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Unveiling the Dynamics of Nonlinear Landau-Ginzburg-Higgs (LGH) Equation: Wave Structures through Multiple Auxiliary Equation Methods

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Abstract — This comprehensive investigation delves deeply into the intricate dynamics governed by the nonlinear Landau-Ginzburg-Higgs equation. It uncovers a diversity of semi-analytical solutions by leveraging three auxiliary equation methods within the traveling wave framework. This article effectively utilizes the improved Kudryashov, Kudryashov's R, and Sardar's subequation methods. The methods discussed are advantageous because they are easy to implement and suitable for use with the Mathematica package program. Each method yields a distinct set of solutions, scrutinized across all cases. We elucidate the complex wave structures through 3D, 2D, and contour graphical representations, providing profound insights into their underlying characteristics. Furthermore, we scrutinize the influence of parameter variations on these wave structures, thereby offering a comprehensive understanding of their dynamic behavior.

Keywords *Landau-Ginzburg-Higgs equation, improved Kudryashov method, Kudryashov's R method, Sardar's subequation method*

Mathematics Subject Classification (2020) 35C07, 35C11

1. Introduction

It has long been known that nonlinear structures are used to model many natural phenomena and basic science fields such as physics, chemistry, and biology. It is very important to obtain the solutions of these models in various engineering fields. Making sense of scientific phenomena and solving the obtained structures with today's knowledge and technology has been the goal of researchers for decades. For this purpose, they have worked on developing different perspectives by applying various analytical and numerical solution methods such as the (G'/G) expansion method [1,2], Bernoulli (G'/G) expansion method [3], sub-equation method [4,5], sine-Gordon expansion method (SGEM) [6], rational sine-Gordon expansion method (rSGEM) [7], exponential function method [8], modified exponential function method [9,10], exponential rational function method [11,12], unified method [13,14], Kudryashov methods [15–19], Khater methods [20], natural decomposition method [21], variational approximation methods, Hirota direct method [22]. Besides, most of the put forward semi-analytical methods are based on the same starting point, it is seen that even small changes in the method steps affect the structure of the solution functions. Considering that a small change causes big consequences, called the butterfly effect in today's age, small changes in the solution structures will allow the scientific phenomenon discussed to be interpreted differently.

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This study discusses the Landau-Ginzburg-Higgs equation (LGHE), created to understand and describe phase transitions, superconductivity condensed matter physics, and the behavior of certain types of fields in high-energy physics. Solutions of this equation can represent the distribution of the superconducting order parameter within the material and provide information on properties, such as the penetration depth of magnetic fields and the critical temperature of the superconducting transition. For this reason, LGHE has attracted the attention of many researchers, and some soliton structures have been obtained by applying various methods. Our aim in doing this study is to add new ones to the solution structures of LGHE and to show the suitability of the methods discussed by comparing them to this equation. In Section 2, we provide the mathematical algorithms of the improved Kudryashov method (IKM), Kudryashov's R method (KRM), and Sardar's subequation method (SSM) to figure out the solitary wave solitons. In section 3, we apply the proposed methods to the nonlinear LGHE. In the last section, we provide the concluding remarks on the obtained solutions.

2. Preliminaries

This section provides the basic steps of IKM, KRM, and SSM.

2.1. Improved Kudryashov Method (IKM)

First, we handle the general expression of a nonlinear partial differential equation in the form [17]:

$$\kappa\left(u, u_x, u_y, u_t, u_{xx}, u_{xy}, ...\right) = 0 \tag{2.1}$$

where κ is polynomial function in u and its partial derivatives are included. Then, the given nonlinear partial differential equation (NPDE) (2.1) can be converted into ordinary differential equation (ODE) by traveling wave transformation as follows:

$$u(x, y, t) = u(\psi), \quad \psi = \mu(x + y - ct) \tag{2.2}$$

where c is an arbitrary constant. After applying the above transformation and the chain rule, we obtain the following equality:

$$K(u, u', u'', ...) = 0 \tag{2.3}$$

Specific items of the method can be applied after that reduction. In this step, according to the proposed method, we assume (2.3) has a solution in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi) \tag{2.4}$$

where

$$\chi(\psi) = \pm(1 + exp(2\psi))^{-1/2}$$

and $\chi(\psi)$ satisfies the following ODE:

$$\chi_\psi^2 = (\chi^2(\chi^4 - 2\chi^2 + 1))^{1/2}$$

After taking this auxiliary differential equation and considering the solution (2.4), we can write the first, second, and third-order derivatives of (2.4) as follows:

$$u_\psi = \sum_{s=0}^{N-1} a_{s+1}(s + 1)(\chi^{s+3}(\psi) - \chi^{s+1}(\psi)) \tag{2.5}$$

$$u_{\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s + 1)(s + 3)\chi^{s+5}(\psi) - 2(s + 1)(s + 2)\chi^{s+3}(\psi) + (s + 1)^2\chi^{s+1}(\psi)\right] \tag{2.6}$$

and

$$u_{\psi\psi\psi} = \sum_{s=0}^{N-1} a_{s+1} \left[(s+1)(s+3)(s+5)\chi^{s+7}(\psi) - 3(s+1)(s+3)^2\chi^{s+5}(\psi) + \left(3(s+1)^2(s+3) + 4(s+1)\right)\chi^{s+3}(\psi) - (s+1)^3\chi^{s+1}(\psi)\right] \tag{2.7}$$

If necessary, other higher-order derivatives can also be written. Substituting (2.4) and (2.5)-(2.7) into (2.3) then, equating degrees of highest order linear term $(u^{(p)}(\chi))^r$ and highest degree nonlinear term $u^l(\chi)u^{(s)}(\chi)$ as required by the principle of homogeneous balance, we can define the pole order N clearly. After this implementation, we obtain an algebraic system. By using the computer package program, we can solve the algebraic system according to degrees of the χ, then the coefficients of the polynomial (2.4) and parameters of (2.2) can be obtained. Finally, substituting the coefficients, parameters, and the traveling wave transformation into the obtained polynomial, the solutions of (2.1) are obtained.

2.2. Kudryashov's R Method (KRM)

The major items of the method proposed above are indicated as follows [19]: In the first item, we handle the general impression of nonlinear partial differential equations in the form:

$$\kappa\left(u, u_x, u_y, u_t, u_{xx}, u_{xy}, ...\right) = 0 \tag{2.8}$$

where κ is a polynomial function in u and its assorted order partial derivatives and nonlinear terms are included. Secondly, we assume that the subsequent traveling wave transformation is done to reduce (2.8) to an ordinary differential equation:

$$u(x, y, t) = u(\psi), \quad \psi = \mu(x + y - ct) \tag{2.9}$$

where μ and c are arbitrary constants. After applying the above transformation and the chain rule, we get the following equality:

$$K(u, u', u'', ...) = 0 \tag{2.10}$$

In this step, according to the proposed method, we assume (2.10) has a solution in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi) \tag{2.11}$$

where

$$\chi(\psi) = \frac{4\alpha}{4\alpha^2 e^{\psi} + \gamma e^{-\psi}}$$

such that $\gamma = 4\alpha\beta$ and $u(\psi)$ adopts the given ordinary differential equation:

$$\chi_\psi^2 = \chi^2(1 - \gamma\chi^2)$$

After taking this auxiliary differential equation and considering the solution (2.11), we can write the first, second, and third-order derivatives of (2.11) as follows:

$$u_\psi = \sum_{s=0}^{N-1} a_{s+1}\chi^s(\psi)\chi_\psi(\psi)$$

$$u_{\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s+1)^2\chi^{s+1}(\psi) - (s+1)^2\gamma\chi^{s+3}(\psi) + (s+1)\gamma\chi^{s+4}(\psi)\right]$$

and

$$u_{\psi\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s+1)^3\chi^s(\psi) - \gamma(s+1)^2(s+3)\chi^{s+2}(\psi) - \gamma(s+1)(s+3)\chi^{s+2}(\psi)\right]\chi_\psi(\psi)$$

By equating the degrees of highest order linear term $(u^{(p)}(\chi))^r$ and highest degree nonlinear term $u^l(\chi)u^{(s)}(\chi)$ as required by the principle of homogeneous balance, we can define the pole order N clearly. After this implementation, we obtain an algebraic system. By using the computer package program, we can solve the algebraic system according to degrees of the χ, then the coefficients of the polynomial (2.11) and parameters of (2.9) can be obtained. Finally, substituting the coefficients, parameters, and the traveling wave transformation into the obtained polynomial, the solutions of (2.8) are obtained.

