http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 09.02.2015 *Year*: **2015**, *Number*: **2** , *Pages*: **43-54**
Accepted: 21.02.2015 *Original Article*[**]

RECENT RESULTS ON THE DOMAIN OF THE SOME LIMITATION METHODS IN THE SEQUENCE SPACES f_0 AND f

Serkan Demiriz[1,*] <serkandemiriz@gmail.com>

[1] *Department of Mathematics, University of Gaziosmanpaşa, 60240 Tokat, Turkey*

Abstract − In the present paper, we summarize the literature on the sequence spaces almost $A-$null and almost $A-$convergent derived by using the domain of the certain $A-$ limitation matrix. Moreover, we introduce the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ and examine some properties of this spaces.

Keywords − *Almost convergence, Matrix domain, Generalized means, Matrix transformations.*

1 Introduction

By a *sequence space*, we mean any vector subspace of ω, the space of all real or complex valued sequences $x=(x_k)$. The well-known sequence spaces that we shall use throughout this paper are as following:

ℓ_∞: the space of all bounded sequences,
c: the space of all convergent sequences,
c_0: the space of all null sequences,
bs: the space of all sequences which forms bounded series,
cs: the space of all sequences which forms convergent series,
ℓ_1: the space of all sequences which forms absolutely convergent series,
ℓ_p: the space of all sequences which forms p-absolutely convergent series,
where $1<p<\infty$.

Let λ,μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k\in\mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A:\lambda\to\mu$, if for every sequence $x=(x_k)\in\lambda$ the sequence $Ax=\{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n=\sum_k a_{nk}x_k, \quad (n\in\mathbb{N}). \tag{1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda:\mu)$, we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda:\mu)$ if and only if the series on the right side of (1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have

[**] *Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief)*.
[*] *Corresponding Author.*

$Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n)\| = 0,$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

The $\beta-$dual of a subset X of ω is defined by

$$X^\beta = \big\{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\big\}.$$

The shift operator P is defined on ω by $(Px)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non-negative linear functional, such that $L(Px) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be *almost convergent* to the generalized limit α if all Banach limits of x are α [1], and denoted by $f - \lim x_k = \alpha$. Let P^j be the composition of P with itself j times and define $t_{mn}(x)$ for a sequence $x = (x_k)$ by

$$t_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} (P^j x)_n \text{ for all } m, n \in \mathbb{N}.$$

Lorentz [1] proved that $f - \lim x_k = \alpha$ if and only if $\lim_{m\to\infty} t_{mn}(x) = \alpha$, uniformly in n. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. By f and f_0, we denote the space of all almost convergent sequences and almost convergent to zero sequences, respectively, i.e.,

$$f = \left\{x = (x_k) \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \alpha \text{ uniformly in } n\right\}$$

and

$$f_0 = \left\{x = (x_k) \in \omega : \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n\right\}.$$

A matrix $A = (a_{nk})$ is called a *triangle* if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for triangle matrices A, B and a sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ that is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$. We write by $\mathcal{U}$ and $\mathcal{U}_0$ for the sets of all sequences with non-zero terms and non-zero first terms, respectively. For $u \in \mathcal{U}$, let $1/u = (1/u_n)$.

Let us give the definition of some triangle limitation matrices which are needed in the text. Let $q = (q_k)$ be a sequence of positive reals and write

$$Q_n = \sum_{k=0}^{n} q_k, \quad (n \in \mathbb{N}).$$

Then the Cesàro mean of order one, Riesz mean with respect to the sequence $q = (q_k)$ and A^r- mean with $0 < r < 1$ are respectively defined by the matrices $C_1 = (c_{nk})$, $R^q = (r^q_{nk})$ and $A^r = (a^r_{nk})$; where

$$c_{nk} = \begin{cases} \dfrac{1}{n+1}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \qquad r^q_{nk} = \begin{cases} \dfrac{q_k}{Q_n}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

