Journal of Universal Mathematics
Vol.7 No.2 pp.113-127 (2024)
ISSN-2618-5660
DOI: 10.33773/jum.1504811

SOME COLORING RESULTS ON SPECIAL SEMIGROUPS OBTAINED FROM PARTICULAR KNOTS

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Abstract. For a coloring set $B \subseteq \mathbb{Z}_n$, by considering the Fox n-coloring of any knot K and using the knot semigroup K_S, we show that the set B is actually the same with the set C in the alternating sum semigroup $AS(\mathbb{Z}_n, C)$. Then, by adapting some results on Fox n-colorings to $AS(\mathbb{Z}_n, B)$, we obtain some new results over this semigroup. In addition, we present the existence of different homomorphisms (or different isomorphisms in some cases) between the semigroups K_S and $AS(\mathbb{Z}_n, B)$, and then obtained the number of homomorphisms is in fact a knot invariant. Moreover, for different knots K^1 and K^2, we establish one can obtain a homomorphism or an isomorphism from the different knot semigroups K^1_S and K^2_S to the same alternating sum semigroup $AS(\mathbb{Z}_n, B)$.

1. Introduction

It is known that the knots are equalivance classes of topological inclusions from $\mathbb{S}^1$ to $\mathbb{S}^3$ under ambient isotopes which these isotopes give the smooth deformations between two knots. We may refer the classical book [8] for the details in knot theory. In here, we will mainly give our interest to Torus knots and Pretzel links during the construction of our theories.

As indicated in [7], the fundamental quandle of a knot was defined in a similar manner to the fundamental group of a knot, which made quandles are important tools in knot theory. The number of homomorphisms from the fundamental quandle to a fixed finite quandle has an interpretation as colorings of knot diagrams by quandle elements, and has been widely used as a knot invariant. Furthermore involutary quandles are defined on a single binary operation ([10]). In detail, they are the algebraic way to represent the Reidemeister movements ([1]) and so they are important to obtain new knot invariants and also important to investigate knots. On the other hand, Fox n-colorings are actually the best known involutary quandles. These colorings will be briefly indicated in coming next subsection, and also one part of the main result will be constructed the base on this subject (see Theorem

Date: **Received:** 2024-06-25; **Accepted:** 2024-07-23.
Key words and phrases. Knots, Quandle, Alternating sum semigroups, Knot semigroups, Homomorphisms.
Mathematics Subject Classification: 05C05, 05C12.

2.1 in below). In fact the other whole main theorems (which are about Pretzel links and Torus knots) given in this paper can be thought as consequences of Theorem 2.1.

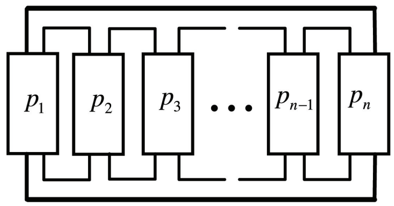
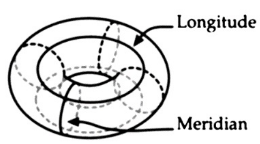


FIGURE 1.

Let $P(p_1, p_2, \cdots, p_n)$ be an n-Pretzel link $\mathbb{S}^3$ in where $p_i \in \mathbb{Z}$ represents the number of half twists (or, we can call it as *regions*) as depicted in Figure 1. If $n = 3$, then it is called a classical pretzel link $P(p, q, r)$. If n is odd, then an n-Pretzel link $P(p_1, p_2, ..., p_n)$ is a knot if and only if none of two p_i's are even. If n is even, then $P(p_1, p_2, ..., p_n)$ is a knot if and only if only one of the p_i's is even. Generally the number of even p_i's is the number of components unless p_i's are all odd. On the other hand, Torus knots are identified by the number of times the strand wraps around the torus meridionally and longitudinally. We speak of a Torus knot $T_{p,q}$, where p and q are relatively prime; when p and q are not relatively prime, we obtain a link of two or more components ([15]).



Let K be an oriented knot (or link) with n crossings. Label those crossings by $1, 2, \cdots, n$ and label the n arcs by $a_1, a_2, \cdots, a_n$. Construct an $n \times n$ matrix M such that each row r corresponds to the crossing labeled by again r and each column s corresponds to the arc labeled by again s. Suppose that at crossing r the over-passing arc is labeled a_i, that the arc a_j ends at crossing r, and that the arc a_k begins at crossing r. Suppose also that i, j and k are mutually distinct. Assume also that crossing r is positive. Then, for a real number t, the entries will be the formed as $M(r, i) = 1 - t$, $M(r, j) = -1$ and $M(r, k) = t$. When crossing r is negative, then $M(r, i) = 1 - t$, $M(r, j) = t$, $M(r, k) = -1$ and other elements of M are zero.

The Alexander matrix A_K is defined as to be the matrix obtained from the matrix M by deleting row n and column n. It is also known that the Alexander polynomial $\Delta_K(t)$ of a knot K is the determinant of it's Alexander matrix (see, for instance, [2, 11]), and the Alexander polynomial at $t = -1$ (and then taking absolute value) defines the determinant of a knot K. We recall that the Alexander polynomial is the first invariant polynomial defined on knots. The invariant property of these polynomials of the knots that belongs to the same equivalence classes are the same. We note that while the Alexander polynomials of a Torus $T_{p,q}$ (cf. [15]) and a

Pretzel link $P(p,q,r)$ (cf. [19]) are

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \quad \text{and}$$
$$\Delta_{P(p,q,r)}(t) = \frac{1}{4}\left[(pq+pr+qr)(t-2+t^{-1})+(t+2+t^{-1})\right] ,$$

respectively, the determinants of them are calculated by

$$\Delta_{T_{p,2}}(-1) = p \quad \text{and} \quad \Delta_{P(p,q,r)}(-1) = pq+pr+qr \,, \tag{1.1}$$

and respectively.

