langle \mathcal{A}(u), v\rangle| \le& \sum_{i=1}^{N} \int_\Omega a_i(x) g_{iR}(u) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} \left|\frac{\partial v}{\partial x_i}\right| dx \\
\le& \sum_{i=1}^{N} \max_{t\in[0,R]} g_i(t) \int_\Omega a_i(x) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1} \left|\frac{\partial v}{\partial x_i}\right| dx \\
\le& G \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}^{p_i-1} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}
\end{aligned}$$

for all $u, v \in W_0^{1,\vec{p}}(\vec{a},\Omega)$, where $G = \max_{1\le i\le N}\left(\max_{t\in[0,R]} g_i(t)\right)$. The operator $\mathcal{A}$ in (4.28) is thus well defined and bounded.

Assume that $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ we can prove that $\mathcal{A}(u_n) \to \mathcal{A}(u)$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)^*$. We observe that

$$\begin{aligned}
&\|\mathcal{A}(u_n) - \mathcal{A}(u)\|_{W_0^{1,\vec{p}}(\vec{a},\Omega)^*} \\
\le& G \sum_{i=1}^{N} \left(\int_\Omega a_i(x) \left| \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \right|^{\frac{p_i}{p_i-1}} dx \right)^{\frac{p_i-1}{p_i}} \\
&+ \sum_{i=1}^{N} \left(\int_\Omega a_i(x) \left|g_{iR}(|u_n|) - g_{iR}(|u|)\right|^{\frac{p_i}{p_i-1}} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \right)^{\frac{p_i-1}{p_i}} \\
=& G \sum_{i=1}^{N} \left\| \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \right\|_{L^{\frac{p_i-1}{p_i}}(a_i,\Omega)} \\
&+ \sum_{i=1}^{N} \left\| \left(g_{iR}(|u_n|) - g_{iR}(|u|)\right)^{\frac{1}{p_i-1}} \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(a_i,\Omega)}^{p_i-1}.
\end{aligned}$$

Applying Lebesgue's Dominated Convergence Theorem on the basis of the continuity of g_i ($i = 1, 2, \ldots, N$) and the strong convergence $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ we have

$$\begin{aligned}
&\left\| \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \right\|_{L^{\frac{p_i}{p_i-1}}(a_i,\Omega)} \to 0 \text{ and} \\
&\left\| \left|g_{iR}(|u_n|) - g_{iR}(|u|)\right|^{\frac{1}{p_i-1}} \left|\frac{\partial u}{\partial x_i}\right| \right\|_{L^{p_i}(a_i,\Omega)} \to 0
\end{aligned}$$

as $n \to \infty$ which establishes the desired conclusion.

Let $\{u_n\} \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ be a sequence which satisfies (4.30). By (4.28) and Hölder's inequality, we have

$$\begin{aligned}
&\langle \mathcal{A}(u_n) - \mathcal{A}(u), u_n - u\rangle \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x) g_{iR}(|u_n|) \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \\
&+ \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|)\right) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx \\
&\geq \alpha^- \sum_{i=1}^{N} \left[(\|u_n\|^{p_i-1}_{W_0^{1,p_i}(a_i,\Omega)} - \|u\|^{p_i-1}_{W_0^{1,p_i}(a_i,\Omega)})(\|u_n\|_{W_0^{1,p_i}(a_i,\Omega)} - \|u\|_{W_0^{1,p_i}(a_i,\Omega)}) \right] \\
&+ \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|)\right) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx,
\end{aligned} \tag{4.31}$$

with $\alpha^- = \min\{\alpha_1, \ldots, \alpha_N\}$. The assumptions $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ and (4.30) imply

$$\limsup_{n\to\infty} \langle \mathcal{A}_R(u_n) - \mathcal{A}_R(u), u_n - u\rangle \leq 0. \tag{4.32}$$

We also have

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|)\right) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx = 0. \tag{4.33}$$

Indeed, through Hölder's inequality and the boundedness of $\{u_n\}$ in $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ (note that $u_n \rightharpoonup u$), there is a constant $\tilde{\alpha} > 0$ such that

$$\begin{aligned}
&\left| \sum_{i=1}^{N} \int_\Omega a_i(x) \left(g_{iR}(|u_n|) - g_{iR}(|u|)\right) \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial (u_n - u)}{\partial x_i} dx \right| \\
&\leq \tilde{\alpha} \sum_{i=1}^{N} \left(\int_\Omega a_i(x) \left|g_{iR}(|u_n|) - g_{iR}(|u|)\right|^{\frac{p_i-1}{p_i}} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \right)^{\frac{p_i-1}{p_i}}.
\end{aligned}$$

