e-ISSN: 2149-4584

Erzincan University Journal of Science and Technology

EJSAT 2026, 19 (1) 51-60

Research Article

On Rings for Which Each Module with the Property $(S_s{}^*)$ Is an Extending Module

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Received: 01/10/2024, **Revised:** 21/03/2025, **Accepted:** 08/04/2025, **Published:** 30/03/2026

Abstract

An S −module H is named *s-cosingular* if $H = Z_s^*(H) = \{h \in H |\, Sh \text{ is a semisimple and small module}\}$. An S −module H is said to satisfy *the property* $(S_s{}^*)$ if H has a direct summand T for any submodule K of H such that $T \leq K$ and K/T is an s-cosingular module. It has been proved that for an extending module H exhibiting $(S_s{}^*)$ if $Z_s{}^*(H)$ is projective, then each submodule of H is an extending module and H is a locally artinian serial module. Semisimple and left ss-Harada rings have also been characterized by the conditions "each extending module exhibits $(S_s{}^*)$" and "each module with $(S_s{}^*)$ is injective".

Keywords: Extending modules, ss-lifting modules, modules having the property $(S_s{}^*)$, left ss-Harada rings.

Her $(S_s{}^*)$ Özelliğe Sahip Modülü Genişletilmiş Modül olan Halkalar Üzerine

Öz

Eğer $H = Z_s^*(H) = \{h \in H |\, Sh \text{ bir yarıbasit ve küçük modüldür}\}$ ise H S −modülüne *s-eştekil* denir. Eğer bir H S −modülü her K alt modülü için $T \leq K$ ve K/T s-eştekil modül olacak şekilde bir T direkt toplam terimine sahip ise H modülüne $(S_s{}^*)$ *özelliğini* sağladığı söylenir. $(S_s{}^*)$ özelliğine sahip bir H genişletilmiş modülü için $Z_s^*(H)$ projektif ise, H modülünün her alt modülünün bir genişletilmiş modül olduğu ve H modülünün bir yerel artin seri modül olduğu kanıtlanmıştır. Yarıbasit ve sol ss-Harada halkaları aynı zamanda "her genişletilmiş modülün $(S_s{}^*)$ özelliğine sahip olduğu" ve "$(S_s{}^*)$ özelliğine sahip her modülün injektif olduğu" koşullarıyla da karakterize edilmiştir.

Anahtar Kelimeler: Genişletilmiş modüller, ss-yükseltilebilir modüller, $(S_s{}^*)$ özelliğine sahip modüller, sol ss-Harada halkalar.

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Cite this Article: Erzincan University Journal of Science and Technology 2026, 19(1)51-60. https://doi.org/10.18185/erzifbed.1513760

1. Introduction

Within this text, S will represent a ring with an identity element, and each module discussed will be a left unitary $S-$module. Consider H as a module of this type. The symbols $E(H)$, $Rad(H), Soc(H), Soc_s(H)$, $Z(H)$ and $End(H)$ will signify the injective hull of H, the radical of H, the socle of H, the sum of simple and small submodules of H, the singular submodule of H and the endomorphism ring of H, respectively (see [1-2-3]). Note that for any module H, it is proved in [4, Lemma 2] that $Soc_s(H) = Rad(H) \cap Soc(H)$. For a submodule K of H, for a direct summand K of H and for an essential submodule K of H, we will use the notations $K \leq H$, $K \leq_{\oplus} H$ and $K \leq_e H$, respectively. By the symbols ${}_SS$ and S_S, we mean that the left $S-$module S and the right $S-$module S. $K \leq H$ is named *closed* when K does not have any proper essential extension in H [5].

