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*- RESEARCH ARTICLE -*

**Relation between Center Coloring and the other Colorings**

Zeynep Ors Yorgancioglu[1*], Pinar Dundar[2], Mehmet Umit Gursoy[2]

[1]Department of Mathematics, Faculty of Science and Letters, Yasar University, Izmir, Turkey.
[2] Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey.

**Abstract**
In this paper, center coloring and center coloring number are defined, some bounds are established for the center coloring number of a graph in terms of other graphical coloring parameters, and a polynomial time algorithm is proposed in order to calculate the center coloring of a graph.

**Keywords:**
Graph Coloring, center coloring, center coloring number

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## Introduction

Over the 150 years, various works have been done on the coloring of graphs such as vertex coloring, edge coloring and etc. A coloring of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different colors. A coloring in which *k* colors are used is a *k*-coloring. The minimum integer *k* for which a graph G is *k*-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$ (Chartrand et. al., 2009).

An assignment of colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an *edge coloring* of G. The graph G is *k*-edge colorable if there exists an $\ell$-edge coloring of G for some $\ell \leq$ k. The minimum integer *k* for which a graph G is *k*-edge colorable is its *edge chromatic number* and is denoted by $\chi_1(G)$ (Chartrand et. al., 2009).

* *Corresponding Author: Zeynep Ors Yorgancioglu, e-mail: zeynep.ors@yasar.edu.tr*

A *total coloring* of a graph G is an assignment of colors to the elements (vertices and edges) of G so that adjacent elements and incident elements of G are colored differently. A *k* total coloring is a total coloring that uses *k* colors. The minimum k for which a graph G admits a *k*-total coloring is called the *total coloring number* of G and is denoted by $\chi_2\ (G)$ (Chartrand et. al., 2009).

*A harmonious coloring* of a simple graph G is proper vertex coloring such that each pair of colors appears together on at most one edge. Formally, a harmonious coloring is a function $c$ from a color set $C$ to the set $V$(G) of vertices of G such that for any edge $e$ of G with end points $x, y$ say $c(x) \neq c(y)$, and for any pair of distinct edges $e, e'$ with end points $x, y$ and $x', y'$ respectively, then $\{c(x), c(y)\} \neq \{c(x'), c(y')\}$. The *harmonious chromatic number* $\chi_h$(G) is the least number of colors in such a coloring (Chartrand & Lesniak, 2005).

For a nontrivial connected graph G, let $c: V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. For a vertex $v$ of G, the neighborhood color set $NC(v)$ is the set of colors of the neighbors of $v$. The coloring $c$ is called a *set coloring* if $NC(u) \neq NC(v)$ for every pair $u, v$ of adjacent vertices of G. The minimum number of colors required of such a coloring is called *the set chromatic number* $\chi_s(G)$ of G (Chartrand et. al., 2009).

Let G be a simple, connected graph with *n* vertices and *m* edges. We define a *k*-coloring of a graph as a mapping *f* from the vertices of G onto the set {1, 2, ..., *k*}. Let $e$ be an edge between vertices *u* and *v*. If *u* and *v* are assigned colors *f* (*u*) and *f* (*v*) respectively, then the color of $e$ is defined by $f(e) = \{f(u), f(v)\}$. *A line-distinguishing coloring* of G is a *k*-coloring of G such that no two edges have the same color. In other words, if $e_1$ and $e_2$ are any two edges in G, then $f(e_1) \neq f(e_2)$. Note that it is not required that each allowable pair of colors appears exactly once. The *line-distinguishing chromatic number* $\lambda(G)$ is defined as the smallest *k* such that G has a line-distinguishing *k*-coloring. Note that two adjacent vertices may have the same color (Immelman, 2007).

Let G be a nontrivial connected graph on which an edge-coloring *c*: *E* (*G*)→ {1,2, · · · ,*n*}, $n \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. An edge-colored graph G is *rainbow connected* if every two distinct vertices are connected by a rainbow path. An edge-coloring under which *G* is rainbow connected is called a *rainbow coloring*. Clearly, if a graph is rainbow connected, it must be connected. Conversely, every connected graph has a trivial edge-coloring that makes it rainbow connected by coloring edges with distinct colors. Thus, we define the *rainbow connection number* of a connected graph G, denoted by *rc* (*G*), as the smallest number of colors that are needed in order to make G rainbow connected. A rainbow coloring using *rc* (*G*) colors is called a *minimum rainbow coloring* (Li & Sun, 2012).