and

$$a^r_{nk} = \begin{cases} \dfrac{1+r^k}{1+n}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$. Additionally, the Euler mean of order r and the weighted mean matrix and the double band matrix are respectively defined by the matrices $E^r = (e^r_{nk})$, $G(u,v) = (g_{nk})$ and $B(r,s) = \{b_{nk}(r,s)\}$; where

$$e^r_{nk} = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k, & (0 \leq k \leq n) \\ 0, & (k > n) \end{cases} \quad \text{and} \quad g_{nk} = \begin{cases} u_n v_k, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases}$$

and

$$b_{nk}(r,s) = \begin{cases} r, & (k = n), \\ s, & (k = n-1), \\ 0, & \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$ and $u, v \in \mathcal{U}$ and $r, s \in \mathbb{R}\backslash\{0\}$.

For a sequence space λ, the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{2}$$

which is a sequence space. Although in the most cases the new sequence space λ_A generated in the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ, it may be observed in some cases that those spaces overlap. Indeed, one can deduce that the inclusions $\lambda_S \subset \lambda$ strictly holds for $\lambda \in \{\ell_\infty, c, c_0\}$. As this, one can deduce that the inclusions $\ell_p \subset bv_p$ and $\lambda \subset \lambda_{\Delta^1}$ also strictly hold for $\lambda \in \{c, c_0\}$, where $1 \leq p \leq \infty$ and the space $(\ell_p)_{\Delta^{(1)}} = bv_p$ has been studied by Başar and Altay [2], (see also Çolak and Et and Malkowsky [3]). However, if we define $\lambda = c_0 \oplus z$ with $z = ((-1)^k)$, that is, $x \in \lambda$ if and only if $x = s + \alpha z$ for some $s \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n = (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, we have $Ae = z \in \lambda$ but $Az = e \notin \lambda$ which lead us to the consequences that $z \in \lambda \setminus \lambda_A$ and $e \in \lambda_A \setminus \lambda$, where $e^{(n)}$ denotes the sequence whose only non-zero term is a 1 in n^{th} place for each $n \in \mathbb{N}$ and $e = (1, 1, 1, ...)$. That is to say that the sequence spaces λ_A and λ overlap but neither contains the other. The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [4], Ng and Lee [5], Aydın and Başar [6], Altay and Başar [7], and Altay et all. [8]. They introduced the sequence spaces $(\ell_\infty)_{N_q}$ and c_{N_q} in [4], $(\ell_p)_{C_1} = X_p$ in [5], $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ in [6], $(c_0)_{E^r} = e^r_0$ and $c_{E^r} = e^r_c$ in [7], $(\ell_p)_{E^r} = e^r_p$ and $(\ell_\infty)_{E^r} = e^r_\infty$ in [8]; where $1 \leq p < \infty$.

In this study, we summarize some knowledge in the existing literature on the almost $A-$null and almost $A-$convergent sequence spaces derived by using the domain $A-$limitation matrix. Additionally, we introduce the new sequence spaces $\bar{f}_0(r,s,t)$ and $\bar{f}(r,s,t)$ and examine some properties of these sequence spaces.

2 Domain of the $A-$limitation matrix in the sequence spaces f_0 and f

In this section, we shortly give the knowledge on the sequence spaces derived by the $A-$limitation matrix from well-known almost convergent and almost null sequence spaces. For the concerning literature about the domain μ_A of an infinite limitation matrix A in a sequence space μ, Table 1 may be useful.

μ	A	μ_A	refer to
f_0, f	$B(r,s)$	$\widehat{f}, \widehat{f_0}$	[9]
f_0, f	C_1	$\widetilde{f}, \widetilde{f_0}$	[14]
f_0, f	R^q	$f_{R^q}, \{f_0\}_{R^q}$	[15]
f_0, f	A^r	$a^r_f, a^r_{f_0}$	[16]
f_0, f	$G(u,v)$	$f_0(G), f(G)$	[17]
f_0, f	E^r	$f(E), f_0(E)$	[18]
f_0, f	$B(r,s,t)$	$f(B), f_0(B)$	[19]
f_0, f	A_λ	$A_\lambda(f_0), A_\lambda(f)$	[20]

Table 1: The domains of the certain $A-$limitation matrix in the sequence spaces f_0 and f

The matrix domain of a certain limitation method on the sequence spaces f_0 and f firstly were studied by Başar and Kirişçi [9].