A module H is named an *extending module* provided its closed submodules are direct summands, or equivalently each submodule of H is essential in a direct summand of H. For modules H_1 and H_2, H_2 is named *essentially* H_1 *−injective*, provided each homomorphism $h: K \longrightarrow H_2$ where K is a submodule of H_1 with $Ker(H) \leq_e K$ can be extended to a homomorphism $\psi: H_1 \longrightarrow H_2$ (see [6]). A ring S is named *left* GV *−ring* provided each simple singular $S-$module is injective (see [7]). A module H is named *locally artinian* provided each finitely generated submodule of H is artinian. H is named a *uniserial module* provided each submodule of H is linearly ordered by inclusion. H is named a *serial module* provided it is a direct sum of uniserial modules. A ring S is named *left serial (right serial)* provided ${}_SS$ (S_S) is a serial module. If a ring S is both left and right serial, S is named *serial ring* (see [1]).

Let $K \leq H$. K is named *small submodule* provided whenever $H = K + L$ for some submodule L of H, we get $L = H$, and for this we use the notation $K \ll H$. The module H is named a *small module* provided H is a small submodule of some modules. H is a small module if and only if $H \ll E(H)$ (see [8]). In [9] the submodule $Z^*(H)$ of a module H consisting of the elements $h \in H$ with the property $Sh \ll E(Sh)$ is introduced. Since $Rad(H)$ is the sum of all small submodules of H, then $Rad(H) \leq Z^*(H)$. In the same paper, H is named a *cosingular module* provided $H = Z^*(H)$ and a ring S is named *left cosingular* provided ${}_SS$ is a cosingular module. The author proved that for a module H, $Z^*(H) = H \cap Rad(E(H))$ and it is showed in [9, Lemma 2.6] that submodules, homomorphic images and direct sums of cosingular modules are also cosingular.

In [10], a module H is named *s-cosingular S-module* provided $H = Z_s^*(H)$ where $Z_s^*(H)$ denotes a submodule of H consisting of elements $h \in H$ such that the cyclic submodule Sh is a semisimple and small module. It is demonstrated in [10, Lemma 2.1] that $Z_s^*(H) = Soc(H) \cap Z^*(H)$. Since $Rad(H) \leq Z^*(H)$, $Soc_s(H) \leq Z_s^*(H)$.

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Cite this Article: Erzincan University Journal of Science and Technology 2025, 18(3)51-60.
https://doi.org/10.18185/erzifbed.1513760

It is well known that a module H is named *lifting module* provided for each submodule K of H, there exists a direct summand H_1 of H such that $H_1 \leq K$ and $K/H_1 \ll H/H_1$ (see [5]). Since small modules are cosingular, Özcan described the property (S^*) in [9], as follows. A module H is named to satisfy *the property* (S^*) when for each submodule K of H, there exists a direct summand H_1 of H such that $H_1 \leq K$ and K/H_1 is a cosingular module. A ring S satisfies (S^*) in case ${}_SS$ satisfies (S^*).

In [11] the author introduced a generalization of lifting modules, namely *ss*-lifting modules, that is, when for each submodule K of a module H, there exists a direct summand H_1 of H such that $H_1 \leq K$ and $K/H_1 \leq Soc_s(H/H_1)$, H is named an *ss-lifting module.* Since $Soc_s(H) \leq Z_s^*(H)$ for any module H, in [10] a module H is named to exhibit *the property* $({S_s}^*)$ provided for each submodule K of H, there exists a direct summand H_1 of H such that $H_1 \leq K$ and K/H_1 is an *s*-cosingular module.

In this paper, we begin with giving some results about an extending module exhibiting $({S_s}^*)$. We demonstrate that an extending module H exhibits $({S_s}^*)$ if and only if its submodules are direct sum of an extending module and an *s*-cosingular module. We show that each module K with ${Z_s}^*(K) = 0$ is H-injective where H is a module with semisimple $H/{Z_s}^*(H)$. As a result of this, we conclude that each module H exhibiting $({S_s}^*)$ has a decomposition $H = H_1 \oplus H_2$ such that H_2 is an H_1 $-$injective module. We exactly prove that for an extending module H exhibiting $({S_s}^*)$, when ${Z_s}^*(H)$ is projective, each submodule of H is extending and H is a locally artinian serial module. We address the characteristics of a ring over which each extending module exhibits $({S_s}^*)$ and each module with $({S_s}^*)$ is extending. We demonstrate that each left injective S $-$module is an ss $-$lifting module if and only if S is a left perfect ring with semisimple radical and each extending S $-$module exhibits $({S_s}^*)$. We prove that a ring S is a semisimple ring if and only if $S/{Z_s}^*({}_SS)$ is semisimple and each S $-$module with $({S_s}^*)$ is injective.