A vertex-colored graph G is *rainbow vertex-connected* if its every two distinct vertices are connected by a path whose *internal* vertices have distinct colors. A vertex-coloring under which G is rainbow vertex-connected is called a *rainbow vertex-coloring*. The *rainbow vertex-connection number* of a connected graph G, denoted by *rvc* (G), is the smallest number of colors that are needed in order to make G rainbow vertex-connected (Li & Sun, 2012).

A *center coloring* of a graph is an assignment of colors to the vertices of G, one color to each vertex so that different distance vertices from the center are assigned different colors. Two adjacent vertices can receive the same color. The number of colors required of such a coloring is called center coloring number $C_c$(G) of G (Yorgancıoğlu et. al., 2015). This coloring can be applied to hierarchy problems to find the number of structures, people, criteria and comparisons, etc. Moreover it can be applied to earthquake motion problems to find the number of settlements that are affected by an earthquake.

The *distance d (u, v)* between two vertices *u* and *v* in a connected graph G is the length of a shortest *u-v* path in G (Buckley & Harary, 1990).

The *eccentricity e* (*v*) of a vertex *v* in a connected graph G is the distance from *v* to a vertex farthest from *v* in G (Buckley & Harary, 1990).

The *radius rad* (G) of a connected graph G is defined as the minimum eccentricity among the vertices of G and the *diameter diam* (G) is a maximum eccentricity among the vertices of G (Buckley & Harary, 1990). And v is a central vertex if *e* (*v*) = *rad* (*v*) and the center *C* (G) is the set of all central vertices (Buckley & Harary, 1990).

Some graphs G have the property that each vertex of G is a central vertex. A graph is *self-centered* if every vertex is in the center (Buckley & Harary, 1990).

***Theorem 1.1. Brooks' Theorem***

Let G be a connected simple graph whose maximum vertex degree is $d$. If G is neither a cycle graph with an odd number of vertices, nor a complete graph, then $\chi(G) \le d$ (Aldous & Wilson, 2006).

***Theorem 1.2. Vizing Theorem***

If $G$ is a nonempty graph, then,

$$\chi_1(G) \le \Delta(G) + 1.$$

***Proposition 1.1. Total Coloring Conjecture***

For every graph $G$,

$\chi_2(G) \le 2 + \Delta(G)$ (Chartrand & Lesniak, 2005).

***Theorem 1.3.***

$\chi_h(G) = n$ for any graph $G$ of a diameter at most 2 (Miller & Pritikin, 1991).

***Proposition 1.2.***

We always have $rvc(G) \le n - 2$ (except for the singleton graph) (Li & Sun, 2012).

***Proposition 1.3.***

$rvc(G) = 0$ if and only if *G* is a complete graph (Li & Sun, 2012).

***Proposition 1.4.***

$rvc(G) \geq diam(G) - 1$ with equality if the diameter of $G$ is 1 or 2 (Li & Sun, 2012).

**Some Bounds For Center Coloring**

In this section, we give some bounds for center coloring number for graphs.

***Theorem 2.1.*** If $G$ is a connected graph with $n$ vertices that is not a tree, then

$$1 \leq C_c(G) \leq \left\lfloor \frac{n+1}{2} \right\rfloor.$$

**Proof:** If the graph is self-centered graph, its center coloring number is 1. To prove the right side of the inequality, the distance from the center vertex of the graph $G$ to the furthest vertices is maximum $\left\lfloor \frac{n-1}{2} \right\rfloor$. So one more color is added for the center coloring to get $\left\lfloor \frac{n+1}{2} \right\rfloor$. ■

***Theorem 2.2.*** If $G$ is a connected graph that is not a tree and not a self-centered graph, then, for $n \geq 3$,

$$2 \leq C_c(G) \leq \Delta(G).$$

**Proof:** The vertex degree of graph $G$ is at most is $n$-1. From theorem 2.1,

$C_c(G) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, $\frac{n+1}{2} \leq n-1$ is obtained. To prove the right side of the inequality, the distance between the center and other vertices can be at least 1, so the center coloring number is at least 2.

***Theorem 2.3.*** If $G$ and $\overline{G}$ is a connected graph of order $n$, then,

(i) $4 \leq C_c(G) + C_c(\overline{G}) \leq n$

(ii) $4 \leq C_c(G).C_c(\overline{G}) \leq \frac{n^2}{4}$.