1.1. **Quandles and Fox n-Coloring.** For any set Q, by defining two binary operations $x \triangleright y$ and $x \triangleright^{-1} y$ which satisfy $(x \triangleright y) \triangleright^{-1} y = x$, one can obtain a *quandle* over Q. If only $(x \triangleright y) \triangleright y = x$ holds, then it is named as *involutory quandle*. On the other hand, the other important quandle is the named as *Alexander quandle* which consists of a quandle with a left action given by $a \triangleright b = ta + (1-t)b$. The importance of Alexander quandle comes from the fact that it is another way the computation of Alexander polynomials. On the other hand, if we take $t = -1$ in an Alexander quandle, then we get the *dihedral quandle*. The dihedral quandles are placed into knot colorings (in some sources, authors use the term Fox n-coloring). We may refer, for instance, [4, 5, 6, 7, 9, 10, 14] for more details on quandles, colorings and some other well known types. In this paper, we will apply Fox n-coloring to the knots in terms of dihedral quandles by following the fact that they are knot invariant and very useful for the characterization of a knot.

At this point let us briefly indicate the meaning of Fox n-coloring. For a knot K and a diagram D of K, let A be the set of arcs in D. Now let us matching (not necessarily one to one) the elements of A by the elements of $\mathbb{Z}_n$. Also, for each matching, let us consider the equivalence

$$a \triangleright b \equiv c \equiv -a + 2b \pmod{n} \;\text{ such that }\; n \geq 2 \tag{1.2}$$

such that a and c represent the numerical values in $\mathbb{Z}_n$ for the bottom arcs, respectively, while b represents the numerical value in $\mathbb{Z}_n$ for the upper arc. After all, if whole equivalences satisfy up to $\mathbb{Z}_n$ then we say that the knot K is named as *Fox n-colorable* (or shortly *n-colorable*). The subject Fox n-coloring is actually correspondent to the involutory quandle ([10]). In here, we strongly note that since the matrix obtained by deleting the last row and column of the coefficient matrix of n-coloring equations and the matrix obtained by replacing $t = -1$ in Alexander matrix of K are the same, we get that the positive integer n is the determinant of K itself (in other words $n = \Delta_K(-1)$) or it is a positive integer that divides this determinant (in other words $n \mid \Delta_K(-1)$).

Now let us denote the number of colorings of K in terms of the quandle Q by $Col_Q(K)$. Then we have the following lemma.

Lemma 1.1 ([7])**.** *The quandle Q distinguishes knots K and K' if $Col_Q(K) \neq Col_Q(K')$.*

1.2. **Semigroups K_S and $AS(G,B)$.** Recently, it has been defined a new semigroup under the name of *knot semigroups* and denoted by K_S (cf. [18]). The elements of K_S are the arcs of the knot K and the relations are every crossings on K. In fact for a single crossing as in Figure 2, we have two relations $xy = yz$ and $zy = yx$, where x, y and z are the generators.

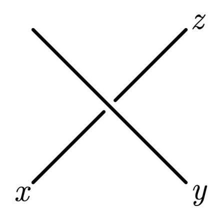


FIGURE 2. Two relations $xy = yz$ and $zy = yx$ obtained from a single crossing

There are some immediate examples that can be given. Firstly, since the *unknot* (or, equivalently, a *circle*), notated by 0_1, contains a unique arc without any crossing, then the knot semigroup of unknot is actually a free semigroup with a single generator which can be expressed as $K_{S(0_1)} = \langle x\,;\,\rangle$. Second example can be given on torus knots $T_{p,q}$, where p and q are relatively prime; when p and q are not relatively prime we obtain a link of two or more components. By taking $q = 2$, we obtain the torus knot semigroups

$$\begin{aligned} K_{ST_{p,2}} = \langle\, a_0, a_1, a_2, ..., a_{p-1} \quad ; \quad & a_0a_1 = a_1a_2,\ a_1a_2 = a_2a_3,\ \cdots,\ a_{p-2}a_{p-1} = a_{p-1}a_0, \\ & a_0a_{p-1} = a_{p-1}a_{p-2},\ a_{p-1}a_{p-2} = a_{p-2}a_{p-3}, \\ & \cdots,\ a_2a_1 = a_1a_0 \,\rangle\ . \end{aligned} \tag{1.3}$$

The diagram for the torus knot semigroup

$$K_{ST_{3,2}} = \langle x, y, z\,;\, xy = yz,\ zy = yx,\ yx = xz,\ zx = xy,\ xz = zy,\ yz = zx\rangle$$

is drawn in Figure 3.

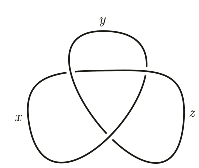


FIGURE 3. The diagram of the knot semigroup $K_{ST_{3,2}}$ and its presentation

In the following, we will give our attention to important terminologies, namely *alternating sum* and *alternating sum semigroups*, for the knot semigroups. The details and some properties on them can be found in [18].