2. Preliminaries

In this part of this paper, we give some known facts for *s*-cosingular modules and the modules exhibiting $({S_s}^*)$ which are the notions we based on for this note.

Proposition 2.1 For a module H, the following statements hold.

1) If $h: H \longrightarrow K$ is a homomorphism for any module K, then $h({Z_s}^*(H)) \leq {Z_s}^*(K)$.
2) If $K \leq H$, then ${Z_s}^*(K) = K \cap {Z_s}^*(H)$.
3) If $H = \oplus_{i\in I} H_i$, then ${Z_s}^*(H) = \oplus_{i\in I} {Z_s}^*(H_i)$.

Proof. By [10, Lemma 2.2].

Corollary 2.2 For any ring S, direct sums, submodules and factor modules of any *s*-cosingular S-module are *s*-cosingular, too.

Proof. By [10, Corollary 2.4].

Theorem 2.3 For any module H, the following statements are equivalent:

1) H exhibits $({S_s}^*)$.

2) H has a decomposition $H = H_1 \oplus H_2$ such that $H_1 \leq K$ and $K \cap H_2$ is an s-cosingular module for any submodule K of H.
3) Each submodule K of H has a decomposition $K = K_1 \oplus K_2$ such that K_1 is a direct summand of H and K_2 is an s-cosingular module.

Proof. By [10, Theorem 3.1].

Proposition 2.4 Each submodule of a module with $(S_s{}^*)$ exhibits $(S_s{}^*)$.

Proof. By [10, Proposition 3.1].

Proposition 2.5 For any injective module H, $Z_s^*(H) = Soc_s(H)$.

Proof. By [10, Proposition 2.5]

3. Main Theorem and Proof

Proposition 3.1 Suppose that H is an extending module. H exhibits $(S_s{}^*)$ if and only if each submodule of H is a direct sum of an extending module and an s-cosingular module.

Proof. ($\Longrightarrow$) Let H be a module with $(S_s{}^*)$ and $K \leq H$. Then $K = K_1 \oplus K_2$ where $K_1 \leq_\oplus H$ and K_2 is an s-cosingular module by [10, Theorem 3.1]. Thus K_1 is an extending module.

($\Longleftarrow$) Let $K \leq H$. By the assumption, K has the decomposition $K = K_1 \oplus K_2$ such that K_1 is an extending module and K_2 is an s-cosingular module. As H is an extending module, then there is $L \leq_\oplus H$ such that $K_1 \leq_e L$. Therefore, $L \cap K_2 = 0$ and $K = L \oplus K_2$. Now by applying [10, Theorem 3.1], we reach at the conclusion that H exhibits $(S_s{}^*)$.

Recall from [1, Section 16] that a module H is named *K-injective* for any module K, provided each diagram with exact row

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d] & K \\ & H & \end{tikzcd}$$

can be extended commutatively by a homomorphism from K to H. H is named a *quasi-injective module* provided H is H-injective.

Proposition 3.2 Let H be a module such that $H/Z_s{}^*(H)$ is semisimple. Then each module T with $Z_s{}^*(T) = 0$ is an H-injective module.

Proof. Suppose that T is a module with $Z_s{}^*(T) = 0$. Then T is clearly $H/Z_s{}^*(H)$-injective. Let $K \leq H$ and $f: K \longrightarrow T$ be any homomorphism. Say $W = Ker(f)$. Therefore, we have that

$$(Z_s{}^*(H) \cap K)/(Z_s{}^*(H) \cap W) \cong (Z_s{}^*(H) \cap K) + W/W \leq K/W \cong Im(f) \leq T.$$