**Proof:** First we verify the upper bound for $C_c(G) + C_c(\overline{G})$ and $C_c(G).\ C_c(\overline{G})$. $C_c(G) \leq n-1$ (Yorgancıoğlu et. al., 2015) but for the connected graph $\overline{G}$, $C_c(G) \leq n-2$. So

$C_c(G) + C_c(\overline{G}) \leq n-2+2 = n$.

Since the arithmetic mean of two positive numbers is always at least as large as their geometric mean, we have

$$\sqrt{C_c(G).C_c(\overline{G})} \leq \frac{C_c(G) + C_c(\overline{G})}{2} \leq \frac{n}{2}$$

$$C_c(G).C_c(\overline{G}) \leq \frac{n^2}{4}.$$

To verify the lower bound for (i) and (ii) $G$ and $\overline{G}$ graphs can not be self-centered graphs so center coloring number of $G$ is at least 2.

So $2+2 \leq C_c(G)+C_c(\overline{G})$ and $2.2 \leq C_c(G)\, C_c(\overline{G})$. ■

***Theorem 2.4.*** For the integer *a, b* pairs in an interval $1 < \log_2(b+1) \leq a$, the center coloring number of a minimum diameter spanning tree is "*a*" and total coloring number is "*b*".

**Proof:** To verify the upper bound; let $C_c(T_{MDST}) = a$ and let there be given set *a*-coloring of $G$ using the colors in $\mathbb{N}_a$. Since there are $(2^a - 1)$ non- empty subsets of $\mathbb{N}_a$ and total coloring number is at most $(2^a - 1)$.

So,

$$b \leq (2^a - 1) \Rightarrow b+1 \leq 2^a \Rightarrow \log_2(b+1) \leq \log_2 2^a \Rightarrow \log_2(b+1) \leq a$$

To verify the lower bound;

$$1 < b \Rightarrow 1+1 < b+1 \Rightarrow \log_2 2 < \log_2(b+1) \Rightarrow 1 < \log_2(b+1)\ .$$

■

**Relations Between Center Coloring and the other colorings**

In this section, we compare the center coloring number with some other coloring numbers.

***Theorem 3.1.*** If $G$ is a connected graph, then

$$C_c(G) \leq \lambda(G)\ .$$

**Proof:** Since $C_c(G) \leq \Delta(G)$ (theorem 2.2) and $\lambda(G) \geq \Delta(G)$ (Immelman, 2007) are given, the inequality $C_c(G) \leq \Delta(G) \leq \lambda(G)$ follows. Then the inequality $C_c(G) \leq \lambda(G)$ is clear.

■

***Theorem 3.2.*** If $G$ is a connected graph that is not a tree, then

$$C_c(G) \leq X_s(G)\,.$$

**Proof:** In the definition of set chromatic number, the neighborhood color sets of each adjacent vertex must be given differently. But in center coloring adjacent vertices may have same color set and adjacent vertices may have the same color. So it is clear from the definition that center coloring number is smaller than the set chromatic number.

■

***Theorem 3.3.*** If $G$ is a connected graph that is not a tree, then

$$C_c(G) \leq \chi(G)\,.$$

**Proof:** Since $\chi_s(G) \le \chi(G)$ (Chartrand et al., 2009) and in theorem 3.2 $C_c(G) \le \chi_s(G)$ are given, the inequality $C_c(G) \le \chi_s(G) \le \chi(G)$ follows, Then the inequality $C_c(G) \le \chi(G)$ is clear.

■

***Theorem 3.4.*** If $G$ is a connected graph, then

$$C_c(G) \le \chi_h(G).$$

**Proof:** Since $\Delta + 1 \le \chi_h(G)$ (Kubale, 2004) and $C_c(G) \le \Delta(G)$ (theorem 2.2) are given, the inequality $C_c(G) + 1 \le \Delta(G) + 1 \le \chi_h(G)$ follows. Then the inequality $C_c(G) \le \lambda(G)$ is clear.

Also in M. Kubale (Kubale, 2004) " For any graph $G$ with diameter at most 2, $\chi_h(G) = n$" is given and for any connected graph with diameter at most 2, the center coloring number of these graphs is at most 3. So it is obvious that center coloring number is smaller than the harmonious chromatic number for any connected graph with diameter at most 2.

■

***Theorem 3.5.*** If $G$ is a connected graph, then

$$C_c(G) \le rc(G) + 1.$$

**Proof:** $C_c(G) \le rad(G) + 1$ [Yorgancıoğlu et al., 2015)] is given and also $rad(G) \le diam(G)$ is known, so $rad(G) + 1 \le diam(G) + 1$ is clear. And since $rc(G) \ge diam(G)$ (Li & Sun, 2012), we can write $rc(G) + 1 \ge diam(G) + 1$. So from $rc(G) + 1 \ge diam(G) + 1 \ge C_c(G)$ it follows, that we have $C_c(G) \le rc(G) + 1$.