Definition 1.2 ([18])**.** Let G be a group as the form of either $\mathbb{Z}_n$ or $\mathbb{Z}$, and let $B \subseteq G$. For any positive word $b_1b_2b_3\cdots b_k \in B^+$, the alternating sum of this word is the value of the expression

$$b_1 - b_2 + b_3 \cdots (-1)^{k+1}b_k$$

that is calculated in G. Further, any such two words $u, v \in B^+$ are in relation $\sim$ if and only if the length of u is equal to the length of v and the alternating sum of u is equal to the alternating sum of v.

Moreover since the relation $\sim$ is a congruence on the set B^+, we then get a factor semigroup $B^+/\sim$. Let us denote it shortly by $AS(G, B)$ and call it an alternating sum semigroup.

Another version of the alternating sum semigroup has also been defined in [18] under the name of strong alternating sum semigroups which will not be needed in this paper.

Since one of our main aim is to obtain a homomorphism (or an isomorphism in some special cases) between knot semigroups and alternating sum semigroups, in the following we will give some fundamental facts about it.

Suppose that A^+/κ is a knot semigroup, where A is the set of arcs and κ is the cancellative congruence on the free semigroup A^+ induced by the defining relations of the knot semigroup. Also similarly as above, let $\sim$ be a congruence on B^+, where B is an alphabet of the same size as A. To obtain an isomorphism between A^+/κ and $B^+/\sim$, the following lemma is useful.

Lemma 1.3 ([18])**.** *Let us consider a bijection $\phi : A \to B$ that in fact induces an isomorphism $\phi : A^+ \to B^+$. Consider a congruence κ on A^+ and a congruence $\sim$ on B^+ such that for each $u, v \in A^+$, if $u\kappa v$ then $\phi(u) \sim \phi(v)$. Then ϕ induces not only a mapping but also a homomorphism $\psi : A^+/\kappa \to B^+/\sim$. Additionally let us suppose that there exists a subset, namely set of canonical words, of B^+ such that in each class of $\sim$ there is exactly one canonical word and at least one word of each class of κ is mapped by ϕ to a canonical word. Then ψ is actually an isomorphism.*

By considering Lemma 1.3, it has been proved the following theory in [18].

Proposition 1 ([18])**.** The knot semigroup $K_{ST_{p,2}}$ of the torus knot diagram $T_{p,2}$ (where p is odd) is isomorphic to the alternating sum semigroup $AS(\mathbb{Z}_p, \mathbb{Z}_p)$.

In this paper, it will be detailed this isomorphism defined in Proposition 1 up to decomposition of p. More clearly, we will say that the set B is changed depends on the value of p or the label corresponding an arc on the diagram of the torus. (See Theorem 2.12, Corollary 3 below).

2. Main Results

Under this section, we will present our main theorems to reach the aim of this paper.

2.1. **Connection Fox n-Coloring and Alternating Sum Semigroup.**

In this first result section, by comparing the Fox n-Coloring which is used for coloring of knots and the alternating sum semigroup, we will get the number of homomorphism from first to second, and also solve a conjecture given in [18, Conjecture 24]. In fact our approximation solve a more general case.

Theorem 2.1. *Let C be a set for using n-coloring of the knot K. Then there exists a homomorphism[1] from the knot semigroup K_S of K to the alternating sum semigroup $AS(G, C)$, where the set G is actually $\mathbb{Z}_n$ that is used for n-coloring. In fact the reverse part is also valid.*

Proof. By the meaning of Fox n-coloring, each arc in the knot was matched with an element of $\mathbb{Z}_n$ and the values obtained after each matching had to be satisfied Equation (1.2) which was written for each crossing of the knot. On the other hand, the relations of the knot semigroup K_S are the relations of the form $xy = yz$ and

[1]We should note that when we define such a homomorphism, we assume that the numerical value of each arc in the knot diagram and the values of these arcs in the semigroup $AS(\mathbb{Z}_n, C)$ are equal.

$zy = yx$ that were written for each crossing. It is easy to see that if we carry these relations to any alternating sum semigroup $AS(G, C)$ such that $C \subseteq G$, then they become the form of $x - y = y - z$ and $z - y = y - x$ since the subset C contains the relations that satisfy the equations $x - y = y - z$, $z - y = y - x$ in $AS(G, C)$.

Now let us rewrite the equation given in (1.2) as $c - b \equiv b - a \,(\text{mod } n)$, and let us renamed the values a, b and c as z, y and x, respectively. Also take $G = \mathbb{Z}_n$. Then the elements used in Fox n-coloring and the elements of B become same. According to the above replacements and equations, since $x - y = y - z = -(z - y) = -(y - x)$, it will enough to obtain the values that satisfy the equation either $x - y = y - z$ or $z - y = y - x$. □

Example 2.2. For a Torus knot $T_{3,2}$, since the determinant $\Delta_{T_{3,2}}(-1) = 3$ by Equation *(1.1)*, the knot $T_{3,2}$ can be colored in terms of $G = \mathbb{Z}_3$. Further, since the number of colors is 9, it can be defined 9 different homomorphisms from $K_{ST_{3,2}}$ to $AS(\mathbb{Z}_3, C)$. Additionally the total number of the set C using the coloring of $T_{3,2}$ is 4 which are defined as

$$C = \{0\}, \quad C = \{1\}, \quad C = \{2\}, \quad C = \{0, 1, 2\}\,.$$

By considering these sets, the 9 homomorphisms defined from $K_{ST_{3,2}}$ to $AS(\mathbb{Z}_3, C)$ are as presented in Table *2.2*. We strictly note that 6 of among these 9 homomorphisms are actually isomorphisms. As a result of this, one can easily say that the homomorphisms defined from $K_{ST_{3,2}}$ to $AS(\mathbb{Z}_3, C)$ are not unique.

