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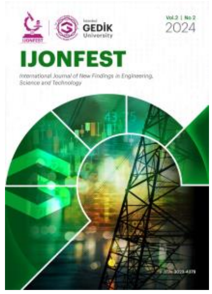

# Investigation of an Exact Solution of a Mixed Boundary Value Problem Using the Residue Method

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**Abstract**

In this study, a finite differences method is proposed for solving a mixed problem, which represents phenomena in hydrodynamics. To evaluate the approximate solution of the problem, its analytical solution is also constructed by the residue method developed by M.L. Rasulov, which is applied to find the solution of the partial differential equations containing time-dependent derivatives at boundary conditions. The formula for expansion of an arbitrary function in a series of residues of the solution of the corresponding spectral problem is used to show that the solution of the mixed boundary-value problem can be represented by the given residue formula. The use of the residual method gives an exact solution for the mixed boundary value problem, represented as a rapidly decreasing series. The derived formula makes it possible to formally establish both the existence and the uniqueness of the solution. Moreover, the derived formula provides a framework for evaluation, allowing a comparative analysis between the exact solution and the numerical approach.

***Keywords:*** Residue method, Residue representation of the solution, Expansion formula, Boundary condition with higher order derivative.

## 1. INTRODUCTION

The Fourier method, also called the method of separation of variables, is one of the basic classical methods for integrating some linear partial differential equations under given boundary and initial conditions. The method is a powerful tool for finding an analytical solution to mixed problems of mathematical physics equations (Fourier, 1822). Other classical methods are the Fourier-Birkhoff method (Birkhoff, 1908a, 1908b), the potential method, the Laplace method, and the residual method, first proposed by Cauchy (1827). The application of the Fourier method to the solution of mixed problems reduces to the problem of the expansion of an arbitrary function from some class of eigenfunctions corresponding to the spectral problem (Fourier, 1822).

For a spectral problem with multiple points with discontinuous coefficients, the concept of regularity was given by M.L. Rasulov; unlike Tamarkin, the formula of multiple expansion was obtained, which is extremely important for the solution of related complex problems for partial differential equations (Rasulov, 1959 and 1963). In fact, he

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has developed the "method of contour integration" and "residual method," which have been applied with success to the solution of problems with both one-dimensional and multidimensional partial differential equations with discontinuous coefficients. Moreover, the residue method can be applied to equations that cannot be separated into their variables and to equations whose operator is not self-adjoint and may even be used in cases where they contain time-dependent derivatives at boundary conditions. It is proven that the formula for the expansion of an arbitrary vector function in the fundamental functions of a one-parameter boundary problem (Rasulov, 1959). The theory of non-self-adjoint operator equations is extensively investigated by M. Keldysh (Keldysh, 1951).

The residual method was applied, respectively, to obtain the exact solution of the boundary value problem with non-classical conditions for one- and two-dimensional heat equations and a one-dimensional linear wave equation in the form of a rapidly decreasing series (Rasulov and Sinsoysal, 2008; Sinsoysal and Rasulov, 2008; Sinsoysal, 2009). Also, it was used the Cauchy problem of a system of ordinary differential equations with constant coefficients to find the exact solution of the constant voltage problem of an RC circuit (Sinsoysal and Rasulov, 2013). Moreover, the solution of the first type of mixed problem for a two-dimensional linear parabolic equation in a bounded cylinder of Euclidean space is found in explicit form by using the residue method (Sinsoysal and Rasulov, 2020).

## 2. THE MAIN PROBLEM

Let $R^2$ be an Euclidean space of points $(x,t)$ and let $Q_T=\{(x,t)\mid 0\le x\le \ell, 0\le t<T\}\subset R^2$. In $Q_T$ we consider the following mixed problem

$$\frac{\partial u(x,t)}{\partial t}=\kappa\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t),\ \ 0\le x\le \ell, t\ge 0, \tag{1}$$

$$u(x,0)=u_0(x), \tag{2}$$

$$A\frac{\partial u(0,t)}{\partial x}+B\frac{\partial u(0,t)}{\partial t}=q(t), \tag{3}$$

$$u(\ell,t)=u_c(t),\ \ t>0 \tag{4}$$

where $\kappa$, $\ell, T, A$ and $B$ are given positive real numbers with physical meaning and, $q(t)$ is a known function for any $t>0$. $t$ and $x$ are time and spatial coordinates, respectively. The problem (1)-(4), which is called the main problem, is often frequently encountered during the theoretical study of many physical processes, both thermo- and hydrodynamic. The theoretical study of many interesting problems of plane-radial filtration theory is also brought to the solution of problems of type (1)-(4).