■

***Theorem 3.6.*** If $G$ is a connected graph, that is not a star or a complete graph, then for *n*>4

$$C_c(G) \le rvc(G).$$

**Proof:** In theorem 2.1 the inequality $1 \le C_c(G) \le \left\lfloor \frac{n+1}{2} \right\rfloor$ and in (Li & Sun, 2012), the inequality $rvc(G) \le n-2$ (except for the singleton graph) are given. For *n*>4, $\left\lfloor \frac{n+1}{2} \right\rfloor \le n-2$ is known. So from $\left\lfloor \frac{n+1}{2} \right\rfloor \le n-2$, we have $C_c(G) \le rvc(G)$.

■

***Theorem 3.7.*** If $G$ is a connected graph that is not a tree, then

$$C_c(G) \le \chi_1(G)$$

**Proof:** Brooks Theorem says $\chi(G) \le \Delta(G)$ and Vizing Theorem for simple graphs say $\Delta(G) \le \chi_1(G) \le \Delta(G) + 1$.

From these two theorems, $\chi(G) \le \chi_1(G)$ is clear. In theorem 3.3 $C_c(G) \le \chi(G)$ is given. So $C_c(G) \le \chi_1(G)$ is proved.

■

***Theorem 3.8.*** If $G$ is a connected graph that is not a tree, then

$$C_c(G) \leq \chi_2(G).$$

**Proof:** Since the inequality $\chi(G) \leq \chi_2(G)$ is clear in (Yorgancıoğlu & Dündar, 2011) and $C_c(G) \leq \chi(G)$ is given in theorem 3.3, by these two theorems, we have $C_c(G) \leq \chi(G) \leq \chi_2(G)$

■

**Computing the center coloring of a graph**

In this section, an algorithm is proposed in order to calculate *the center coloring* for any simple finite undirected graph without loops and multiple edges by using Floyd-Warshall shortest-path algorithm. It gives definite results for all given data.

---

**Algorithm:** The Center Coloring

---

```
Begin
Floyd-Warshall (G);
        Input A adjacency (or W weighted ) matrix of graph G and find D distance matrix
  from A(W)  by using Floyd-Warshall algorithm above.
Ec-rad-center (D);
  Find eccentricities of all vertices (ec 1xn matrix )
          Find radius value and center vertices (Center 1xn matrix )
k←1;
j←1;
  for i=1 to n do
      if ec[i]=r then
        Centercoloring[k,j]←i;
        j←j+1;
       endif
  endfor
   k←2 ;
for d=1to n/2+1 do
  t←1;
 for m=1to n do
  if Centercoloring[1,m]<>0 then
    for i=1to n do
       if  D[i,Centercoloring[1,m]]=d then
           questioning←true;
            for s=1 to k-1 do
               for j=1 to n do
                 if  Centercoloring[s,j]<>0 then
                   if Centercoloring[s,j] =i then
                     questioning ←false ;
```

```
                endif
               endif
              endfor
            endfor
           if questioning = true then
            Centercoloring[k,t]←i;
            t←t+1;
           endif
          endif
        endfor
       endif
      endfor
       k←k+1;
      endfor
      End.
```

The above function is Floyd-Warshall which returns the distances matrix from adjacency matrix of graph $G$.

```
    Function Floyd-Warshall (G);
    Begin
 if i=j then w_ij ← 0 ;

 if v_i disjoint v_j then w_ij ← ∞ ;

 for k =1 to n do

    for i=1 to n do

      for j=1 to n do    { for j=1 to i do }

        if w_ij > w_ik + w_kj then

                          w_ij ← w_ik + w_kj ;

        endif

     endfor

    endfor

 endfor

 D ← W;

 End;
```

The above function is finding eccentricities, radius and center vertices which is used by the distances matrix of graph *G*.

```
Function Ec-rad-center (D);
Begin
min← ∞;

  for i=1 to  n  do

      max←0;
    for  j=1 to n do

      if  max < D[i,j]  then

            max ← D[i,j];

      endif

    endfor

    ec[i]←max;

      if ec[i]<min then
         min←ec[i];
    endif

  endfor

r ← min;

j←1;

  for i=1 to n do
      if ec[i]=r then
        Center[j]←i;
       j←j+1;
      endif
    endfor
End;
```

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