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Research Article

p-Summable sequence spaces with inner products

Şükran Konca [1*], Mochammad Idris [2] and Hendra Gunawan [2]

[1] Bitlis Eren University, Faculty of Science and Art, Department of Mathematics, Bitlis - Turkey
[2] Department of Mathematics, Institute of Technology Bandung, Bandung, 40132, Indonesia
[*] Corresponding author: skonca@beu.edu.tr

Abstract

We revisit the space ℓ^p of p-summable sequences of real numbers. In particular, we show that this space is actually contained in a (weighted) inner product space. The relationship between ℓ^p and the (weighted) inner product space that contains ℓ^p is studied. For $p>2$ we also obtain a result which describe how the weighted inner product space is associated to the weights.

Keywords: inner product spaces, normed spaces, p-summable sequences, weights.

1. Introduction

By $\ell^p=\ell^p(\mathbb{R})$ we denote the space of all p-summable sequences of real numbers. We know that for $p\neq 2$, the space ℓ^p is not an inner product space, since the usual norm $\|x\|_p=\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}}$ on ℓ^p does not satisfy the parallelogram law.

As an infinite dimensional normed space, ℓ^p can be equipped with another norm which is not equivalent to the usual norm. For example, $\|x\|_* := \left(\sum_{k=1}^{\infty}|x_k/k|^p\right)^{\frac{1}{p}}$, is such a norm (Idris et al. 2013). However, one may observe that this norm does not satisfy the parallelogram law too for $p\neq 2$.

One question arises: can we define a norm on ℓ^p which satisfies the parallelogram law? The reason why we are interested in the parallelogram law is because we eventually wish to define an inner product, possibly with weights, on ℓ^p, so that we can define orthogonality and many other notions on this space. One alternative is to define a semi-inner product on ℓ^p as in (Miličić, 1987), but having a semi-inner product is not as nice as having an inner product.

The reader might agree that inner product spaces, which were initially introduced by D. Hilbert in 1912, have been up to now the most useful spaces in practical applications of functional analysis.

In this paper, we introduce a (weighted) inner product on ℓ^p where $1\le p<\infty$. We discuss the properties of the induced norm and its relationship with the usual norm on ℓ^p. We also find that the inner product is actually defined on a larger space. We study the relationship between ℓ^p and this larger space, and found many interesting results, which we shall present in the following sections.

Throughout the paper, we assume that X is a real vector space. As in (Kreyszig, 1978), the norm on X is a mapping $\|.\|: X\to\mathbb{R}$ such that for all vectors $x,y\in X$ and scalars $\alpha\in\mathbb{R}$ we have:

1) $\|x\|\ge 0$ and $\|x\|=0$ if and only if $x=0$,
2) $\|\alpha x\|=|\alpha|\|x\|$,
3) $\|x+y\|\le\|x\|+\|y\|$.

The inner product on X is a mapping $\langle .,.\rangle$ from $X\times X$ into $\mathbb{R}$ such that for all vectors $x,y,z\in X$ and scalars α we have:

1) $\langle x,x\rangle\ge 0$ and $\langle x,x\rangle=0$ if and only if $x=0$,
2) $\langle x,y\rangle=\langle y,x\rangle$,
3) $\langle \alpha x,y\rangle=\alpha\langle x,y\rangle$,
4) $\langle x_1+x_2,y\rangle=\langle x_1,y\rangle+\langle x_2,y\rangle$.

As we work with sequence spaces of real numbers, we will use the sum notation $\sum_k$ instead of $\sum_{k=1}^{\infty}$, for brevity.

2. RESULTS FOR $1\le p\le 2$

In this section, we let $1\le p\le 2$, unless otherwise stated. The results presented here are mostly known; we write them for convenience. First, we observe that $\ell^p\subseteq\ell^2$ (as sets). Indeed, if $x=(x_k)$ is a sequence in ℓ^p, then

$$\|x\|_2^2=\sum_k|x_k|^2=\sum_k|x_k|^{2-p}|x_k|^p\le\sup_k|x_k|^{2-p}\sum_k|x_k|^p$$
$$\le\left[\sum_k|x_k|^p\right]^{\frac{2-p}{p}}\sum_k|x_k|^p=\left[\sum_k|x_k|^p\right]^{\frac{2}{p}}.$$

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Taking the square roots of both sides, we get

$$\left[\sum_k |x_k|^2\right]^{\frac{1}{2}} \le \left[\sum_k |x_k|^p\right]^{\frac{1}{p}} ,$$

which means that x is in ℓ^2.