		Homomorphism								
		ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
Elements	x	0	1	2	0	0	1	1	2	2
	y	0	1	2	1	2	0	2	0	1
	z	0	1	2	2	1	2	0	1	0

The first consequence of Theorem 2.1 is the following.

Corollary 1. *For the value t in the homomorph semigroup $AS(\mathbb{Z}_t, B)$ of K_S, we have either $t = \Delta_K(-1)$ or $t \mid \Delta_K(-1)$.*

Proof. According to Theorem 2.1, the knot K can be colored in terms of the subset B in the semigroup $AS(\mathbb{Z}_t, B)$. However it is well known that to a knot K be colored by modulo n, the value n must satisfy $n = \Delta_K(-1)$ or $n \mid \Delta_K(-1)$. Thus it is seen that $t = n$ or $t \mid n$ which implies that $t = \Delta_K(-1)$ or $t \mid \Delta_K(-1)$. □

Depends on the above corollary, if a knot K can be colored by modulo t then it can be colored by modulo kt as well. In fact the importance of this theory for us is the values of t which satisfies $t \leq \Delta_K(-1)$.

The following lemma is important for the characterization of a knot.

Proposition 2 ([3])**.** If a knot can be colored by modulo $n > 2$, then it cannot be deformed to an unknotted curve.

Now by considering Corollary 1 and Proposition 2 together, one can decide whether a knot can be deformed to an unknot via homomorphisms.

Corollary 2. *If there exists a homomorphism from the knot semigroup K_S of a knot K to any alternating sum semigroup $AS(G,B)$, then K cannot be deformed to an unknot. In here, K_S and $AS(G,B)$ are different than $K_{S(0_1)}$ and $\mathbb{N}$, respectively.*

Proof. Assume that such a homomorphism exists with the certain rule of not every element of K_S mapped to a single element in $AS(G,B)$. Under this rule, since the images of all values are same then the knot is colored by a unique color obviously. On the other hand, by Theorem 2.1, the subset B can be used for Fox n-coloring as well. Then, by Proposition 2, K cannot be unknot, as required.

Note that, since $K_{S(0_1)}$ is actually an unknot (circle), the homomorphism $K_{S(0_1)} \to \mathbb{N}$ obviously cannot imply a deformation as required. □

In [18, Conjecture 24], it has been recently stated that a knot diagram has the knot semigroup isomorphic to $\mathbb{N}$ if and only if it is a diagram of the trivial knot. In the following, by considering a splittable knot, we present a more effective situation.

Lemma 2.3 ([16])**.** *If a link is splittable then it can be colored by modulo $n \geq 2$.*

Therefore we have the following result which has a direct proof by Lemma 2.3 and Theorem 2.1.

Theorem 2.4. *Suppose K is a splittable knot. Then one can define a non-trivial homomorphism from the knot semigroup K_S to the alternating sum semigroup $AS(\mathbb{Z}_n, B)$.*

2.2. **Results on the links $P(u,m,1)$, $P(-u,-u,-u)$ and $T_{p,2}$.** In this section, by obtaining knot semigroups of some special Pretzel and Torus links, we will formulate how one can establish the elements of the alternating sum semigroups $AS(G,B)$ that are homomorph of the knot semigroups of these links. Moreover, depends on these formulas, we will give another formulate concerning about the number of homomorphisms from the knot semigroups of these links to the related semigroups $AS(G,B)$.

Unless stated otherwise throughout this section $n, m, p \in \mathbb{Z}^+$.

First of all, we should note that the diagram of the Pretzel link $P(u_1,u_2,u_3)$ can be drawn as in Figure 4 according to the famous book [12]. Thus, by considering the crossing as indicated in Section 1.2 over the diagram in Figure 4, we obtain the following lemma. In fact the proof of it will be omitted since it is basically based on the idea in Section 1.2.

Lemma 2.5. *The knot semigroup for the Pretzel link $P(u_1,u_2,u_3)$ is defined as $K_{SP(u_1,u_2,u_3)} = \langle A\ ;\ R\rangle$, where $A = \{a_0, a_1, a_2, \cdots, a_{u_1+u_2+u_3-1}\}$ and the relation set R is*

(2.1)

$$\left.\begin{array}{ll}
\textit{From regions } u_1: & a_{u_1+1}a_{u_1} = a_{u_1}a_{u_1-1} = \cdots = a_1a_0\,, \\
 & a_0a_1 = a_1a_2 = \cdots a_{u_1-1}a_{u_1} = a_{u_1}a_{u_1+1}\,, \\
 & \\
\textit{From regions } u_2: & a_{u_1}a_{u_1+2} = a_{u_1+2}a_{u_1+3} = \cdots = a_0a_{u_1+u_2+1}\,, \\
 & a_{u_1+u_2+1}a_0 = a_0a_{u_1+u_2} = \cdots = a_{u_1+3}a_{u_1+2} = a_{u_1+2}a_{u_1}\,, \\
 & \\
\textit{From regions } u_3: & a_{u_1+2}a_{u_1+1} = a_{u_1+1}a_{u_1+u_2+2} = \cdots = a_{u_1+u_2+1}a_1\,, \\
 & a_1a_{u_1+u_2+1} = a_{u_1+u_2+1}a_{u_1+u_2+u_3-1} = \cdots = a_{u_1+1}a_{u_1+2}\,.
\end{array}\right\}$$

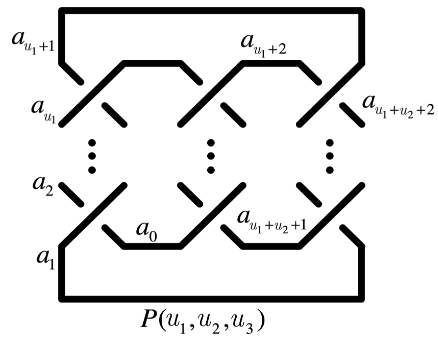


FIGURE 4. The diagram for the Pretzel link $P(u_1, u_2, u_3)$.