It is known that the information necessary for the determination of some hydrodynamic indicators of oil fields can be obtained only when the well is suddenly closed and opened (Charny, 1963). In this case, the time derivative includes in the boundary condition, which causes certain difficulties in the application of classical solution methods.

### *2.1 Finite Differences Schemes for the Main Problem*

An algorithm for finding an approximate solution to the dimensionless problem (1)-(4) by the following finite difference method is proposed

$$\frac{\partial u(\xi,\tau)}{\partial\tau}=\kappa\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2}+f(\xi,\tau),\quad 0\le\xi\le 1, \tag{5}$$

$$u(\xi,0)=\ 1, \tag{6}$$

$$\delta_0\frac{\partial u(0,\tau)}{\partial\tau}+\frac{\partial u(0,\tau)}{\partial\xi}=0, \tag{7}$$

$$u(1,\tau)=\frac{u_c(\tau)}{u_0} \tag{8}$$

Here $\kappa$ and $\delta_0$ are given positive parameters having physical sense related of anayzed problem. Now, we will explain the finite difference algorithm to compute a numerical solution to the main problem (5)-(8). For this purpose, a uniform grid covering the region $Q_T^{(\xi,\tau)}$ in which the problem is discretized is created

$$\Omega_{h_\xi,h_\tau}=\{(\xi_j,\tau_n)\mid \xi_j=jh_\xi,\tau_n=nh_\tau,h_\xi>0,h_\tau>0,j=0,1,2,\ldots N,n=0,1,2,\ldots\}$$

and the finite difference approximation of the problem (5)-(8) is constructed at an arbitrary $(\xi_j,\tau_n)$ node point of the grid as

$$\frac{U_{j,n+1}-U_{j,n}}{h_\tau}=\frac{1}{h_\xi^2}\left(U_{j+1,n+1}-2U_{j,n+1}+U_{j-1,n+1}\right), \tag{9}$$

where $j=1,2,\ldots,N-1;\ n=0,1,2,\ldots,$

$$U_{j,0}=1,(j=0,2,\ldots,N), \tag{10}$$

$$\delta_0\frac{U_{0,n+1}-U_{0,n}}{h_\tau}+\frac{U_{1,n+1}-U_{0,n+1}}{h_\xi}=0, \tag{11}$$

$$U_{N,n+1}=\phi(\tau_{n+1}),\ (n=0,2\ \ldots.). \tag{12}$$

Here, $\phi(\tau_n)=\frac{u_{\tau n}}{u_0}$, the grid function $U_{j,n}$ represents approximate values of the function $u(x,t)$ at point $(i,k)$. In order to obtain the solution of the main problem by using algorithm (9)-(12) we must define value of $U_{0,n+1}$. From (11) we have: $U_{0,n+1}=U_{0,n}-\frac{\delta_0 h_\tau}{h_\xi}\left(U_{1,n}-2U_{0,n}+U_{-1,n}\right)$.

It should be note that the order of approximation of schema (9)-(12) has $O(h_\tau+h_\xi)$ of accuracy. The question of increase the order of accuracy of this schema we will consider later. Now the algorithm for obtaining of the unknown $U_{j,n+1}$ is purposed. On the basis of this proposed algorithm, a computer experiment was carried out and the result is shown in Figure 1.