Thus we realize that ℓ^p can actually be considered as a subspace of ℓ^2, equipped with the inner product

$$\langle x, y\rangle := \sum_k x_k y_k , \qquad (2.1)$$

and the norm

$$\|x\|_2 := \left[\sum_k |x_k|^2\right]^{\frac{1}{2}}. \qquad (2.2)$$

Being an induced norm from the inner product, the norm $\|.\|_2$ of course satisfies the parallelogram law:

$$\|x+y\|_2^2 + \|x-y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2,$$

for every $x, y \in \ell^p$.

A more general result is formulated in the following proposition, which describes the monotonicity property of the norms on ℓ^p spaces.

Proposition 2.1. If $1 \le p \le q \le \infty$, then $\ell^p \subseteq \ell^q$, with $\|x\|_q \le \|x\|_p$ for every $x \in \ell^p$. Moreover, the inclusion is strict: if $1 \le p < q \le \infty$, then we can find a sequence X in ℓ^q which is not in ℓ^p.

Proof. Let $1 \le p \le q \le \infty$. For every $x \in \ell^p$, we have

$$\|x\|_q^q = \sum_k |x_k|^q = \sum_k |x_k|^{q-p}|x_k|^p \le \sup_k |x_k|^{q-p} \sum_k |x_k|^p$$
$$\le \left[\sum_k |x_k|^p\right]^{\frac{q-p}{p}} \sum_k |x_k|^p = \left[\sum_k |x_k|^p\right]^{\frac{q}{p}}.$$

Taking the q-th roots of both sides, we get

$$\|x\|_q \le \|x\|_p,$$

which tells us that $\ell^p \subseteq \ell^q$.

To show that the inclusion is strict, one may take $x = (x_k) := \left(\frac{1}{k^{1/p}}\right)$, where $1 \le p < \infty$. Clearly $x \in \ell^q$ for $p < q \le \infty$, but $x \notin \ell^p$.

When we equip ℓ^p, where $1 \le p < 2$, with $\|.\|_2$, one might ask whether $\|.\|_2$ is equivalent with the usual norm $\|.\|_p$. The answer is negative. We already have $\|x\|_2 \le \|x\|_p$ for every $x \in \ell^p$. The following proposition tells us that we cannot control $\|x\|_p$ by $\|x\|_2$ for every $x \in \ell^p$.

Proposition 2.2. Let $1 \le p < 2$. There is no constant $C > 0$ such that $\|x\|_2 \ge C\|x\|_p$ for every $x \in \ell^p$.

Proof. For each $n \in \mathbb{N}$, take $x^{(n)} := \left(\frac{1}{k^{\frac{1}{p}+\frac{1}{n}}}\right)$. Then,

$$\|x^{(n)}\|_2^2 = \sum_k \frac{1}{k^{\frac{2}{p}+\frac{2}{n}}} \le \sum_k \frac{1}{k^{\frac{2}{p}}} < \infty$$

while

$$\|x^{(n)}\|_p^p = \sum_k \frac{1}{k^{1+\frac{p}{n}}} < \infty .$$

The first sum is bounded by a fixed number independent of n, while the second one is dependent on n and tends to ∞ as $n \to \infty$. Hence

$$\frac{\|x^{(n)}\|_2}{\|x^{(n)}\|_p} \to 0, \text{ as } n \to \infty .$$

So, there is no constant $C > 0$ such that $\|x\|_2 \ge C\|x\|_p$ for every $x \in \ell^p$.

Proposition 2.3. Let $1 \le p < 2$. As a set in $(\ell^2, \|.\|_2)$, ℓ^p is not closed but dense in ℓ^2.

Proof. To show that ℓ^p is not closed in ℓ^2, for each $n \in \mathbb{N}$ we take $x^{(n)} := \left(1, \frac{1}{2^{1/p}}, \ldots, \frac{1}{n^{1/p}}, 0, 0, \ldots\right)$. Clearly $(x^{(n)})$ converges to $x = (x_k) := \left(\frac{1}{k^{1/p}}\right)$ in $\|.\|_2$. But $x^{(n)} \in \ell^p$ for each $n \in \mathbb{N}$, while $x \notin \ell^p$. Therefore ℓ^p is not closed in ℓ^2.