Then (4.33) is achieved by applying Lebesgue's Dominated Convergence Theorem on the basis of the continuity of g_i $(i = 1, 2, \ldots, N)$. Combining (4.31), (4.32) and (4.33) we have

$$\lim_{n\to\infty} \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|^{p_i-1}_{L^{p_i}(a_i,\Omega)} - \left\|\frac{\partial u}{\partial x_i}\right\|^{p_i-1}_{L^{p_i}(a_i,\Omega)} \right) \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)} - \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)} \right) = 0.$$

Then

$$\lim_{n\to\infty} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)} = \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(a_i,\Omega)}, \quad i = 1, 2, \ldots, N.$$

Since the space $W_0^{1,\vec{p}}(\vec{a}, \Omega)$ is uniformly convex (see Proposition 2.2), we obtain $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a}, \Omega)$, this proves the S_+-property of the operator $\mathcal{A}$. □

Proposition 4.4. *Assume that hypotheses (H_1)-(H_3) hold. Then the map $\mathcal{N} : W_0^{1,\vec{p}}(\vec{a}, \Omega) \to L^{q'}(\Omega)$ in (4.29) is well defined, continuous and bounded.*

Proof. Assumption (H_2) yields

$$\int_\Omega |f(x,u,\nabla u)|^{q'}dx \leq \theta\left(\int_\Omega \sum_{i=1}^N \left(a_i(x)|\frac{\partial u}{\partial x_i}|^{p_i}\right)dx + \int_\Omega |u|^q dx + 1\right)$$
$$\leq \theta\left(\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} + \|u\|^q_{L^q(\Omega)} + 1\right)$$

for all $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ with a constant θ. Hence $\mathcal{N}(u) \in L^{q'}(\Omega)$ whenever $u \in W_0^{1,\vec{p}}(\vec{a},\Omega)$, Thus the previous estimate shows that

$$\mathcal{N} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to L^{q'}(\Omega)$$

is well defined and bounded. Therefore the mapping $\mathcal{N}$ is continuous. Let $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$, so $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^{p_i}(a_i,\Omega)$ $(i = 1,2,\ldots,N$ and $u_n \to u$ in $L^q(\Omega)$. Hypothesis (H_2) and Krasnoseleskii's theorem on Nemitskii operator assure that $f(x,u_n,\nabla u_n) \to f(x,u,\nabla u)$ in $L^{q'}(\Omega)$ whence $\mathcal{N}$ is a continuous operator. □

Now we are able to prove the solvability of auxiliary problem (4.26).

Theorem 4.5. *Assume that the weights $\nu_i : \Omega \times \mathbb{R} \to \mathbb{R}$ has the structure in (2.4) with a positive functions $a_i \in L^1_{loc}(\Omega)$ satisfying the condition (H_1) and continuous functions $g_{iR} : [0,+\infty) \to [\alpha_i,+\infty)$ with $\alpha_i > 0$ and $i = 1,2,\ldots,N$. If $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the conditions (H_2) and (H_3), then problem (4.26) has a weak solution $u_R \in W_0^{1,\vec{p}}(\vec{a},\Omega)$ for every $R > 0$.*

Proof. We are going to apply Theorem 2.3 to the operator A_R in (4.27) with any fixed $R > 0$. By Propositions 4.3 and 4.4 it is known that the mapping A_R is bounded. Let us show that A_R is a pseudomonotone operator. To this end, let $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$ and

$$\limsup_{n\to\infty}\langle A_R(u_n), u_n - u\rangle \leq 0. \tag{4.34}$$

There holds

$$\limsup_{n\to\infty}\langle \mathcal{N}(u_n), u_n - u\rangle = 0 \tag{4.35}$$

as can be noticed from Proposition 4.3 since

$$|\langle \mathcal{N}(u_n), u_n - u\rangle| \leq \|\mathcal{N}(u_n)\|_{L^{q'}(\Omega)}\|u_n - u\|_{L^q(\Omega)}$$

and $u_n \to u$ in $L^q(\Omega)$ (refer to the compact embedding of $W_0^{1,\vec{p}}(\vec{a},\Omega)$ into $L^q(\Omega)$ and that $\mathcal{N}(u_n)$ is bounded in $L^{q'}(\Omega)$).

On the basis of (4.27) and (4.35), we note that (4.34) reduces to (4.30). We are thus enabled to apply Proposition 4.3 obtaining the strong convergence $u_n \to u$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)$. In view of the continuity of the maps $\mathcal{A} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to W_0^{1,\vec{p}}(\vec{a},\Omega)^*$ and $\mathcal{N} : W_0^{1,\vec{p}}(\vec{a},\Omega) \to L^{q'}(\Omega)$ for which we address to Proposition 4.3 and Proposition 4.4, we infer that $A_R(u_n) \to A_R(u)$ in $W_0^{1,\vec{p}}(\vec{a},\Omega)^*$ and $\langle A_R(u_n), u_n\rangle \to \langle A_R(u), u\rangle$, thus A_R is pseudomonotone.