Although Lemma 2.5 will not be directly needed in our theories, it will be used as an adaption to the cases $P(u, m, 1)$ and $P(-u, -u, -u)$ in below. By replacing the link $P(u_1, u_2, u_3)$ to the link $P(u, m, 1)$, the first related result is obtained as in the following.

Theorem 2.6. *The knot semigroup $K_{SP(u,m,1)}$ of the Pretzel Link $P(u, m, 1)$ is homomorphic to the semigroup $AS(\mathbb{Z}_t, B)$, where*

(2.2)

$$B = \{\, x_0 + rk \;;\; r = 0, 1, 2, \cdots, u+1 \,\} \cup \{\, x_0 + [s(u+1) - 1]\, k \;;\; s = 2, 3, 4, \cdots, m \,\}$$

such that $x_0, k \in \mathbb{Z}_t$ are arbitrary elements and

$$\text{either } t = (m+1)(u+1) - 1 \text{ or } t \mid (m+1)(u+1) - 1\,. \tag{2.3}$$

Remark 2.7. The set B in *(2.2)* is the same set with C in Fox n-coloring (used in Theorem *2.1*), and the number t in *(2.3)* is giving the number n in the Fox n-colorings. These correspondents are also valid for Theorems *2.9* and *2.12*.

Proof. By Lemma 2.5, it is clear that the generating set is given as $A = \{a_0, a_1, a_2, \ldots, a_{u+m}\}$. On the other hand, by considering the diagram in Figure 5 and then replacing the equations in (2.1) to the case $P(u, m, 1)$, the relation set R can be obtained as in Eq. (2.4) below.

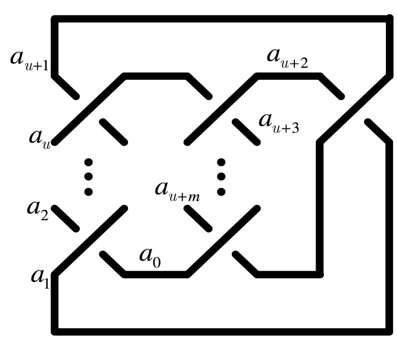


FIGURE 5. Diagram for the Pretzel link $P(n, m, 1)$.

$$
\left.\begin{array}{lll}
 & \text{From the region } u: & a_{u+1}a_u = a_u a_{u-1} = \cdots = a_1 a_0\,, \\
 & & a_0 a_1 = a_1 a_2 = \cdots = a_{u-1}a_u = a_{u_1}a_{u+1}\,, \\
 & & \\
(2.4) & \text{From the region } m: & a_u a_{u+2} = a_{u+2}a_{u+3} = \cdots = a_{u+m}a_0 = a_0 a_{u+1}\,, \\
 & & a_{u+1}a_0 = a_0 a_{u+m} = \cdots = a_{u+2}a_u\,, \\
 & & \\
 & \text{From the region } 1: & a_{u+2}a_{u+1} = a_{u+1}a_1\,, \\
 & & a_{u+1}a_{u+2} = a_1 a_{u+1}\,.
\end{array}\right\}
$$

Let us match each a_i with an element $x_i \in \mathbb{Z}_t$. Now, by the diagram in Figure 5, if we translate a general relation $a_i a_j = a_j a_k$ (where i, j and k are the elements of the set $\{a_0, a_1, \cdots, a_{u+1}\}$) to the alternating sum, then we clearly get

$$x_i - x_j = x_j - x_k\,.$$

Thus, if we apply same translation to the first row of "From the region u" in Eq. (2.4), then we get

$$x_{u+1} - x_u = x_u - x_{u-1} = \cdots = x_1 - x_0\,. \tag{2.5}$$

To simplify of the calculation, let us equalize the equation in (2.5) to an arbitrary value $k \in \mathbb{Z}_t$. After that, by assuming the initial value as $x_0 = x_0$, we have

$$x_1 = x_0 + k\,, \quad x_2 = x_0 + 2k\,, \quad \cdots, \quad x_u = x_0 + uk\,, \quad x_{u+1} = x_0 + (u+1)k\,. \tag{2.6}$$

Similarly as in (2.5), by applying the alternating sum to the first row of "From the region m" in Eq. (2.4) and by the last term

$$x_0 - x_{u+1} = -(u+1)k \tag{2.7}$$

of Eq. (2.6), we clearly have

$$x_u - x_{u+2} = x_{u+2} - x_{u+3} = \cdots = x_0 - x_{u+1} = -(u+1)k\,.$$

In the last equality, let us think each difference pairs separately as in Eq. (2.7). In that case, we obtain the following systematical equations.