To show that ℓ^p is dense in ℓ^2, we observe that every $x = (x_k) \in \ell^2$ can be approximated arbitrarily close by $x^{(n)} := (x_1, \ldots, x_n, 0, 0, \ldots) \in \ell^p, n \in \mathbb{N}$.

Remark 2.4. Proposition 2.2 and Proposition 2.3 warn us that when we use the ℓ^2-inner product and its induced norm on ℓ^p for $1 \le p < 2$, we have to be careful especially when we deal with the topology.

3. RESULTS FOR $2 < p < \infty$

Throughout this section, we let $2 < p < \infty$, unless otherwise stated. As we have seen in Proposition 2.1, the space ℓ^p is now larger than ℓ^2. Thus the usual inner product and norm on ℓ^2 are not defined for all sequences in ℓ^p.

To define an inner product and a new norm on ℓ^p which satisfies the parallelogram law, we have to use weights. Let us choose $v = (v_k) \in \ell^p$ where $v_k > 0,\ k \in \mathbb{N}$.

Next we define the mapping $\langle .,. \rangle_v$ which maps every pair of sequences $x = (x_k)$ and $y = (y_k)$ to

$$\langle x, y\rangle_v := \sum_k v_k^{p-2} x_k y_k , \qquad (3.1)$$

and the mapping $\|.\|_{2,v}$ which maps every sequence $x = (x_k)$ to

$$\|x\|_{2,v} := \left[\sum_k v_k^{p-2}|x_k|^2\right]^{\frac{1}{2}}. \qquad (3.2)$$

We observe that the mappings are well defined on ℓ^p. Indeed, for $x = (x_k)$ and $y = (y_k)$ in ℓ^p, it follows from Hölder's inequality that

$$\sum_k v_k^{p-2} x_k y_k \le \left[\sum_k v_k^p\right]^{\frac{p-2}{p}} \left[\sum_k |x_k|^p\right]^{\frac{1}{p}} \left[\sum_k |y_k|^p\right]^{\frac{1}{p}}. \qquad (3.3)$$

Thus the two mappings are defined on ℓ^p. Moreover, we have the following proposition, whose proof is left to the reader.

Proposition 3.1. The mappings in (3.1) and (3.2) define a weighted inner product and a weighted norm, respectively, on ℓ^p.

From (3.3), we see that the following inequality

$$\|x\|_{2,v} \leq \|v\|_p^{\frac{p-2}{2}} \|x\|_p$$

holds for every $x \in \ell^p$. It is then tempting to ask whether the two norms are equivalent on ℓ^p. The answer, however, is negative, due to the following result.

Proposition 3.2. There is no constant $C = C_v > 0$ such that

$$\|x\|_p \leq C\|x\|_{2,v},$$

for every $x \in \ell^p$.

Proof. Suppose that such a constant exists. Then, for $x := e_n = (0,\ldots,0,1,0,\ldots)$, where the 1 is the n^{th}-term, we have

$$1 \leq C v_n^{\frac{p-2}{2}}.$$

But this cannot be true, since $v_n \to 0$ as $n \to \infty$.

According to Proposition 3.2, it is possible for us to find a sequence in ℓ^p which is divergent with respect to the norm $\|\cdot\|_p$, but convergent with respect to the norm $\|\cdot\|_{2,v}$.

Example 3.3. Let $x^{(n)} := e_n \in \ell^p$, where $e_n = (0,\ldots,0,1,0,\ldots)$ (the 1 is the n^{th} term), then $\|x^{(m)} - x^{(n)}\|_p = 2^{\frac{1}{p}} \nrightarrow 0$ as $m,n \to \infty$. Since $(x^{(n)})$ is not a Cauchy sequence with respect to $\|\cdot\|_p$, it is not convergent with respect to $\|\cdot\|_p$. However, $\|x^{(n)}\|_{2,v} = v_n^{\frac{p-2}{2}} \to 0$ as $n \to \infty$, since $v_n \to 0$ as $n \to \infty$. Hence, $(x^{(n)})$ is convergent with respect to the norm $\|\cdot\|_{2,v}$.