Başar and Kirişçi introduced the sequence spaces $\widehat{f_0}$ and $\widehat{f}$ in [9] as follows:

$$\widehat{f_0} : = \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} \sum_{j=0}^{m} \frac{sx_{k-1+j} + rx_{k+j}}{m+1} = 0 \quad \text{uniformly in} \quad k \right\},$$

$$\widehat{f} : = \left\{ x = (x_k) \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{j=0}^{m} \frac{sx_{k-1+j} + rx_{k+j}}{m+1} = \alpha \quad \text{uniformly in} \quad k \right\}.$$

It is trivial that the sequence spaces $\widehat{f_0}$ and $\widehat{f}$ are the domain of the matrix $B(r,s)$ in the spaces f_0 and f, respectively. Thus, with the notation of (2) we can redefine the spaces $\widehat{f_0}$ and $\widehat{f}$ by

$$\widehat{f_0} := \{f_0\}_{B(r,s)} \qquad \text{and} \qquad \widehat{f} := \{f\}_{B(r,s)}.$$

Define the sequence $y = (y_k)$ by the $B(r,s)-$transform of a sequence $x = (x_k)$, i.e.,

$$y_k := sx_{k-1} + rx_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{3}$$

Since the matrix $B(r,s)$ is triangle, one can easily observe that $x = (x_k) \in \widehat{X}$ if and only if $y = (y_k) \in X$, where the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (3), and X denotes any of the sequence spaces f_0 and f. Therefore, one can easily see that the linear operator $T : \widehat{X} \to X$, $Tx = y = B(r,s)x$ which maps every sequence x in $\widehat{X}$ to the associated sequence y in X, is bijective and norm preserving, where $\|x\|_{\widehat{X}} = \|B(r,s)x\|_X$. This gives the fact that $\widehat{X}$ and X are norm isomorphic.

Başar and Kirişçi [9] proved that the sequence space f is a $BK-$space with the norm $\|.\|_\infty$ and non-separable closed subspace of ℓ_∞. So, the sequence space f has no Schauder basis. Jarrah and Malkowsky [12] showed that the matrix domain λ_A of a normed sequence space λ has a basis whenever $A = (a_{nk})$ is triangle. Then; our corollary concerning the space $\widehat{f_0}$ and $\widehat{f}$ is about their Schauder basis:

Corollary 2.1. [9, Corollary 4.2] The space $\widehat{f}$ has no Schauder basis.

The gamma- and beta-duals of the spaces $\widehat{f_0}$ and $\widehat{f}$ are determined. Also, some matrix transformations on these sequence spaces are characterized.

Quite recently, E. E. Kara and K. Elmaağaç [21] introduced the sequence space $\hat{c}^u$ as follows:

$$\hat{c}^u = \left\{ x = (x_k) \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{j=0}^{m} \frac{u_{k+j}x_{k+j} + u_{k-1+j}x_{k-1+j}}{m+1} = \alpha \quad \text{uniformly in} \quad k \right\}.$$

It is trivial that the sequence space $\hat{c}^u$ is the domain of the matrix $A^u = (a^u_{nk})$ in the space f, where the matrix $A^u = (a^u_{nk})$ is defined by

$$a^u_{nk} = \begin{cases} (-1)^{n-k}u_k, & n-1 \le k \le n, \\ 0, & 0 \le k < n-1 \text{ or } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Also, they show that $\hat{c}^u$ is linearly isomorphic to the space $\hat{c}$. Further, they compute the $\beta-$dual of the space $\hat{c}^u$ and characterize the classes of infinite matrices related to sequence space $\hat{c}^u$.

3 Spaces of $\bar{A}(r,s,t)-$almost null and $\bar{A}(r,s,t)-$almost convergent sequences

In this section, we study some properties of the spaces of the $\bar{A}(r,s,t)-$almost null and $\bar{A}(r,s,t)-$almost convergent sequences.