Thus $Z_s{}^*((Z_s{}^*(H) \cap K)/(Z_s{}^*(H) \cap W)) = 0$. Moreover, $(Z_s{}^*(H) \cap K)/(Z_s{}^*(H) \cap W)$ is s-cosingular due to $Z_s{}^*\big(Z_s{}^*(H)\big) = Z_s{}^*(H)$ and s-cosingular modules being closed under submodules and homomorphic images by Corollary 2.2. Now we readily have that $Z_s{}^*(H) \cap K = Z_s{}^*(H) \cap W$. The map $\alpha: \big(K + Z_s{}^*(H)\big)/Z_s{}^*(H) \longrightarrow T$ defined by $\alpha\big(k + Z_s{}^*(H)\big) = f(k)$ for all $k \in K$ is a homomorphism. By the assumption, α can be extended to a homomorphism $\beta: H/{Z_s}^*(H) \longrightarrow T$. Here we can define $\psi: H \longrightarrow T$ via $\psi = \beta\pi$ where $\pi: H \longrightarrow H/{Z_s}^*(H)$ is the canonical projection. It easily can be checked that ψ extends f. Hence T is an H −injective module.

Corollary 3.3 Let H be a module exhibiting $({S_s}^*)$. Then $H = H_1 \oplus H_2$ where H_2 is an H_1 −injective module.

Proof. As H exhibits $({S_s}^*)$, then by [10, Corollary 3.1], $H/{Z_s}^*(H)$ is semisimple. Therefore, $H = H_1 \oplus H_2$ where H_1 and $H_2/Soc(H_2)$ are semisimple modules and $Soc(H_2) \leq_e H_2$ by [10, Proposition 3.2]. Thus H_2 is an H_1 −injective module due to semisimplicity of H_1.

Proposition 3.4 Let H and T be modules such that ${Z_s}^*(H)$ is projective and T exhibits $({S_s}^*)$. Then H is an essentially T −injective module.

Proof. Suppose that $K \leq T$ and $\alpha: K \longrightarrow H$ is a homomorphism with $Ker(\alpha) \leq_e K$. As T exhibits $({S_s}^*)$, $T = T_1 \oplus T_2$ where $T_1 \leq K$ and $K \cap T_2$ is an *s*-cosingular module. Thus $K = T_1 \oplus (K \cap T_2)$. The map $\beta: T \longrightarrow H$ defined by $\beta(t) = \beta(t_1 + t_2) = \alpha(t_1)$ for all $t = t_1 + t_2 \in T$ such that $t_1 \in T_1$, $t_2 \in T_2$ is a homomorphism. For any $k \in K$, let $k = t_1 + t_2$ where $t_1 \in T_1$, $t_2 \in T_2$. Then $\beta(k) = \alpha(t_1)$. Since $\alpha(K \cap T_2) \leq {Z_s}^*(H)$ and ${Z_s}^*(H)$ is semisimple projective, then

$$\alpha(K \cap T_2) \cong (K \cap T_2)/\big(K \cap T_2 \cap Ker(\alpha)\big) \cong \big((K \cap T_2) + Ker(\alpha)\big)/Ker(\alpha)$$

is a semisimple projective module. Hence $Ker(\alpha)$ is a direct summand of $(K \cap T_2) + Ker(\alpha)$. Moreover, we have that $(K \cap T_2) + Ker(\alpha) = K \cap (T_2 + Ker(\alpha))$, $Ker(\alpha) \leq_e K$ and $Ker(\alpha) \leq K \cap (T_2 + Ker(\alpha)) \leq K$. Thus $Ker(\alpha) \leq_e (K \cap T_2) + Ker(\alpha)$. This follows that $K \cap T_2 \leq Ker(\alpha)$. As $t_2 \in K \cap T_2$, then $\alpha(t_2) = 0$. Hence $\beta(k) = \alpha(k)$ for all $k \in K$, and so α extends to β.

In [12] it is proved that when S is a left GV −ring, then $Z^*(H)$ is a semisimple projective submodule for each left S −module H. By combining this fact with Proposition 3.4, we obtain the following result.

Corollary 3.5 Let S be a left GV −ring and H be an S −module exhibiting $({S_s}^*)$. Then each left S −module is essentially H −injective.