If we wish, we can also define another weighted norm $\|\cdot\|_{\beta,v}$ on ℓ^p, where $1 \leq \beta \leq p < \infty$, by

$$\|x\|_{\beta,v} := \Big[\sum_k v_k^{p-\beta} | x_k |^\beta\Big]^{\frac{1}{\beta}}.$$

Here p may be less than 2. Note that if $\beta = p$, then $\|\cdot\|_{\beta,v} = \|\cdot\|_p$.

The following proposition gives a relationship between two such weighted norms on ℓ^p.

Proposition 3.4. Let $1 \leq \beta < \gamma \leq p$. Then we have $\|x\|_{\beta,v} \leq \|v\|_p^{\frac{p(\gamma-\beta)}{\gamma}} \|x\|_{\gamma,v}$ for every $x \in \ell^p$.

Proof. Suppose that $x \in \ell^p$. We compute

$$\|x\|_{\beta,v} = \Big[\sum_k v_k^{p-\beta} | x_k |^\beta\Big]^{\frac{1}{\beta}} = \Big[\sum_k v_k^{\frac{p(\gamma-\beta)}{\gamma}} v_k^{\frac{(p-\gamma)\beta}{\gamma}} | x_k |^\beta\Big]^{\frac{1}{\beta}}$$

$$\leq \Big[\sum_k v_k^p\Big]^{\frac{\gamma-\beta}{\gamma\beta}} \Big[\sum_k v_k^{p-\gamma} | x_k |^\gamma\Big]^{\frac{1}{\gamma}} = \|v\|_p^{\frac{p(\gamma-\beta)}{\gamma\beta}} \|x\|_{\gamma,v}$$

as desired.

Corollary 3.5. If $1 \leq \beta < 2 < \gamma \leq p$, then there are constants $C_{1,v}, C_{2,v} > 0$ such that

$$C_{1,v}\|x\|_{\beta,v} \leq \|x\|_{2,v} \leq C_{2,v}\|x\|_{\gamma,v}$$

for every $x \in \ell^p$.

4. FURTHER RESULTS

Let $2 < p < \infty$. As we have seen in the previous section, every sequence $x \in \ell^p$ has a finite norm in ℓ_v^2, namely $\|x\|_{2,v} < \infty$. This suggests that ℓ^p lives inside a larger space, consisting all sequences x with $\|x\|_{2,v} < \infty$. Let us denote this space by ℓ_v^2. We shall now discuss some properties of this space. First, we have the following proposition, which describes the relationship between ℓ_v^2 and ℓ^p.

Proposition 4.1. As sets, we have $\ell^p \subset \ell_v^2$ and the inclusion is strict.

Proof. Let $x \in \ell^p$. It follows from (3.3) that

$$\|x\|_{2,v} \leq \|v\|_p^{\frac{p-2}{2}} \|x\|_p,$$

which means that $x \in \ell_v^2$.

To show that the inclusion is strict, we need to find $x = (x_k)$ such that $\sum_k v_k^{p-2} | x_k |^2 < \infty$ but $\sum_k |x_k|^p = \infty$. We know that $v_k > 0$ for all $k \in \mathbb{N}$, and $v_k \to 0$ as $k \to \infty$. So, choose $m_1 \in \mathbb{N}$ such that $v_{m_1}^{p-2} < \frac{1}{2}$, $m_2 > m_1$ such that $v_{m_2}^{p-2} < \frac{1}{2^2}$, $m_3 > m_2$ such that $v_{m_3}^{p-2} < \frac{1}{2^3}$, and so on. Since the process never stops, we obtain an increasing sequence of nonnegative integers (m_j) such that $v_{m_j}^{p-2} < 2^{-j}$ for every $j \in \mathbb{N}$. Now put $x_k := 1$ for $k = m_1, m_2, m_3, \ldots$ and $x_k := 0$ otherwise. Hence

$$\sum_k v_k^{p-2} | x_k |^2 = \sum_j v_{m_j}^{p-2} < \sum_j \frac{1}{2^j} = 1,$$

while $\sum_k |x_k|^p = \sum_j 1 = \infty$. This means that x is in ℓ_v^2 but not in ℓ^p.

Theorem 4.2. The space $(\ell_v^2, \|\cdot\|_{2,v})$ is complete. Accordingly, $(\ell_v^2, \langle \cdot,\cdot \rangle_v)$ is a Hilbert space.