2.3. Subequation Method in Sardar's Sense (SSM)

The major items of the method proposed above are indicated as follows [24]: In the first item, we handle the general impression of nonlinear partial differential equations in the form:

$$\kappa\left(u, u_x, u_y, u_t, u_{xx}, u_{xy}, ...\right) = 0 \tag{2.12}$$

where κ is a polynomial function in u and its assorted order partial derivatives and nonlinear terms are included. Secondly, we assume that the subsequent traveling wave transformation is done to reduce (2.12) to an ordinary differential equation:

$$u(x,y,t) = u(\psi), \quad \psi = \mu(x+y-ct) \tag{2.13}$$

where μ and c are arbitrary constants. After applying the above transformation and the chain rule, we get the following equality:

$$K(u, u', u'', ...) = 0 \tag{2.14}$$

In this step, according to the proposed method, we assume (2.14) has a solution in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi) \tag{2.15}$$

where $\chi(\psi)$ is the solution of the following differential equation

$$\chi_\psi^2 = \eta + \gamma\chi^2(\psi) + \chi^4(\psi) \tag{2.16}$$

and the solutions of (2.16) has four cases of solutions:

Case 1: If $\gamma > 0$ and $\eta = 0$, then

$$\chi_1(\psi) = \pm\sqrt{-\alpha\beta\gamma}\,\mathrm{sech}_{\alpha\beta}\left(\sqrt{\gamma}\psi\right)$$

and

$$\chi_2(\psi) = \pm\sqrt{\alpha\beta\gamma}\,\mathrm{csch}_{\alpha\beta}\left(\sqrt{\gamma}\psi\right)$$

where $\mathrm{sech}_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{\psi}+\beta e^{-\psi}}$ and $\mathrm{csch}_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{\psi}-\beta e^{-\psi}}$.

Case 2: If $\gamma < 0$ and $\eta = 0$, then

$$\chi_3(\psi) = \pm\sqrt{-\alpha\beta\gamma}\,\sec_{\alpha\beta}\left(\sqrt{-\gamma}\psi\right)$$

and

$$\chi_4(\psi) = \pm\sqrt{-\alpha\beta\gamma}\,\csc_{\alpha\beta}\left(\sqrt{-\gamma}\psi\right)$$

where $\sec_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{i\psi}+\beta e^{-i\psi}}$ and $\csc_{\alpha\beta}(\psi) = \frac{2}{\alpha e^{i\psi}-\beta e^{-i\psi}}$.

Case 3: If $\gamma < 0$ and $\eta = \frac{\gamma^2}{4\beta}$, then

$$\chi_5(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\tanh_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{2}}\psi\right)$$

$$\chi_6(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\coth_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{2}}\psi\right)$$

$$\chi_7(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\left(\tanh_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right) \pm i\sqrt{\alpha\beta}\mathrm{sech}_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right)\right)$$

$$\chi_8(\psi) = \pm\sqrt{-\frac{\gamma}{2}}\left(\coth_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right) \pm \sqrt{\alpha\beta}\mathrm{csch}_{\alpha\beta}\left(\sqrt{-2\gamma}\psi\right)\right)$$

and

$$\chi_9(\psi) = \pm\sqrt{-\frac{\gamma}{8}}\left(\tanh_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{8}}\psi\right) + \coth_{\alpha\beta}\left(\sqrt{-\frac{\gamma}{8}}\psi\right)\right)$$

where $\tanh_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi}-\beta e^{-\psi}}{\alpha e^{\psi}+\beta e^{\psi}}$ and $\coth_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi}+\beta e^{-\psi}}{\alpha e^{\psi}-\beta e^{\psi}}$.

Case 4: If $\gamma > 0$ and $\eta = \frac{\gamma^2}{4}$, then

$$\chi_{10}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\tan_{\alpha\beta}\left(\sqrt{\frac{\gamma}{2}}\psi\right)$$

$$\chi_{11}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\cot_{\alpha\beta}\left(\sqrt{\frac{\gamma}{2}}\psi\right)$$

$$\chi_{12}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\left(\tan_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right) \pm \sqrt{\alpha\beta}\sec_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right)\right)$$

$$\chi_{13}(\psi) = \pm\sqrt{\frac{\gamma}{2}}\left(\cot_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right) \pm \sqrt{\alpha\beta}\csc_{\alpha\beta}\left(\sqrt{2\gamma}\psi\right)\right)$$

and

$$\chi_{14}(\psi) = \pm\sqrt{\frac{\gamma}{8}}\left(\tan_{\alpha\beta}\left(\sqrt{\frac{\gamma}{8}}\psi\right) + \cot_{\alpha\beta}\left(\sqrt{\frac{\gamma}{8}}\psi\right)\right)$$

where $\tanh_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi}-\beta e^{-\psi}}{\alpha e^{\psi}+\beta e^{\psi}}$ and $\coth_{\alpha\beta}(\psi) = \frac{\alpha e^{\psi}+\beta e^{-\psi}}{\alpha e^{\psi}-\beta e^{\psi}}$. After taking this auxiliary differential equation and considering the solution (2.15), we can write the first, second, and third-order derivatives of (2.15) as follows:

$$u_{\psi} = \sum_{s=0}^{N-1}(s+1)a_{s+1}\chi^{s}(\psi)\chi_{\psi}(\psi)$$

$$u_{\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[s(s+1)\eta\chi^{s-1}(\psi) + (s+1)(s\gamma+1)\chi^{s+1}(\psi) + (s+1)(s+2)\chi^{s+3}(\psi)\right]$$

and

$$u_{\psi\psi\psi} = \sum_{s=0}^{N-1} a_{s+1}\left[(s^2-1)s\eta\chi^{s-2}(\psi) + (s+1)^2(s\gamma+1)\chi^{s}(\psi) + (s+1)(s+2)(s+3)\chi^{s+2}(\psi)\right]\chi_{\psi}(\psi)$$

By equating the degrees of highest order linear term $(u^{(p)}(\chi))^r$ and highest degree nonlinear term $u^l(\chi)u^{(s)}(\chi)$ as required by the principle of homogeneous balance, we can define the pole order N clearly. After this implementation, we obtain an algebraic system. By using the computer package program, we can solve the algebraic system according to degrees of the χ, then the coefficients of the polynomial (2.15) and parameters of (2.13) can be obtained. Finally, substituting the coefficients, parameters, and the traveling wave transformation into the obtained polynomial, the solutions of (2.12) are obtained.

3. Solutions of LGHE

The Landau-Ginzburg-Higgs equation (LGHE) is a physics equation that arises in the fields of condensed matter physics and high energy physics, especially in the study of phase transitions, superconductivity, cosmology, optics and the behavior of certain field theories and it can be represented as follows [23]:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} - g^2 U + h^2 U^3 = 0. \tag{3.1}$$

Here, $\frac{\partial^2 U}{\partial t^2}$ represents the second partial derivative of a field U with respect to time t, which describes the time evolution of the field. Moreover, $\frac{\partial^2 U}{\partial x^2}$ represents the second partial derivative of the field U with respect to space x, which describes how the field varies in space. Besides, g is a constant that determines the strength of the linear term, and h is another constant that determines the strength of the quadratic nonlinear term. Using the following transformation

$$U(x,t) = u(\psi), \quad \psi = \mu x - ct$$

we can reduce (3.1) into the following ODE:

$$(c^2 - \mu^2)\frac{d^2 u}{d\psi^2} - g^2 u + h^2 u^2 = 0 \tag{3.2}$$

After this reduction, we can use analytical approaches as follows:

3.1. IKM Sense

According to the IKM, we can think (3.2) has polynomial solution as:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi)$$

where

$$\chi(\psi) = \pm(1 + \exp(2\psi))^{-1/2}$$

and $u(\psi)$ adopts the given ordinary differential equation:

$$\chi_\psi^2 = (\chi^2(\chi^4 - 2\chi^2 + 1))^{1/2}$$

Then, according to the previous adoption and using the homogeneous balance principle, we obtain pole order $N = 2$ and thus

$$u(\psi) = a_0 + a_1\chi(\psi) + a_2\chi^2(\psi) \tag{3.3}$$

After determining the quadratic polynomial,

$$u''(\psi) = a_1\chi(\psi) + 4a_2\chi^2(\psi) - 4a_1\chi^3(\psi) - 12a_2\chi^4(\psi) + 3a_1\chi^5(\psi) + 8a_2\chi^6(\psi) \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.1), we obtain four cases solutions of LGHE.

Case 1:

$$a_0 = -\frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = \frac{2g}{h^{3/2}}, \qquad c = -\frac{2\mu^2 - g^2}{2}$$

$$U_1 = -\frac{g}{h^{3/2}}\tanh\left(\mu x - \sqrt{\frac{2\mu^2 - g^2}{2}}t\right) \tag{3.5}$$

Case 2:

$$a_0 = -\frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = \frac{2g}{h^{3/2}}, \qquad c = \frac{2\mu^2 - g^2}{2}$$

$$U_2 = -\frac{g}{h^{3/2}} \tanh\left(\mu x + \sqrt{\frac{2\mu^2 - g^2}{2}} t\right)$$

Case 3:

$$a_0 = \frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = -\frac{2g}{h^{3/2}}, \qquad c = -\frac{2\mu^2 - g^2}{2}$$

$$U_3 = \frac{g}{h^{3/2}} \tanh\left(\mu x - \sqrt{\frac{2\mu^2 - g^2}{2}} t\right)$$