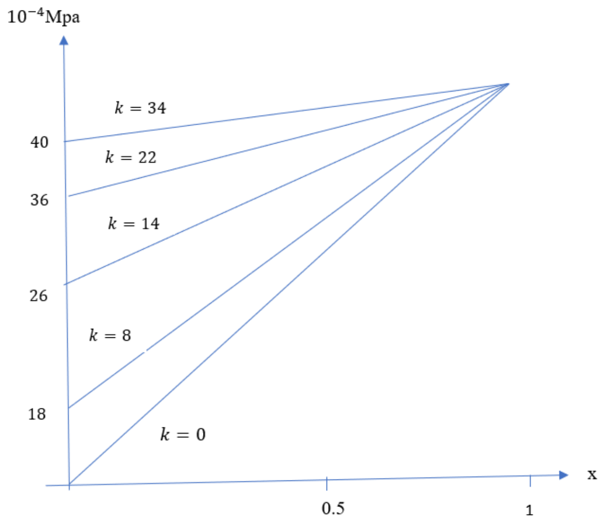


**Figure 1.** Numerical solution

### *2.2 Analytical Solution of the Main Problem*

Now let us construct the analytical solution of the main problem by residue method. The aim here is to compare the analytical solution with the solution given by the numerical algorithm. In order to perform the residue method, by replacing with $v(\xi,\tau) = u(\xi,\tau) - g(\xi,\tau)$, $g(\xi,\tau) = e^{\frac{\tau}{\delta_0}}\xi + 1 - e^{\frac{\tau}{\delta_0}}$, $f(\xi,\tau) = \frac{1}{\delta_0}(1-\xi)e^{\frac{\tau}{\delta_0}}$, $v_0(\xi) = 1 - \xi$ the problem is reduced to a homogeneous boundary condition problem sas follows

$$\frac{\partial v(\xi,\tau)}{\partial \tau} = \frac{\partial^2 v(\xi,\tau)}{\partial \xi^2} + f(\xi,\tau), \tag{13}$$

$$v(\xi,0) = v_0(\xi), \tag{14}$$

$$\delta_0 \frac{\partial v(0,\tau)}{\partial \tau} + \frac{\partial v(0,\tau)}{\partial \xi} = 0, \tag{15}$$

$$v(1,t) = 0. \tag{16}$$

According of the residue method, we associate the following two problems with complex parameter corresponding to problem (13)-(16) as

2.2.1. *Spectral problem*

$$y'' - \lambda^2 y = h(\xi), \tag{17}$$

$$\lambda^2 y(\xi) + \frac{1}{\delta_0} y'(0) = -h(0), \quad y(1,\lambda) = 0, \tag{18}$$

2.2.2. *Chauchy's problem*

$$\frac{dz}{dt} - \lambda^2 z = f, \tag{19}$$

$$z(\xi, 0) = u_0(\xi), \tag{20}$$

where $h(\xi)$ is an arbitrary continuous function having the derivative of first order on [0,1].

It is clearly that $z(\xi,\tau,\lambda) = u_0(\xi)\, e^{\lambda^2\tau} + \int_0^\tau f(\xi,\theta) e^{\lambda^2(\tau-\theta)} d\theta$ is solution of problem (19), (20). According to the general theory of boundary problem of ordinary differential equations the soltion of (17), (18) is constructed as follows

$$y(\xi,\lambda,h(\xi)) = \int_0^1 G(\xi,\eta,\lambda) h(\xi)\, d\eta + \frac{Y(\xi,\lambda,h(\xi))}{\Delta(\lambda)}, \tag{21}$$

where

$$G(\xi,\eta,\lambda) = \frac{\Delta(\xi,\eta,\lambda)}{\Delta(\lambda)} \tag{22}$$

which is called Green's function of problem (17), (18),

$$\Delta(\lambda) = (\delta_0\lambda^2 + \lambda)e^{-\lambda} - (\delta_0\lambda^2 - \lambda)e^{\lambda},$$

$$\Delta(\xi,\eta,\lambda) = \begin{vmatrix} g(\xi,\eta,\lambda) & e^{\lambda\xi} & e^{-\lambda\xi} \\ \delta_0\lambda^2 g(0,\eta,\lambda) + g'(0,\eta,\lambda) & \delta_0\lambda^2 + \lambda & \delta_0\lambda^2 - \lambda \\ g(1,\eta,\lambda) & e^{\lambda} & e^{-\lambda} \end{vmatrix},$$

$$g(\xi,\eta,\lambda) = \begin{cases} \frac{1}{2\lambda}\sinh\lambda(\xi-\eta), & 0 \le \eta \le \xi \le 1, \\ -\frac{1}{2\lambda}\sinh\lambda(\xi-\eta), & 0 \le \xi \le \eta \le 1. \end{cases}$$

Because of (17), (18) is *R* regular in sense (Rasulov, 1963), that the following two statements take place:

**1.** Under fulfillment the condition *R* regularity for the Green's function in domain $R_\delta^*$ takes place the estimate $G(x,\xi,\lambda) = O\left(\frac{1}{\lambda^{n-1}}\right)$,

where $R_\delta^* = R_\delta - \cup_\nu K_\delta(\lambda_\nu)$, $K_\delta(\lambda_\nu)$ is a circle with radius of δ and in center of the roots of the equation $\Delta(\lambda) = 0$.