Proof. It is easy to see that the space $(\ell_v^2, \|\cdot\|_{2,v})$ is a linear normed space, so we omit the details. To prove the completeness, let $(x^{(n)})$ be any Cauchy sequence in the space ℓ_v^2, where $x^{(n)} = (x_k^{(n)}) = (x_1^{(n)}, x_2^{(n)}, \ldots)$. Then for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n,m > n_0$

$$\|x^{(n)} - x^{(m)}\|_{2,v} = \Big[\sum_k v_k^{p-2} | x_k^{(n)} - x_k^{(m)} |^2\Big]^{\frac{1}{2}} < \varepsilon. \qquad (4.1)$$

It follows that for each $k \in \mathbb{N}$ we have

$$v_k^{p-2} | x_k^{(n)} - x_k^{(m)} |^2 \leq \sum_k v_k^{p-2} | x_k^{(n)} - x_k^{(m)} |^2 = \|x^{(n)} - x^{(m)}\|_{2,v}^2 < \varepsilon^2.$$

Then

$$| x_k^{(n)} - x_k^{(m)} | < \varepsilon v_k^{\frac{2-p}{2}}$$

for all $m,n > n_0$. Thus, for each fixed $k \in \mathbb{N}$, $(x_k^{(n)})$ is a Cauchy sequence of real numbers. Hence, it is convergent, say $x_k^{(n)} \to x_k$ as $n \to \infty$.

Using these infinitely many limits $x_1, x_2, x_3, \ldots$, we define the sequence $x := (x_1, x_2, x_3, \ldots)$. Then we have constructed a candidate limit for the sequence $(x^{(n)})$. However, so far, we only have that each individual component of $x^{(n)}$ converges to the corresponding component of x, i.e., $(x^{(n)})$ converges componentwise to x. To prove that $(x^{(n)})$ converges to X in norm, we go back to (4.1) and pass it to the limit $m \to \infty$. We obtain

$$\left\| x^{(n)} - x \right\|_{2,v} = \left[\sum_k v_k^{p-2} | x_k^{(n)} - x_k |^2 \right]^{\frac{1}{2}} < \varepsilon ,$$

for all $n > n_0$. Since the space ℓ_v^2 is linear, we also get $x = (x - x^n) + x^n \in \ell_v^2$. This completes the proof.

The following proposition tells us that ℓ^p is "not too far" from ℓ_v^2.

Proposition 4.3. As a set in $(\ell_v^2, \|.\|_{2,v})$, ℓ^p is not closed but dense in ℓ_v^2.

Proof. As in the proof of Proposition 4.1, we construct an increasing sequence of nonnegative integers (m_j) such that $v_{m_j}^{p-2} < 2^{-j}$ for every $j \in \mathbb{N}$. Next, for each $j \in \mathbb{N}$, we define $x^{(n)} = (x_k^{(n)})$ by $x_k^{(n)} := 1$ for $k = m_1, m_2, \ldots, m_n$ and $x_k^{(n)} := 0$ otherwise. Then we see that $(x^{(n)})$ converges in ℓ_v^2 and the limit is the sequence $x = (x_k)$ where $x_k := 1$ for $k = m_1, m_2, m_3, \ldots$ and $x_k := 0$ otherwise. While $x^{(n)} \in \ell^p$ for every $n \in \mathbb{N}$, we find that the limit $x \notin \ell^p$. This shows that ℓ^p is not closed in ℓ_v^2.

The fact that ℓ^p is dense in ℓ_v^2 is easy to see, since every $x = (x_k) \in \ell_v^2$ can be approximated by $x^{(n)} := \{x_1, \ldots, x_n, 0, ..$ $0, \ldots)$ for sufficiently large $n \in \mathbb{N}$.

Proposition 4.3 motivates us to study ℓ_v^2 further as the ambient space, replacing ℓ^p. So far, we have fixed the weight $v = (v_k)$. We now would like to know how the space ℓ_v^2 depends on the choice of v.

Let V_p be the collection of all sequences $v = (v_k) \in \ell^p$ with $v_k > 0$ for every $k \in \mathbb{N}$. Let $v = (v_k), w = (w_k) \in V_p$. We say that V and W are equivalent and write $v \sim w$ if and only if there exists a constant $C > 0$ such that

$$\frac{1}{C} v_k \le w_k \le C v_k ,$$

for every $k \in \mathbb{N}$. Our final theorem is the following.

Theorem 4.4. Let $v, w \in V_p$. Then, the following statements are equivalent:

(1) $v \sim w$.