Proposition 3.6 Let $H = H_1 \oplus H_2$ where ${Z_s}^*(H_1)$ is projective and H_2 is an *s*-cosingular module. Then H_2 is an extending module if and only if ${Z_s}^*(H)$ is an extending module.

Proof. (⟹) As $H = H_1 \oplus H_2$, then ${Z_s}^*(H) = {Z_s}^*(H_1) \oplus H_2$. Then by Proposition 3.4, ${Z_s}^*(H_1)$ is essentially H_2 −injective, since H_2 exhibits $({S_s}^*)$ and ${Z_s}^*({Z_s}^*(H_1)) = {Z_s}^*(H_1)$ is projective. Note that H_2 is ${Z_s}^*(H_1)$ −injective and ${Z_s}^*(H_1)$ is extending. Therefore, since H_2 is an extending module, then ${Z_s}^*(H)$ is an extending module by [13, Theorem 8].

(⟸) Since ${Z_s}^*(H)$ is an extending module, then as a direct summand of ${Z_s}^*(H)$, H_2 is an extending module.

Proposition 3.7 Suppose that $H = H_1 \oplus H_2$ such that H_1 is semisimple and H_2 is an extending module. When ${Z_s}^*(H_1)$ is projective and H_2 exhibits $({S_s}^*)$, then H is an extending module.

Proof. As ${Z_s}^*(H_1)$ is projective and H_2 exhibits $({S_s}^*)$, then H_1 is essentially H_2 −injective by Proposition 3.4. Note that H_2 is H_1 −injective. By applying [13, Theorem 8] we conclude that H is an extending module.

Corollary 3.8 Suppose that S is a left GV −ring and H is an S −module. Then $H = H_1 \oplus H_2$ is an extending module for each semisimple module H_1 and each extending module H_2 with $({S_s}^*)$.

Theorem 3.9 Let H be an extending module exhibiting $({S_s}^*)$. When ${Z_s}^*(H)$ is projective, each submodule of H is extending and H is a locally artinian serial module.

Proof. Let $K \leq H$. Then $K = K_1 \oplus K_2$ where K_1 is an extending module and K_2 is an s-cosingular module by Proposition 3.1. As ${Z_s}^*(H)$ is projective, K_2 is projective. Hence by Proposition 3.7, K is an extending module, and thus each submodule of H is an extending module. On the other hand, let G be a finitely generated submodule of H. As G exhibits $({S_s}^*)$, $G/{Z_s}^*(G)$ is semisimple. Then $G/Soc(G)$ is a semisimple module by [14, 8.1.5]. By [6, 5.15], G has descending chain condition on essential submodules. Therefore, G is an artinian module by [6, 18.7], and so H is a locally artinian module.

To show the extending locally noetherian module H being serial, we need to prove that it is a direct sum of uniserial modules. H is a direct sum of uniform submodules K by [6, 8.3]. Now assume that $T \leq K$ is nonzero finitely generated. Let $Rad(T) \leq X \leq T$ such that $Rad(T)$ is a proper submodule of X. Since T exhibits $({S_s}^*)$, there exists a direct summand T' of T such that $T' \leq X$ and X/T' is an s-cosingular module. Here two cases arise. If $T' = 0$, then X is an s-cosingular module. By the assumption, X is projective. If $Rad(T) = 0$, then T is semisimple by [1, 31.2]. If $Rad(T) \neq 0$, since K is a uniform module $Rad(T) = X$, but this is a contradiction. Therefore, $X = T$.

If $T' \neq 0, T' = T = X$ as T is uniform. Then $Rad(T)$ is a maximal submodule of T. In all cases, we reach at the conclusion that $T/Rad(T)$ is a simple module. Thus K is a uniserial module. Hence H is a direct sum of these uniserial submodules K, and so H is a serial module.

Corollary 3.10 Suppose that S is a left GV −ring. When H is an extending S −module exhibiting $({S_s}^*)$, each submodule of H is extending and H is a locally artinian serial module.

Corollary 3.11 Suppose that S is a left extending ring exhibiting $({S_s}^*)$. When ${Z_s}^*({}_S S)$ is projective, each left ideal of S is extending and S is a left artinian serial ring.