**2.** For any continuously differentiable on [0,1] function $h(\xi)$ take place the following expansion

$$h(\xi) = -\frac{1}{2\pi\sqrt{-1}}\sum_{\nu}\int_{C_\nu}\lambda\, y(\xi,\lambda,h)\, d\lambda$$

where the sum with respect to $\nu$ extend on all the poles of the integrand function.

Under these assumptions and according to the general theory of residue method (Rasulov, 1963), the analytical solution of problem (1)-(4) is obtained in the form of a rapidly decreasing series

$$u(\xi,\tau) = \frac{-1}{2\pi\sqrt{-1}}\sum_{\nu}\int_{C_\nu}\lambda e^{\lambda^2\tau}\int_0^1 G(\xi,\eta,\lambda)\left[u_0(\xi) + \int_0^\tau e^{\lambda^2(\tau-\theta)}f(\eta,\theta)d\theta\right] d\eta d\lambda +$$

$$\frac{-1}{2\pi\sqrt{-1}}\sum_{\nu}\int_{C_\nu}\lambda_\nu \frac{Y(\xi,\lambda,h)}{\Delta(\lambda)}d\lambda \tag{23}$$

where $C_V$ is a simple closed contour enclosing only one of the poles of $\lambda_\nu$ of integrand function.

As it is seen from (22) computation of this integral be reduced to calculation of the roots of the characteristic determinant $\Delta(\lambda)$. It is clear that the equation (22) is transcendental, therefore necessary to find of this roots numerically, or for roots it possible asymptotic representations. Transforming equation (22) we have $e^{2\lambda} = \frac{\delta_0+\frac{1}{\lambda}}{\delta_0-\frac{1}{\lambda}}$.

It is seen from this the relation of the right side tend to one if $\lambda \to \infty$. Therefore the roots of $\Delta(\lambda) = 0$ by $\lambda \to \infty$ tend to roots of the $\Delta(\lambda) = 1$, that is $\lambda_\nu = 2\pi\nu\sqrt{-1}$, $\nu = 0, \pm1, \pm2, ....$

For these values of $\lambda_\nu$ the function defined by (23) is computed easily. Taking into account the above substitutions for the function $v(\xi,\tau)$ we have the following representation as

$$v(\xi,\tau) = \sum_{\nu=1}^{\infty}\frac{1}{\pi\nu(\delta_0\pi^2\nu^2+1)}\left\{\sin\pi\nu\xi(\cos\pi\nu - 1) - \pi\nu\cos\pi\nu\xi\left[e^{-\pi^2\nu^2\tau} + \frac{e^{\frac{\tau}{\delta_0}} - e^{-\pi^2\nu^2\tau}}{1+\delta_0\pi^2\nu^2}\right]\right\} +$$

$$+1 - e^{\frac{\tau}{\delta_0}}\xi - e^{\frac{\tau}{\delta_0}}.$$

and its graph is shown in Figure 2.

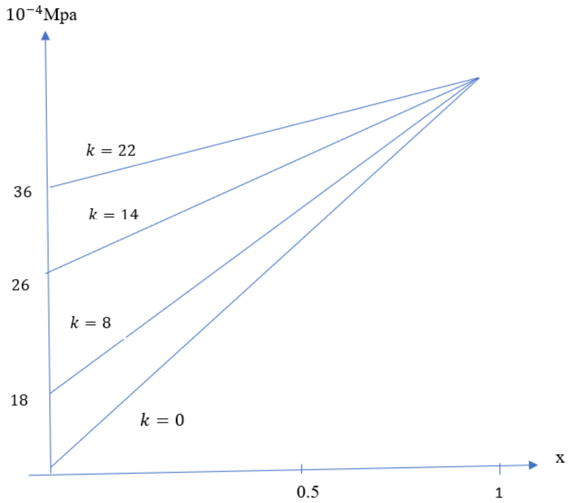


**Figure 2.** Result obtained by analytical solution

## 3. CONCLUSION

By finite differences method the numerical solution of the main problem is obtained with order $O(h_\tau + h_\xi)$. It is also possible to write higher order difference schema of the finite difference system with respect to τ. The residual method is used to find the exact solution of the mixed boundary value problem in the form of a rapidly decreasing series. Through the derived formula, the existence and uniqueness of the solution can be proved. Furthermore, this formula allows us to compare the exact solution with the approximate solution calculated by using numerical methods.


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### Declaration of Competing Interest

There is no conflict of interest in this study.

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