(2) There exists a constant $C > 0$ such that

$$\|x\|_{2,w} \le C \|x\|_{2,v}, \quad x \in \ell_v^2,$$

and

$$\|x\|_{2,v} \le C \|x\|_{2,w}, \quad x \in \ell_w^2.$$

(3) $\ell_v^2 = \ell_w^2$ as sets, and the two norms $\|.\|_{2,v}$ and $\|.\|_{2,w}$ are equivalent there.

(4) There exists a constant $C > 0$ such that

$$\frac{1}{C} \|x\|_{2,v} \le \|x\|_{2,w} \le C \|x\|_{2,v}, \quad x \in \ell^p.$$

Proof. The chain of implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are clear. Hence it remains only to show that $(4) \Rightarrow (1)$. Assume that there exists a constant $C > 0$ such that

$$\frac{1}{C} \|x\|_{2,v} \le \|x\|_{2,w} \le C \|x\|_{2,v}, \quad x \in \ell^p.$$

Take $x := e_n$, where $n \in \mathbb{N}$ is fixed but arbitrary. Then $x \in \ell^p$, so that x is in ℓ_v^2 as well as in ℓ_w^2. Moreover,

$$\|x\|_{2,v} = v_n^{\frac{p-2}{2}} \quad \text{and} \quad \|x\|_{2,w} = w_n^{\frac{p-2}{2}}.$$

Hence, from our assumption, we obtain

$$\frac{1}{C} v_n^{\frac{p-2}{2}} \le w_n^{\frac{p-2}{2}} \le C v_n^{\frac{p-2}{2}},$$

and this holds for every $n \in \mathbb{N}$. Taking the $\left(\frac{p-2}{2}\right)$-th roots, we conclude that $v \sim w$. This completes the proof.

Remark 4.5. Theorem 4.4 is interesting in the sense that the two spaces $(\ell_v^2, \|.\|_{2,v})$ and $(\ell_w^2, \|.\|_{2,w})$ are identical if and only if $\|.\|_{2,v}$ and $\|.\|_{2,w}$ are equivalent on ℓ^p, and this can be checked through the equivalence of v and w. Thus, for example, the space associated to $u := (1/k)$ is identical with that associated to $v := (k/k^2 + 1)$, but is different from the one associated to $w := (1/k^2)$. We also note that, according to Proposition 4.3, even though ℓ_v^2 and ℓ_w^2 may be different, both spaces contain ℓ^p as a dense subset.

5. CONCLUDING REMARKS

We have shown that the space ℓ^p can be equipped with a (weighted) inner product and its induced norm. Using the inner product, one may define orthogonality on ℓ^p, carry out the Gram-Schmidt process to get an orthogonal set, define the volume of an n-dimensional parallelepiped on ℓ^p (Gunawan et al., 2005), and so on. When we have to deal with topology, however, we are suggested to consider the (weighted) inner product space that contains ℓ^p, namely ℓ^2 for $1 \le p \le 2$ or ℓ_v^2 for $2 < p < \infty$ (where $v \in \ell^p$ with $v_k > 0$ for every $k \in \mathbb{N}$), which are complete.

There we might also be interested in bounded linear functionals. For example, for $2 < p < \infty$, the functional $f_y := \langle \cdot, y \rangle_v$ is linear and bounded on ℓ_v^2, and its norm can be given by

$$\| f_y \| = \sup_{\|x\|_{2,v} \ne 0} \frac{| f_y(x) |}{\|x\|_{2,v}} = \sup_{\|x\|_{2,v} \ne 0} \frac{| \langle x, y \rangle_v |}{\|x\|_{2,v}}.$$

Clearly $\| f_y \| \le \|y\|_{2,v}$, and by taking $x := \frac{y}{\|y\|_{2,v}}$ we obtain $\| f_y \| = \|y\|_{2,v}$. Moreover, we can prove an analog of the Riesz-Fréchet Theorem (Berberian, 1961), which states that for any bounded linear functional g on ℓ_v^2, there exists a unique $y \in \ell_v^2$ such that $g(x) = \langle x, y \rangle_v$ for every $x \in \ell_v^2$ and $\|g\| = \|y\|_{2,v}$.

All the results for $2 < p < \infty$ also hold for $L^p(\mathbb{R})$, the space of p -integrable functions on $\mathbb{R}$. For $1 \le p < 2$, however, different situations take place.

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