When an extending module H with projective ${Z_s}^*(H)$, H may not exhibit $({S_s}^*)$.

Example 3.12 Suppose that V is an infinite dimensional left vector spaces over a field Q. Let $S = End({}_Q V)$. S is a regular left self-injective ring (see [15, Proposition 2.23]). Therefore, S

is a left extending ring and $Z_s{}^*({}_SS) = Soc_s({}_SS) = Soc({}_SS) \cap Rad({}_SS) = Soc({}_SS) \cap 0 = 0$ but ${}_SS$ does not exhibit $(S_s{}^*)$.

Lemma 3.13 The statements listed below are equivalent for a ring S:

1) Each S −module exhibits $(S_s{}^*)$.
2) Each extending S −module exhibits $(S_s{}^*)$.
3) Each quasi-injective S −module exhibits $(S_s{}^*)$.
4) Each injective S −module exhibits $(S_s{}^*)$.
5) Each S −module is a direct sum of an extending module and an s-cosingular module.
6) Each S −module is a direct sum of an injective module and an s-cosingular module.

Proof. The proof of $(1) \Leftrightarrow (6)$ is proved in [10, Theorem 3.3]**.** The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are evident. The equivalence $(2) \Leftrightarrow (5)$ is Proposition 3.1.

$(4) \Rightarrow (6)$ Let H be an injective S-module. By the assumption H is an ss-lifting module according to [10, Corollary 3.2]. Thus each injective module is an ss-lifting module. Hence (6) holds by [10, Theorem 3.3].

A ring S is named *left ss-Harada ring* provided each injective left S-module is an ss-lifting module (see [16]).

Theorem 3.14 The statements listed below are equivalent for a ring S:

1) S is a left ss −Harada ring.
2) S is a left perfect ring with semisimple radical and each extending S −module exhibits $(S_s{}^*)$.
3) S is a left perfect ring with semisimple radical and the injective hull of each semisimple module exhibits $(S_s{}^*)$.
4) S is a left perfect ring with semisimple radical and $E((S/Rad(S))^{(\mathbb{N})})$ exhibits $(S_s{}^*)$.

Proof. $(1) \Rightarrow (2)$ Since S is a left ss-Harada ring, S is a left artinian ring by [16, Lemma 2.3]. Then S is a left perfect ring with semisimple radical by [10, Corollary 3.3]. On the other hand, each injective module is ss-lifting, and hence each injective module exhibits $(S_s{}^*)$. By Lemma 3.13, we conclude that each extending module exhibits $(S_s{}^*)$.

The implications $(2) \Rightarrow (3) \Rightarrow (4)$ are evident.

$(4) \Rightarrow (1)$ Since $E((S/Rad(S))^{(\mathbb{N})})$ is an injective module, $E((S/Rad(S))^{(\mathbb{N})})$ is ss-lifting by [10, Corollary 3.2], and so a lifting module. Therefore S is a left H −ring by [17, Theorem 2.14]. Consequently, S is a left ss-Harada ring by [10, Corollary 3.3].

Recall from [1, Section 23] that a ring S is named *left V-ring* provided each simple S −module is injective. S is a left V-ring if and only if $Rad(H) = 0$ for each S −module H (see [1, 23.1]).

A projective module H together with an epimorphism $h: H \longrightarrow K$ such that $Ker(h) \ll H$ is named *a projective cover* of K (see [1, 19.4]). When each (simple) S −module has a projective cover, S is named *a* (*left semiperfect ring*) *left perfect ring* (see [1, 42.6 and 43.9]).

Theorem 3.15 Consider the statements listed below for a ring S:

1) $S/Z_s{}^*({}_SS)$ is semisimple and each S −module with $(S_s{}^*)$ is injective.
2) S is a left perfect ring and each S −module with $(S_s{}^*)$ is quasi-injective.
3) S is a left perfect ring and each *ss*-lifting S −module is quasi-injective.
4) S is a semisimple ring.