$$
\begin{array}{rcl}
x_u - x_{u+2} = -(u+1)k & \Rightarrow & x_{u+2} = x_0 + 2\left[(u+1)-1\right]k \\
 & & \text{by the equality in (2.7)} \\
x_{u+2} - x_{u+3} = -(u+1)k & \Rightarrow & x_0 + 2\left[(u+1)-1\right]k - x_{u+3} = -(u+1)k \\
 & \Rightarrow & x_{u+3} = x_0 + 3\left[(u+1)-1\right]k \\
 & & \text{by iteratively using of the equality in (2.7)} \\
\vdots & \vdots & \vdots \\
x_{u+m-1} - x_{u+m} = -(u+1)k & \Rightarrow & x_{u+m} = x_{u+m-1} + (u+1)k \Rightarrow \\
 & \Rightarrow & x_{u+m} = x_0 + m\left[(u+1)-1\right]k \qquad (2.8) \\
 & & \text{by iteratively using of the equality in (2.7)}
\end{array}
$$

$$x_{u+m} - x_0 = -(u+1)k \quad \Rightarrow \quad x_{u+m} = x_0 - (u+1)k\,. \tag{2.9}$$

Now, by equalizing the values of the term x_{u+m} in Eqs. (2.8) and (2.9), we obtain

$$(mu + m + u)k = 0 \quad \text{or equivalently} \quad (mu + m + u)k \equiv 0 \pmod{t} \tag{2.10}$$

In here, the congruence $(mu+m+u)k \equiv 0 \pmod{t}$ gives the correctness of equations in (2.3), as required.

At this point we should note that one can also take $t = k$ or $t \mid k$ to be held the congruency in (2.10). So this will also give that since all x_i's are equal to each other, the equations for alternating sum semigroup still hold. However, since such these solutions will imply infinite number of homomorphisms, we only consider the cases $t = \Delta_K(-1)$ or $t \mid \Delta_K(-1)$.

To end up the proof, let us express how can one define a homomorphism as required in theorem. For the semigroups $K_{SP(u,m,1)} = A^+/\kappa$ and $AS(\mathbb{Z}_t, B) = B^+/\sim$, where $A = \{a_0, a_1, a_2, \ldots, a_{u+m}\}$ (which is the set of arcs), B is as in the expression of theorem, κ is the set of relations as given in (2.4) and $\sim$ is the set of relations correspond to the relations in (2.4) which we have already obtained in above. Now, since for each a_i $(0 \leq i \leq u+m)$ we obtain a different corresponds value x_i up to choosing of x_0, k and t, this will imply that we have a finite number of different functions $\phi_j : A \to B$ with the rule $a_i \to x_i$. Thus, by Lemma 1.3, there must exists a unique homomorphism from A^+/κ to $B^+/\sim$ for each of these different functions. In fact the number of such these different homomorphisms is defined in Theorem 2.16 below.

Hence the result. □

Example 2.8. For $P(u, m, 1)$, if one choose $x_0 = 0$, $k = 1$ and $t = (m+1)(u+1)-1$, then the number of elements in sets A and B become equal. Therefore we have a one-to-one matching between each a_i and x_i which implies that we obtain not only a homomorphism from $K_{SP(u,m,1)}$ to $AS(\mathbb{Z}_t, B)$ but also an isomorphism. In here, the set B is defined as

$$\{0,\, 1,\, 2,\, \cdots,\, u,\, u+1,\, 2(u+1)-1,\, 3(u+1)-1,\, \cdots,\, m(u+1)-1\}\ .$$

By applying a quite similar progress as in the case of $P(u, m, 1)$, we can obtain similar results for the Pretzel link $P(-u, -u, -u)$ and the Torus knot $T_{p,2}$. In the following, by omitting the proofs but considering Lemma 1.3, we will indicate the existence of homomorphisms from the knot semigroup $K_{SP(-u,-u,-u)}$ to $AS(\mathbb{Z}_t, B)$ as in the coming result which is another version of Theorem 2.6. We first note that, by [12], the diagram of the Pretzel link $P(-u, -u, -u)$ is drawn as in Figure 6, and so as a consequence of Lemma 2.5 one can easily obtain the generating set $A = \{a_0, a_1, a_2, \ldots, a_{3u-1}\}$ while the set of relations R as defined in Eq. (2.11) below.

(2.11)

$$\left.\begin{array}{rl}
\text{From the first region } -u: & a_{u+1}a_u = a_u a_{u-1} = \cdots = a_1 a_0\ , \\
& a_0 a_1 = a_1 a_2 = \cdots = a_{u-1}a_u = a_u a_{u+1}\ , \\[1ex]
\text{From the second region } -u: & a_{u+2}a_{u+1} = a_{u+1}a_{u+3} = \cdots = a_{2u}a_{2u+1} = a_{2u+1}a_2\ , \\
& a_2 a_{2u+1} = a_{2u+1}a_{2u} = \cdots = a_{u+3}a_{u+1} = a_{u+1}a_{u+2}\ , \\[1ex]
\text{From the third region } -u: & a_u a_{u+2} = a_{u+2}a_{2u+2} = \cdots = a_{3u-1}a_0 = a_0 a_{2u+1}\ , \\
& a_{2u+1}a_0 = a_0 a_{3u-1} = \cdots = a_{2u+2}a_{u+2} = a_{u+2}a_u\ .
\end{array}\right\}$$

Thus the other main result of this paper is the following.

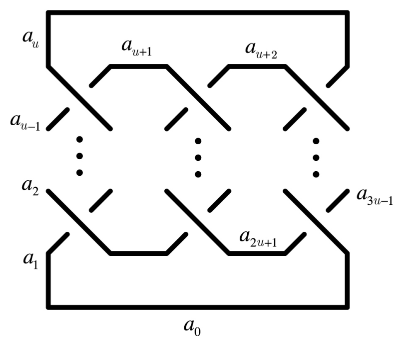


FIGURE 6. Diagram for the Pretzel link $P(-u,-u,-u)$.