Case 4:

$$a_0 = \frac{g}{h^{3/2}}, \qquad a_1 = 0, \qquad a_2 = -\frac{2g}{h^{3/2}}, \qquad c = -\frac{2\mu^2 - g^2}{2}$$

$$U_4 = \frac{g}{h^{3/2}} \tanh\left(\mu x + \sqrt{\frac{2\mu^2 - g^2}{2}} t\right)$$

3D surface, 2D plots, and contour plot of the kink type solution (3.5) are shown in Figure 1:

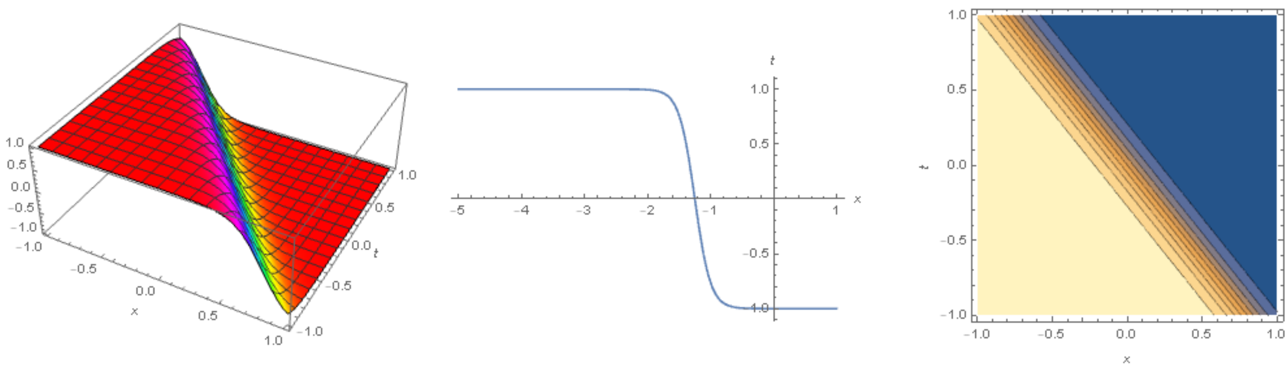

Figure 1. 3D surface, 2D plots, and contour plot of the kink type solution (3.5) for $g = 3\sqrt{2}$, $h = 2$, and $\mu = 5$

3.2. KRM Sense

Supposing the solution of (3.1) in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1} \chi^{s+1}(\psi)$$

where

$$\chi(\psi) = \frac{4\alpha}{4\alpha^2 e^{\psi} + \gamma e^{-\psi}}$$

such that $\gamma = 4\alpha\beta$ and $u(\psi)$ adopts the given ordinary differential equation:

$$\chi_\psi^2 = \chi^2(1 - \gamma\chi^2)$$

Then, according to the previous adoption and using the homogeneous balance principle, we obtain pole order $N = 1$ and thus

$$u(\psi) = a_0 + a_1\chi(\psi). \tag{3.6}$$

After determining the first-degree polynomial,

$$u''(\psi) = a_1\chi(\psi)(1 - 2\gamma\chi^2(\psi)) \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.1), we obtain four cases solutions of LGHE.

Case 1:

$$a_0 = 0, \qquad a_1 = -\frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = \sqrt{\mu^2 + g^2}$$

$$U_1 = -\frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x - \sqrt{\mu^2 + g^2}t\right)$$

Case 2:

$$a_0 = 0, \qquad a_1 = \frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = -\sqrt{\mu^2 + g^2}$$

$$U_2 = \frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x + \sqrt{\mu^2 + g^2}t\right)$$

Case 3:

$$a_0 = 0, \qquad a_1 = -\frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = \sqrt{\mu^2 + g^2}$$

$$U_3 = \frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x - \sqrt{\mu^2 + g^2}t\right)$$

Case 4:

$$a_0 = 0, \qquad a_1 = -\frac{g\sqrt{2\gamma}}{h^{3/2}}, \qquad c = -\sqrt{\mu^2 + g^2}$$

$$U_4 = -\frac{2g\sqrt{2\alpha\beta}}{h^{3/2}} \operatorname{sech}_{\alpha\beta}\left(\mu x + \sqrt{\mu^2 + g^2}t\right) \tag{3.8}$$

3D surface, 2D plots, and contour plot of the bell shaped bright soliton solution (3.8) are shown in Figure 2:

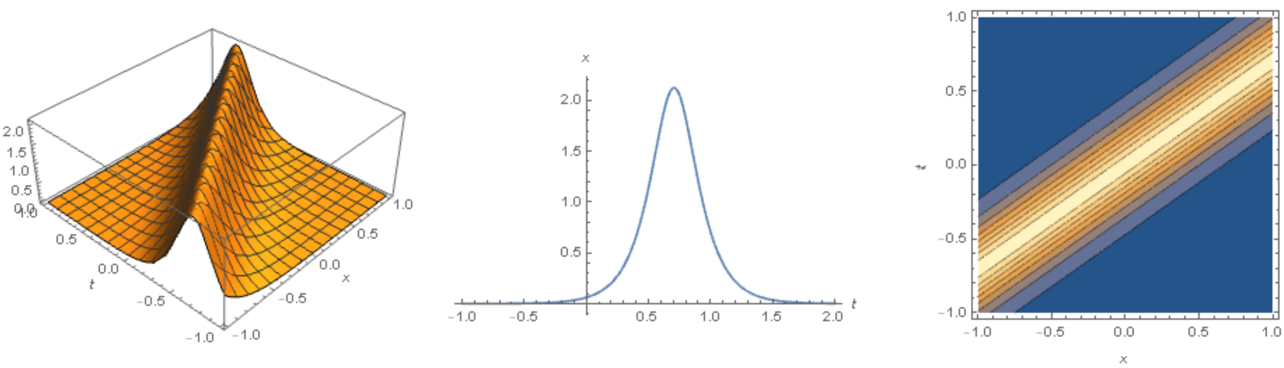

Figure 2. 3D surface, 2D plots, and contour plot of the bell shaped bright soliton solution (3.8) for $g = 3\sqrt{2}$, $h = 2$, $\mu = 3\sqrt{2}$, α=1, and β=1

3.3. SSM Sense

Supposing the solution of (3.1) in the form:

$$u(\psi) = \sum_{s=-1}^{N-1} a_{s+1}\chi^{s+1}(\psi)$$

where $\chi(\psi)$ is the solution of (2.16). Then, according to the previous adoption and using the homogeneous balance principle, we obtain pole order $N = 1$ and thus

$$u(\psi) = a_0 + a_1\chi(\psi) \tag{3.9}$$

After determining the first-degree polynomial,

$$u''(\psi) = a_1\gamma\chi(\psi) + 2a_1\chi^3(\psi) \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.1), we obtain four cases solutions of LGHE where

$$a_0 = 0, \qquad a_1 = \pm\frac{\sqrt{2(\mu^2 - c^2)}}{h^{3/2}}, \quad \text{and} \quad \gamma = \frac{g^2}{c^2 - \mu^2}$$

Case 1: If $|c| > |\mu|$ and $\eta = 0$, then

$$U_{1,1} = \pm\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\text{sech}_{\alpha\beta}\left(\frac{g}{\sqrt{c^2 - \mu^2}}(\mu x - ct)\right)$$

and

$$U_{1,2} = \pm i\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\text{csch}_{\alpha\beta}\left(\frac{g}{\sqrt{c^2 - \mu^2}}(\mu x - ct)\right) \tag{3.11}$$

3D surface, 2D plots, and contour plot of the kink type solution (3.11) are shown in Figure 3:

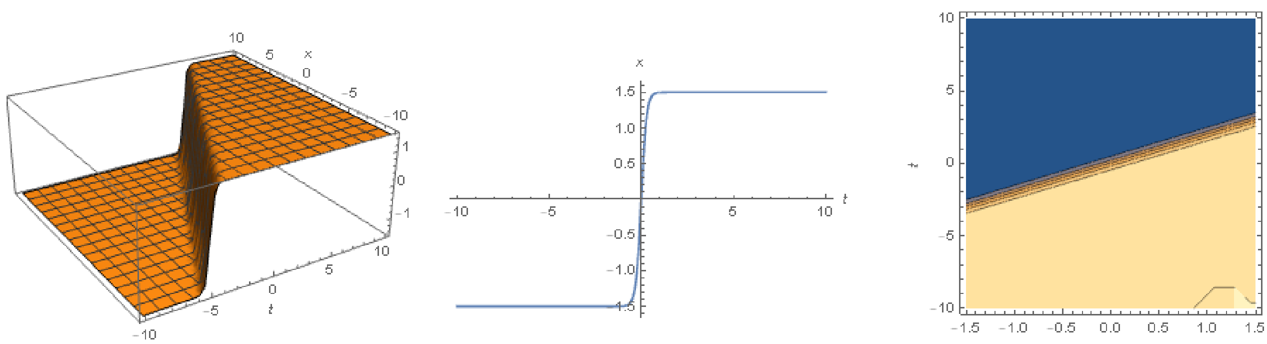

Figure 3. 3D surface, 2D plots, and contour plot of the kink type solution (3.11) for $g = 3\sqrt{2}$, $h = 2$, $\mu = 2$, α=1, β=1, and $c = 1$ and 2D plot for $t = 0$