For any sequences $s, t \in \omega$, the convolution $s * t$ is a sequence defined by

$$(s * t)_n = \sum_{k=0}^{n} s_{n-k} t_k; \quad (n \in \mathbb{N}).$$

Throughout this section, let $r, t \in \mathcal{U}$ and $s \in \mathcal{U}_0$. For any sequence $x = (x_n) \in \omega$, we define the sequence $\bar{x} = (\bar{x}_n)$ of generalized means of x by

$$\overline{x}_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k; \quad (n \in \mathbb{N}), \tag{4}$$

that is $\overline{x}_n = (s * tx)_n / r_n$ for all $n \in \mathbb{N}$. Further, we define the infinite matrix $\bar{A}(r, s, t)$ of generalized means by

$$\{\bar{A}(r, s, t)\}_{nk} = \begin{cases} \dfrac{s_{n-k} t_k}{r_n}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases} \tag{5}$$

for all $n, k \in \mathbb{N}$. Then, it follows by (4) that $\bar{x}$ is the $\bar{A}(r, s, t)-$transform of x, that is $\bar{x} = (\bar{A}(r, s, t)x)$ for all $x \in \omega$.

It is obvious by (5) that $\bar{A}(r, s, t)$ is a triangle. Moreover, it can easily be seen that $\bar{A}(r, s, t)$ is regular if and only if $s_{n-i} = o(r_n)$ for each $i \in \mathbb{N}$, $\sum_{k=0}^{n} |s_{n-k} t_k| = O(|r_n|)$ and $(s*t)_n / r_n \to 1$ $(n \to \infty)$.

The above definition of the matrix $\bar{A}(r, s, t)$ of generalized means given by (5) includes the following special cases:

(1) If $r_n = (s*t)_n \neq 0$ for all n, then $\bar{A}(r, s, t)$ reduces to the matrix (N, s, t) of generalized Nörlund means [22, 23]. In particular, if $t = e$ then $\bar{A}(r, s, t)$ reduces to the familiar matrix of Nörlund means [30, 4].

(2) If $\alpha > 0$, $r_k = \frac{\Gamma(\alpha+k+1)}{k!\Gamma(\alpha+1)}$, $s_k = \frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)}$ and $t_k = 1$ for all k, then $\bar{A}(r, s, t)$ reduces to the matrix (C, α) of Cesàro means of order α [24, 25]. In particular, if $\alpha = 1$ then $\bar{A}(r, s, t)$ reduces to the famous matrix $(C, 1)$ of arithmetic means [5, 26].

(3) If $0 < \alpha < 1$, $r_k = \frac{1}{k!}$, $s_k = \frac{(1-\alpha)^k}{k!}$ and $t_k = \frac{\alpha^k}{k!}$ for all k, then $\bar{A}(r, s, t)$ reduces to the matrix (E, α) of Euler means of order α [7, 10, 8].

(4) If $t_n > 0$ and $r_n = \sum_{k=0}^{n} t_k$ for all n, then $\bar{A}(r, s, t)$ reduces to the matrix $(\overline{N}, t)$ of weighted means [12, 27].

(5) If $0 < \alpha < 1$, $r_k = k + 1$, $s_k = 1$ and $t_k = 1 + \alpha^k$ for all k, then $\bar{A}(r, s, t)$ reduces to the matrix A^α studied by Aydın and Başar [6, 28].

(6) If $s = e^{(0)}$ and $t = e$, then $\bar{A}(r, s, t)$ reduces to the diagonal matrix $D_{1/r}$ studied by de Malafosse [29].

Now, since $\bar{A}(r, s, t)$ is a triangle, it has a unique inverse which is also a triangle. More precisely, by making a slight generalization of a work done in [30], we put $D_0^{(s)} = 1/s_0$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & s_0 & 0 & \cdots & 0 \\ s_3 & s_2 & s_1 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{vmatrix}; \qquad (n = 1, 2, ...).$$

Then the inverse of $\bar{A}(r, s, t)$ is the triangle $\bar{B} = (\bar{b}_{nk})_{n,k=0}^{\infty}$ defined by

$$\bar{b}_{nk} = \begin{cases} (-1)^{n-k} D_{n-k}^{(s)} r_k \dfrac{1}{t_n}, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases}$$

for all $n, k \in \mathbb{N}$. For an arbitrary subset X of ω, the set $X(r, s, t)$ has recently been introduced in [31] as the matrix domain of the triangle $\bar{A}(r, s, t)$ in X.