Theorem 2.9. *We always have a finite number of homomorphisms from $K_{SP(-u,-u,-u)}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$, where*

$$B = \{\, x_0 + rk \;;\; r = 0, 1, \cdots, u+1 \,\} \cup \{\, x_0 + (u+2s)k \;;\; s = 1, 2, \cdots, u-1 \,\}$$

such that x_0, k are arbitrary elements of $\mathbb{Z}_t$ and either $t = 3u$ or $t \mid 3u$.

On the other hand the existence of isomorphism is defined as follows.

Theorem 2.10. *For only $u = 1$, there exists $K_{SP(-u,-u,-u)} \cong AS(\mathbb{Z}_t, B)$.*

Proof. If $u = 1$, then the diagram and knots of the $P(-1,-1,-1)$ are the same with the diagram and knots of Torus knot $T_{3,2}$. So, by Proposition 1, we have $K_{ST_{3,2}} \cong AS(\mathbb{Z}_3, \mathbb{Z}_3)$. On the other hand, if $u \neq 1$, then the number of arcs in the diagram of $P(-u,-u,-u)$ is $3u$ (which gives the cardinality of the generating set A) and so the number of elements in the set B is $(u+2)+(u-1) = 2u+1$. However, for all $u > 1$, since it is always true that $3u > 2u+1$, we obtain the number of arcs in $P(-u,-u,-u)$ is greater than the number of elements of B which implies that it cannot be defined an isomorphism. □

It is known that tricolorability (i.e. Fox n-coloring when $u = 3$) is an invariant under Reidemeister moves (cf. [1]). Since invariant property is an important tool in every branch of mathematics, it is good enough to study tricolorability for our cases. In fact, by the condition $t = 3u$ or $t \mid 3u$ in Theorem 2.9, it is not hard to see that t can be choosed as 3. That means there exists a homomorphism from the semigroup $K_{SP(-u,-u,-u)}$ to $AS(\mathbb{Z}_3, B)$. Therefore we have the following result.

Theorem 2.11. *All Pretzel links $P(-u,-u,-u)$ are tricolorability.*

Now let us give our attention to the Torus knot. In the remaining part of this section, we will adapt the theories on $P(u,m,1)$ and $P(-u,-u,-u)$ to the Torus knot $T_{p,2}$. Recall that the case $p = 3$ in Torus knot gives $P(-1,-1,-1)$ and so there is nothing to do since we have already obtained previously. Therefore in the following result the case $p = 3$ coincides with Theorems 2.9, 2.10 and 2.11.

Theorem 2.12. *Consider the Torus knot $T_{p,2}$ (where p and 2 are relatively prime) as defined in (1.3). Then we have a homomorphism from the Torus knot semigroup $K_{ST_{p,2}}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$, where*

$$B = \{\, x_0 + rk \;;\; r = 0, 1, 2, \cdots, p-1 \,\},$$

x_0, $k \in \mathbb{Z}_t$ and either $t \mid \Delta_{T_{p,2}}(-1)$ or $t = \Delta_{T_{p,2}}(-1)$ such that $\Delta_{T_{p,2}}(-1) = p$ by (1.1).

Proof. In the proof, we will actually follow a similar way as in the proof of Theorem 2.6. Now if we translate the relations defined in (1.3) to the relations of $AS(\mathbb{Z}_t, B)$, then we have

$$x_0 - x_1 = x_1 - x_2, x_1 - x_2 = x_2 - x_3, \cdots, x_{p-2} - x_{p-1} = x_{p-1} - x_0 \,. \tag{2.12}$$

By rearranging and then equalizing a constant k, we also get

$$x_0 - x_1 = x_1 - x_2 = x_2 - x_3 = \cdots = x_{p-2} - x_{p-1} = x_{p-1} - x_0 = k \,,$$

which can be clearly written as

$$x_1 = x_0 + k, x_2 = x_0 + 2k, \cdots, x_{p-1} = x_0 + (p-1)k \,,$$

In (2.12), as the general term, let us take $x_{p-2} - x_{p-1} = x_{p-1} - x_0 = k$ and then replace the x_i values all the related places. So

$$\begin{aligned} & x_0 + (p-2)k - (x_0 + (p-1)k) = x_0 + (p-1)k - x_0 = -k = (p-1)k \\ & \Longrightarrow pk \equiv 0 \pmod{t} \,. \end{aligned}$$

Therefore, by this last congruence, we must have $t \mid p$ or $t = p$, where $p = \Delta_{T_{p,2}}(-1)$.

The set of arcs (or equivalently the generating set) is defined as $A = \{ a_0, a_1, a_2, \cdots, a_{p-1}\}$ while the set of x_i values is given by $B = \{ x_0 + rk \; ; \; r = 0, 1, 2, ..., p-1 \}$. Hence, by applying Lemma 1.3, we reached that there exists a homomorphism from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_t, B)$, as required. □

Example 2.13. In Theorem *2.12*, if we choose $x_0 = 0$, $k = 1$ and $t = \Delta_{T_{p,2}}(-1) = p$, then the set B is given by $\{0, 1, 2, \cdots p-2, p-1\}$. Therefore the number of arcs in Torus knot $T_{p,2}$ and the cardinality of B are both p, and so there is a one-to-one correspondence between each arc in A and each element in B. So, by Lemma *1.3*, we obtain an isomorphism $K_{ST_{p,2}} \cong AS(\mathbb{Z}_p, B)$.