Then $(1) \Leftrightarrow (4) \Longrightarrow (2) \Longrightarrow (3)$. If either each left S −module has semisimple radical or each lifting module is an *ss*- lifting module, then $(3) \Longrightarrow (4)$ holds.

Proof. $(1) \Longrightarrow (4)$ Let H be a simple S −module. Assume that H is a small module. Then H exhibits $(S_s{}^*)$, as it is a simple small module. By the assumption, H is an injective module. But this is a contradiction. Hence H is not small, and so H is an injective module. Therefore, S is a left V −ring. Thus we conclude that $Z_s{}^*({}_SS) = Soc({}_SS) \cap Z^*({}_SS) = Soc({}_SS) \cap S \cap Rad\left(E({}_SS)\right) = 0$ by [1, 23.1]. Then by assumption, S is a semisimple ring.

The implications $(4) \Longrightarrow (1)$ and $(4) \Longrightarrow (2)$ are evident.

$(2) \Longrightarrow (3)$ As *ss*-lifting modules exhibit $(S_s{}^*)$, the result is evident.

$(3) \Longrightarrow (4)$ Note that for an S −module with semisimple radical being an *ss*-lifting module is equivalent to be lifting by [11, Lemma 4]. Thus by the assumption, each lifting S −module is quasi-injective. Therefore, by (3) and [17, Proposition 2.12], S is a semisimple ring.

Proposition 3.16 Let S be a semiperfect ring with $Rad(S) \leq Soc({}_SS)$ such that each S −module with $(S_s{}^*)$ is extending. Assume that H is a semisimple S −module such that T is a projective cover of H. Then T is an extending module.

Proof. Since S is a semiperfect ring, then each simple S −module has a projective cover by [1, 42.6]. Then T is semiperfect by [1, 42.4(4)], as each simple direct summand of H has a projective cover. Thus T is a lifting module by [18, Corollary 4.43]. Since T is a projective S −module and $Rad(S) \leq Soc({}_SS)$, then we conclude that $Rad(T) = Rad(S)T \leq Soc({}_SS)T = Soc(T)$. Therefore, according to [11, Corollary 2], T is an *ss*-lifting module. Hence by the hypothesis, T is an extending module.

Theorem 3.17 S is a left perfect ring with $Rad(S) \leq Soc({}_SS)$ and each S −module with $(S_s{}^*)$ is an extending module if and only if S is a semiperfect ring with $Rad(S) \leq Soc({}_SS)$, $Rad(S^{(\mathbb{N})}) \ll S^{(\mathbb{N})}$ and each S −module with $(S_s{}^*)$ is extending.

Proof. The necessity of the proof is evident. For the sufficiency of the proof, we have that $S/Rad(S)$ is semisimple as S is a semiperfect ring. By the hypothesis, we conclude that $S^{(\mathbb{N})}$ is a projective cover of $(S/Rad(S))^{(\mathbb{N})} \cong S^{(\mathbb{N})}/Rad(S)^{(\mathbb{N})} = S^{(\mathbb{N})}/Rad(S^{(\mathbb{N})})$. Thus by Proposition 3.16, $S^{(\mathbb{N})}$ is an extending module. Hence by [6, 11.13] S is a left perfect ring.

4. Conclusion

In this paper, we present some relations between the extending modules and the modules exhibiting $(S_s{}^*)$ and explore the properties of rings over which each module with $(S_s{}^*)$ is extending and each extending module exhibits $(S_s{}^*)$. We prove that an extending module H exhibits $(S_s{}^*)$ if and only if each submodule of H is a direct sum of an extending module and an *s*-cosingular module. We establish that if H is a module with semisimple $H/Z_s{}^*(H)$, then each module T such that $Z_s{}^*(T) = 0$ is H-injective. Furthermore, we demonstrate that S is a semisimple ring if and only if $S/Z_s{}^*({}_SS)$ is semisimple and each $S-$module with $(S_s{}^*)$ is injective.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

All authors contributed at all stages of the manuscript and reviewed the published version.

Acknowledgements

The authors would like to thank the referees for their careful review of the manuscript and their helpful comments.

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