We introduce the sequence spaces $\bar{f}(r, s, t)$ and $\bar{f}_0(r, s, t)$ as the sets of all sequences whose $\bar{A}(r, s, t)-$transforms are in the spaces f_0 and f, that is

$$\bar{f}_0(r,s,t) = \left\{x=(x_k)\in\omega : \lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n}=0 \text{ uniformly in } n\right\},$$

$$\bar{f}(r,s,t) = \left\{x=(x_k)\in\omega : \exists l\in\mathbb{C}\ni\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n}=l \text{ uniformly in } n\right\}.$$

With the notation of (2), we can redefine the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ as follows:

$$\bar{f}(r,s,t)=\{f\}_{\bar{A}(r,s,t)} \quad\text{and}\quad \bar{f}_0(r,s,t)=\{f_0\}_{\bar{A}(r,s,t)}.$$

It is worth mentioning that the general forms of the well-known matrices of Nörlund, Cesàro, Euler and weighted means can be obtained as special cases of the matrix $\bar{A}(r,s,t)$ of generalized means. Therefore, all of the sequence spaces in Tablo 1 can be obtained by special choice from the sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ which are defined by using matrix domain of the matrix $\bar{A}(r,s,t)$.

Theorem 3.1. The sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are $BK-$spaces with the same norm given by

$$\|x\|_{\bar{f}(r,s,t)}=\|\bar{A}(r,s,t)x\|_f=\sup_{m,n\in\mathbb{N}}|t_{mn}(\bar{A}(r,s,t)x)|, \tag{6}$$

where

$$\begin{aligned} t_{mn}(\bar{A}(r,s,t)x) &= \frac{1}{m+1}\sum_{j=0}^{n}(\bar{A}(r,s,t)x)_{n+j} \\ &= \frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n} \end{aligned}$$

for all $m,n\in\mathbb{N}$.

Proof. f_0 and f endowed with the norm $\|.\|_\infty$ are $BK-$spaces [24, Example 7.3.2 (b)] and $\bar{A}(r,s,t)$ is a triangle matrix, Theorem 4.3.2 of Wilansky [32, p.61] gives the fact that $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are $BK-$spaces with the norm $\|.\|_{\bar{f}(r,s,t)}$. □

Remark 3.2. It can easily be seen that the absolute property does not hold on the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$, that is $\|x\|_{\bar{f}(r,s,t)}\neq\||x|\|_{\bar{f}(r,s,t)}$ for at least one sequence x in each of these spaces, where $|x|=(|x_k|)$. Thus, the spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are $BK-$spaces of non-absolute type.

Theorem 3.3. The sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ are norm isomorphic to the spaces f and f_0, respectively.

Proof. Since the fact $\bar{f}_0(r,s,t)\cong f_0$ can be similarly proved, we consider only the case $\bar{f}(r,s,t)\cong f$. To prove this, we should show the existence of a linear bijection between the spaces $\bar{f}(r,s,t)$ and f which preserves the norm. Consider the transformation T defined, with the notation of (4), from $\bar{f}(r,s,t)$ to f by $x\mapsto\bar{x}=Tx=\bar{A}(r,s,t)x$. The linearity of T is clear. Further, it is trivial that $x=\theta$ whenever $Tx=\theta$ and hence T is injective.