Remark 2.14. We strictly note that a similar situation in Example *2.13* (which is an example of Theorem *2.12*) was given as a result in the paper [18, Theorem 3] by considered with only a unique isomorphism. Nevertheless, Example *2.13* actually shows that different choices for arbitrary x_0, k and p will imply different isomorphisms between $K_{ST_{p,2}}$ and $AS(\mathbb{Z}_p, B)$.

The situation depicted in Example 2.13 and Remark 2.14 can be summarized with the following theorem.

Theorem 2.15. *To define an isomorphisms between $K_{ST_{p,2}}$ and $AS(\mathbb{Z}_t, B)$, it must be held $k \neq 0$, $k \nmid p$ and $t = p$.*

Proof. Without loss of the generality, let us investigate the cases as $k = 0$, $k \mid p$ and $t \neq p$, respectively.

- Let $k = 0$. If we write 0 instead of k in the set B in Theorem 2.12, then we have $B = \{x_0\}$. But, in this case, whole elements of $T_{p,2}$ map to a single element in the homomorphism from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_t, B)$ which clearly breaks down the isomorphism.
- Assume $k \mid p$. Let us reconsider the set B in Theorem 2.12. By the assumption, for any $r_i \neq 0$ $(0 \leq i \leq p-1)$, we get $r_i k \equiv 0 \pmod{p}$. But, since this will imply that $x_0 + 0k = x_0 + r_i k$ (as the meaning of congruence classes), we cannot reach the isomorphism.

- Suppose $t < p$ and $t \mid p$. Remember that the set B in Theorem 2.12 was obtained by considered the equivalence over modulo $t = p$. However, when we take it as $t < p$ and $t \mid p$, clearly the cardinality t of B will be definitely less than p. On the other hand, the number of arcs in the knot diagram (or equivalently, the number of generators in the knot semigroup) is still p. This means that we cannot define an isomorphism between $K_{ST_{p,2}}$ and $AS(\mathbb{Z}_t, B)$ since the cardinality of B is less than p.

As a result of these above facts, we say that to define an isomorphism from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_t, B)$ (or vice versa), all conditions in theorem must be satisfied. □

Finally, we can bring together Theorems 2.6, 2.9 and 2.12 in a common point as in the following.

Theorem 2.16. *For simplicity, let $\mathcal{N}$ denotes one of $P(u, m, 1)$, $P(-u, -u, -u)$ or $T_{p,2}$. Then the number of homomorphisms from each of the knot semigroups $K_{S\mathcal{N}}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$ is*

$$\sum_{i=1}^{\chi-1} t_i^2$$

such that $t_i \mid \Delta_{\mathcal{N}}(-1)$ and χ is the number of t_i's that divides t.

Proof. In Theorems 2.6, 2.9 and 2.12, we established that if one wants to define a homomorphism from one of the knot semigroups of $P(u, m, 1)$, $P(-u, -u, -u)$ ve $T_{p,2}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$, then the value t must be satisfied $t \mid \Delta_{\mathcal{N}}(-1)$ or $t = \Delta_{\mathcal{N}}(-1)$, and additionally, for each of these theorems, we presented the related B set while $t = \Delta_{\mathcal{N}}(-1)$. Remember that the elements $x_0, k \in \mathbb{Z}_t$ were chosen arbitrarily in these B sets. It easy to verify that each of x_0 and t can be chosen t different ways from $\mathbb{Z}_t$ which imply that the values of x_0 and t can be totally chosen as t^2 different options. On the other hand, since we obtain different B sets up to for each different choices of x_0 and k, we get t^2 different homomorphisms that can be defined on these B sets.

For $t = \Delta_{\mathcal{N}}(-1)$, now let us consider the $t_i \mid t$ values and say χ to the number of such t_i's. In here we must consider 1 does not count in χ since Fox n-colorings start always from $n \geq 2$ (by Lemma 2.3 or more generally Equation (1.2)) and so $t_i \neq 1$. Let B_i denotes a congruence class of the elements in B depends on the value t_i. According to Theorems 2.6, 2.9 and 2.12, one can define a homomorphism from the knot semigroup to the semigroup $AS(\mathbb{Z}_{t_i}, B_i)$ in which x_0, $k \in \mathbb{Z}_{t_i}$. With the same idea as in the above paragraph, ${t_i}^2$ different choices can be applied to x_0 and k in $\mathbb{Z}_{t_i}$, and since each of those gives a new homomorphism, we get total ${t_i}^2$ different homomorphisms for each t_i from the knot semigroup to the semigroup $AS(\mathbb{Z}_{t_i}, B)$. Hence, since this situation can be seen for all $t_i \mid t$, we say that the total number of homomorphisms is $\sum_{i=1}^{\chi-1} t_i^2$, as required. □

Remember that the number of colorings of a knot K in terms of the quandle Q was denoted by $Col_Q(K)$. By considering Lemma 1.1, we can give the following result as a consequence of Theorems 2.1 and 2.16.

Theorem 2.17.

$$Col_Q(\mathcal{N}) = \sum_{i=1}^{\chi-1} t_i^2 \,.$$

One may also present the following particular corollary as a consequence of Theorems 2.1, 2.12 and 2.16.

Corollary 3. *For a prime p, there are total p^2 homomorphisms and $p^2 - p$ isomorphisms from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_p, B)$.*

3. CONCLUSION

In this study, the homomorphism relations between the nodal semigroups and the alternative total semigroups of some pretzel chains and torus chains are investigated and the number of homomorphisms and isomorphisms in some special cases are given.

4. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The authors declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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