We turn our attention to show that the operator A_R in (4.27) is coercive which reads as

$$\lim_{\|u\|\to\infty}\frac{\langle A_R(u), u\rangle}{\|u\|} = +\infty. \tag{4.36}$$

The proof is carried out by making use of hypothesis (H_3) that implies for $\|u\| > 1$ we have

$$\begin{aligned}
&\langle A_R(u), u\rangle \\
&= \sum_{i=1}^{N} \int_\Omega \nu_{iR}(x,u)|\frac{\partial u}{\partial x_i}|^{p_i}dx - \int_\Omega f(x,u,\nabla u)udx \\
&\geq \sum_{i=1}^{N} \alpha_i \int_\Omega a_i(x)|\frac{\partial u}{\partial x_i}|^{p_i}dx - c_1\sum_{i=1}^{N} \int_\Omega a_i(x)|\frac{\partial u}{\partial x_i}|^{p_i}dx - c_2\int_\Omega |u|^{p^-}dx - \int_\Omega \varrho(x)dx \\
&= \sum_{i=1}^{N} \alpha_i \left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} - c_1\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} - c_2\|u\|^{p^-}_{L^{p^-}(\Omega)} - \|\varrho\|_{L^1(\Omega)} \\
&\geq \sum_{i=1}^{N}(\alpha_i - c_1 - c_2\eta^{p^-}N^{p^- -1}) \left\|\frac{\partial u}{\partial x_i}\right\|^{p_i}_{L^{p_i}(a_i,\Omega)} - c_2\eta^{p^-}N^{p^-} - \|\varrho\|_{L^1(\Omega)}.
\end{aligned}$$

Since $\alpha_i - c_1 - c_2\eta^{p^-}N^{p^- -1} > 0$ for all $i = 1, 2, \dots, N$ and $p^- > 1$, we infer that (4.36) holds true. Therefore it is allowed to apply Theorem 2.3, which provides a solution $u_R \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ for the operator equation $A_R(u_R) = 0$. Invoking Remark 4.2, u_R represents a weak solution to equation (4.26). The proof is complete. □

Now we can state the proof of the main result, i.e. the proof of Theorem 1.1.

Proof. Theorem 3.4 ensures that the entire set of solutions of problem (1.3) is uniformly bounded, that is, there exists a constant $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq C$ for all weak solutions $u \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ to problem (1.3). The truncated problem (4.26) satisfies exactly the same hypotheses, and with the same coefficients, as the original problem (1.3) with g_{iR} in place of g_i. It is essential to note that the inequality $\alpha_i - c_1 - c_2\eta^{p^-}N^{p^- -1} > 0$ for all $i = 1, 2, \dots, N$, assumed in hypothesis (H_3) is independent of $R > 0$. Consequently, Theorem 3.4 applies to the truncated problem (4.26) involving the truncation g_{iR} and produces the same uniform bound $C > 0$ for the solution set of (4.26) with any $R > 0$. Actually, the statements of Theorem 3.4 and Lemma 3.1 show that the uniform bound $C > 0$ for the solution set depends on the function g_i only through the lower bound α_i of g_i, which is the same for each truncation g_{iR}. In particular, we have that the solution $u_R \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ to problem (4.26) provided by Theorem 4.5 satisfies the estimate $\|u_R\|_{L^\infty(\Omega)} \leq C$ whenever $R > 0$.

Now choose $R \geq C$. Then the estimate $\|u_R\|_{L^\infty(\Omega)} \leq C$ and (4.25) imply

$$g_{iR}(|u_R(x)|) = g_i(|u_R(x)|) \text{ for all } x \in \Omega \text{ and } i = 1, 2, \dots, N,$$

hence due to (2.4),

$$\nu_{iR}(x, u_R(x)) = \nu_i(x, u_R(x)) \text{ for all } x \in \Omega \text{ and } i = 1, 2, \dots, N.$$

It follows that the solution $u_R \in W_0^{1,\vec{p}}(\vec{a}, \Omega)$ to the auxiliary problem (4.26) is a bounded weak solution to the original problem (1.3), which completes the proof of Theorem 1.1. □

Remark 4.3. *We end this section by observing that, in the case of variable exponents, that is when $p_j = p_j(x)$, there are many applications to electrorheological fluids, thermorheological fluids, elastic materials, and image restoration. Thus (1.3) can be studied in the case of variable exponents, for the future studies.*

5. Conclusion

We study the nonlinear elliptic problem (1.3) characterized by an anisotropic leading differential operator that includes unbounded coefficients, with the nonlinear component being a convection term. This problem is a new extension of problems (1.1) and (1.2) to a degenerate one in the anisotropic setting. Our result shows the solvability of the degenerate Dirichlet problem associated with convection, demonstrating the existence of at least one bounded weak solution. This is achieved through the application of the theory of pseudomonotone operators, the Nemytskii-type operator, and a priori estimates within the framework of degenerate anisotropic Sobolev spaces. To the best of our knowledge, our result represents the first result in this literature.

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