Let us take any $\bar{x}=(\bar{x}_k)\in f$ and define the sequence $x=(x_n)$ by

$$x_n=\frac{1}{t_n}\sum_{k=0}^{n}(-1)^{n-k}D^{(s)}_{n-k}r_k\bar{x}_k;\quad \text{for all } n\in\mathbb{N}. \tag{7}$$

Then, it is immediate that

$$\begin{aligned} \sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n} &= \sum_{k=0}^{n+j}\frac{s_{n+j-k}t_k}{r_n}\frac{1}{t_k}\sum_{i=0}^{k}(-1)^{k-i}D^{(s)}_{k-i}r_i\bar{x}_i \\ &= \bar{x}_{n+j} \end{aligned}$$

which gives by a short calculation that

$$\frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j}\frac{s_{n+j-k}t_kx_k}{r_n}=\frac{1}{m+1}\sum_{j=0}^{m}\bar{x}_{n+j}.$$

Therefore, we have

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\{\bar{A}(r,s,t)x\}_{n+j}=\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\bar{x}_{n+j}=l \quad \text{uniformly in } n.$$

This means that $x\in\bar{f}(r,s,t)$ and hence T is surjective. Thus, one can easily see from (6) that T is a norm preserving transformation. This completes the proof. □

Remark 3.4. It is known from Corollary of Başar and Kirişçi [9] that the Banach space f has no Schauder basis. It is also known from Theorem 2.3 of Jarrah and Malkowsky [12] that the domain λ_A of a matrix A in a normed sequence space λ has a basis if and only if λ has a basis whenever $A=(a_{nk})$ is a triangle. Combining these two facts one can immediately conclude that both the space $\bar{f}(r,s,t)$ and the space $\bar{f}_0(r,s,t)$ have no Schauder basis.

Now, we give the beta- and gamma-duals of the sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$. For this, we need the following lemmas:

Lemma 3.5. [11] $A=(a_{nk})\in(f:\ell_\infty)$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|<\infty. \tag{8}$$

Lemma 3.6. [11] $A=(a_{nk})\in(f:c)$ if and only if (8) holds, and there are $\alpha_k,\alpha\in\mathbb{C}$ such that

$$\lim_{n\to\infty}a_{nk}=\alpha_k \text{ for each } k\in\mathbb{N}, \tag{9}$$

$$\lim_{n\to\infty}\sum_k a_{nk}=\alpha, \tag{10}$$

$$\lim_{n\to\infty}\sum_k\left|\Delta(a_{nk}-\alpha_k)\right|=0. \tag{11}$$

Theorem 3.7. Define the sets $F_1(r,s,t),F_2(r,s,t),F_3(r,s,t),F_4(r,s,t),F_5(r,s,t)$ as follows:

$$\begin{aligned}
F_1(r,s,t) &= \left\{a=(a_k)\in\omega:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_ka_j\right|<\infty\right\},\\
F_2(r,s,t) &= \left\{a=(a_k)\in\omega:\lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_ka_j \quad \text{exists}\right\},\\
F_3(r,s,t) &= \left\{a=(a_k)\in\omega:\lim_{n\to\infty}\sum_{k=0}^{n}\left[\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_ka_j\right]-\text{exists}\right\},\\
F_4(r,s,t) &= \left\{a=(a_k)\in\omega:\lim_{n\to\infty}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\frac{1}{t_j}(-1)^{j-k}D^{(s)}_{j-k}r_ka_j\right|=0\right\},\\
F_5(r,s,t) &= \left\{a=(a_k)\in\omega:\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\left|\sum_{j=n+1}^{\infty}(\Delta\bar{a}_{jk}-\alpha_k)\right|=0\right\}.
\end{aligned}$$

Then, the β−dual of the sequence space $\bar{f}(r,s,t)$ is

$$\bigcap_{i=1}^{5}F_i(r,s,t).$$

Proof. Let $a = (a_k) \in \omega$ and consider the equality

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\frac{1}{t_k} \sum_{j=0}^{k} (-1)^{k-j} D_{k-j}^{(s)} r_j \bar{x}_j \right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{1}{t_j} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j \right] \bar{x}_k = \{\bar{F}(r,s,t)\bar{x}\}_n, \end{aligned} \tag{12}$$