Case 2: If $|c| < |\mu|$ and $\eta = 0$, then

$$U_{2,1} = \pm\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\sec_{\alpha\beta}\left(\frac{g}{\sqrt{\mu^2 - c^2}}(\mu x - ct)\right)$$

and

$$U_{2,2} = \pm\frac{g\sqrt{2\alpha\beta}}{h^{3/4}}\csc_{\alpha\beta}\left(\frac{g}{\sqrt{\mu^2 - c^2}}(\mu x - ct)\right)$$

Case 3: If $|c| < |\mu|$ and $\eta = \frac{g^4}{4\beta(c^2-\mu^2)^2}$, then

$$U_{3,1} = \pm\frac{g}{h^{3/4}}\tanh_{\alpha\beta}\left(\frac{g}{\sqrt{2(\mu^2 - c^2)}}(\mu x - ct)\right)$$

$$U_{3,2} = \pm\frac{g}{h^{3/4}}\coth_{\alpha\beta}\left(\frac{g}{\sqrt{2(\mu^2 - c^2)}}(\mu x - ct)\right) \tag{3.12}$$

$$U_{3,3} = \pm\frac{g}{h^{3/4}}\left[\tanh_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}(\mu x - ct)\right) \pm i\sqrt{\alpha\beta}\text{sech}_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}(\mu x - ct)\right)\right]$$

$$U_{3,4} = \pm\frac{g}{h^{3/4}}\left[\coth_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}(\mu x - ct)\right) \pm \sqrt{\alpha\beta}\text{csch}_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2 - c^2}}(\mu x - ct)\right)\right]$$

and

$$U_{3,5} = \pm\frac{g}{2h^{3/4}}\left[\tanh_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2 - c^2)}}(\mu x - ct) + \coth_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2 - c^2)}}(\mu x - ct)\right]$$

3D surface, 2D plots, and contour plot of the multiple singular soliton type solution (3.12) are shown in Figure 4:

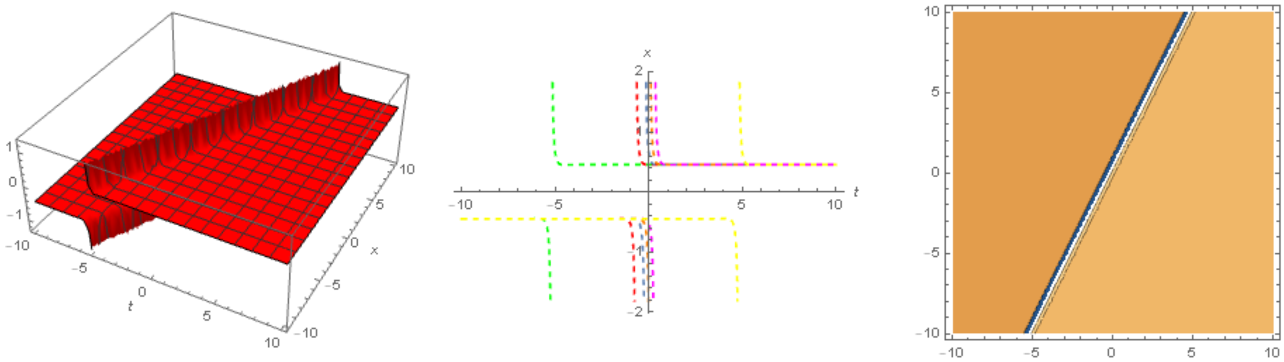

Figure 4. 3D surface, 2D plots, and contour plot of the multiple singular soliton type solution (3.12) for $g = 3\sqrt{2}$, $h = 16$, $\mu = 2$, α=1, β=0.5, and $c = 1$ and 2D plot when $t = -10$, $t = -1$, $t = 0$, $t = 0.5$, $t = 1$, and $t = 10$, respectively

Case 4: If $|c| > |\mu|$ and $\eta = \frac{g^4}{4(c^2-\mu^2)^2}$, then

$$U_{4,1} = \pm i\frac{g}{h^{3/4}} \tan_{\alpha\beta}\left(\frac{g}{\sqrt{2\,(c^2-\mu^2)}}\,(\mu x - ct)\right)$$

$$U_{4,2} = \pm i\frac{g}{h^{3/4}} \cot_{\alpha\beta}\left(\frac{g}{\sqrt{2\,(c^2-\mu^2)}}\,(\mu x - ct)\right) \tag{3.13}$$

$$U_{4,3} = \pm i\frac{g}{h^{3/4}}\left[\tan_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}\,(\mu x - ct)\right) \pm \sqrt{\alpha\beta}\sec_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}\,(\mu x - ct)\right)\right]$$

$$U_{4,4} = \pm i\frac{g}{h^{3/4}}\left[\cot_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}\,(\mu x - ct)\right) \pm \sqrt{\alpha\beta}\csc_{\alpha\beta}\left(g\sqrt{\frac{2}{\mu^2-c^2}}\,(\mu x - ct)\right)\right]$$

and

$$U_{4,5} = \pm i\frac{g}{2\sqrt{2}h^{3/2}}\left[\tan_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2-c^2)}}\,(\mu x - ct) + \cot_{\alpha\beta}\frac{g}{2\sqrt{2(\mu^2-c^2)}}\,(\mu x - ct)\right]$$

3D surface, 2D plots, and contour plot of the singular periodic soliton type solution (3.13) are shown in Figure 5:

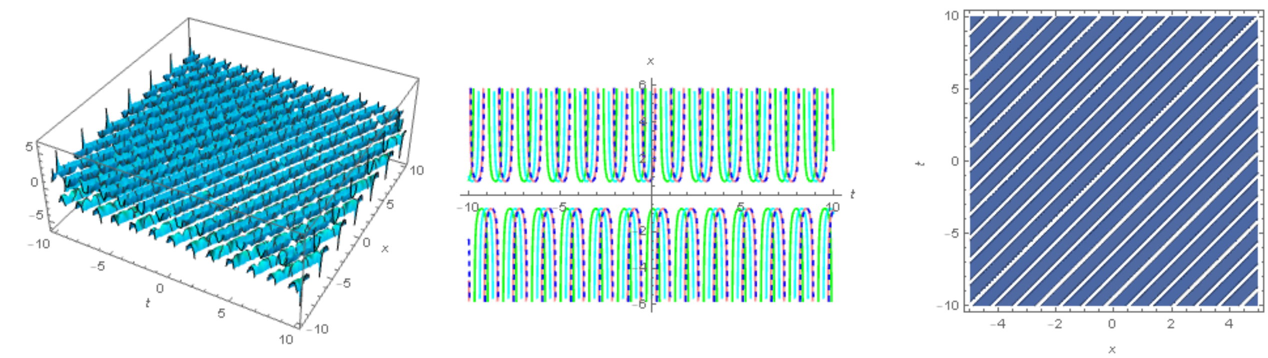

Figure 5. 3D surface, 2D plots, and contour plot of the singular periodic soliton type solution (3.13) for $g = 3\sqrt{2}$, $h = 16$, $\mu = 2$, α=1, β=1, and $c = 1$ and 2D plot when $t = -10$, $t = 0$, $t = 0.5$, and $t = 10$, respectively

4. Conclusion

In conclusion, our investigation of the solitary wave solutions of (3.1) has provided intriguing insights through three distinct approaches. We have identified four distinct solution scenarios using the IKM and the KRM. The IKM approach revealed the emergence of hyperbolic-type solutions, as shown in

Figure 1, featuring non-breaking, smooth traveling wave structures. Moreover, the KRM approach led to the discovery of special hyperbolic-type solutions, exemplified by Figure 2. Furthermore, employing the SSM enabled us to derive several solitary wave solutions, including generalized hyperbolic and trigonometric function solutions. It is worth noting that while these waves progress smoothly, they exhibit a distinct turning point transition within this context. These findings provide valuable contributions to understanding the intricate dynamics governed by (3.1). Given the observed effectiveness of the methods on the double-order model, these methods can also be extended to different nonlinear models. All the methods we have discussed offer the advantage of a systematic algorithmic structure, enabling diverse solution forms through polynomial-type auxiliary equations and supporting the use of computer software packages. However, their sole limitation is that they are only applicable to equations of even order and those with nonlinearity involving the square of the first derivative. Additionally, employing various numerical methods from the literature to obtain different solution structures of the LGH model will be beneficial for comparing the solutions. Furthermore, these methods can be applied to the fractional LGH equation, a more generalized form of the LGH equation discussed in [21]. This allows for easy comparison of the similarities and differences in solution structures.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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