where $\bar{F}(r,s,t) = \{\bar{f}_{nk}(r,s,t)\}$ is defined by

$$\bar{f}_{nk}(r,s,t) = \begin{cases} \sum_{j=k}^{n} \frac{1}{t_j} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j & (0 \le k \le n), \\ 0 & (k > n) \end{cases} \tag{13}$$

for all $n, k \in \mathbb{N}$. Thus, we deduce from Lemma 3.6 with (12) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \bar{f}(r,s,t)$ if and only if $\bar{F}(r,s,t)\bar{x} \in c$ whenever $\bar{x} = (\bar{x}_k) \in f$, where $\bar{F}(r,s,t) = \{\bar{f}_{nk}(r,s,t)\}$ is defined by (13). Therefore, we derive from (8), (9), (10) and (11) that

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{n} \frac{1}{t_j} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j \right| < \infty,$$

$$\lim_{n \to \infty} \sum_{j=k}^{n} \frac{1}{t_j} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j = \alpha_k \quad \text{for each fixed } k \in \mathbb{N},$$

$$\lim_{n \to \infty} \sum_k \sum_{j=k}^{n} \frac{1}{t_j} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j = \alpha,$$

$$\lim_{n \to \infty} \sum_k \left| \Delta \left[\sum_{j=k}^{n} \frac{1}{t_j} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j \right] \right| = 0$$

which shows that

$$\{\bar{f}(r,s,t)\}^{\beta} = \bigcap_{i=1}^{5} F_i(r,s,t).$$

□

Theorem 3.8. The $\gamma-$dual of the sequence spaces $\bar{f}(r,s,t)$ and $\bar{f}_0(r,s,t)$ is the set $F_1(r,s,t)$.

Proof. This is similar to the proof of Theorem 3.7 with Lemma 3.5 instead of Lemma 3.6. So, we omit the detail. □

4 Matrix Transformations Related to The Sequence Space $\bar{f}(r,s,t)$

In the present section, we characterize the matrix transformations from $\bar{f}(r,s,t)$ into any given sequence space μ.

Since $\bar{f}(r,s,t) \cong f$, it is trivial that the equivalence "$x \in \bar{f}(r,s,t)$ if and only if $\bar{x} \in f$" holds.

Theorem 4.1. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation

$$e_{nk} = \sum_{j=k}^{\infty} \frac{1}{t_j} (-1)^{j-k} D_{j-k}^{(s)} r_k a_{nj} \tag{14}$$

for all $n, k \in \mathbb{N}$ and μ is any given sequence space. Then $A \in (\bar{f}(r,s,t) : \mu)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\bar{f}(r,s,t)\}^{\beta}$ for all $n \in \mathbb{N}$ and $E \in (f : \mu)$.

Proof. Let μ be any given sequence space. Suppose that (14) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $\bar{f}(r,s,t)$ and f are linearly isomorphic.

Let $A \in (\bar{f}(r,s,t) : \mu)$ and take any $\bar{x} = (\bar{x}_k) \in f$. Then $E\bar{A}(r,s,t)$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in \cap_{i=1}^{5} F_i(r,s,t)$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, $E\bar{x}$ exists and thus

$$\sum_k e_{nk}\bar{x}_k = \sum_k a_{nk}x_k$$

for all $n \in \mathbb{N}$. We have that $E\bar{x} = Ax$ which leads us to the consequence $E \in (f : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^{\beta}$ for each $n \in \mathbb{N}$ and $E \in (f : \mu)$ hold, and take any $x = (x_k) \in \bar{f}(r,s,t)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m}\left[\sum_{j=k}^{m}\frac{1}{t_j}(-1)^{j-k}D_{j-k}^{(s)}r_k a_{nj}\right]\bar{x}_k$$

for all $n \in \mathbb{N}$, as $m \to \infty$ that $E\bar{x} = Ax$ and this shows that $A \in (\bar{f}(r,s,t) : \mu)$. This step completes the proof. □

By changing the roles of the spaces $\bar{f}(r,s,t)$ and μ in Theorem (4.1), we have:

Theorem 4.2. Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $C = (c_{nk})$ are connected with the relation

$$c_{nk} = \frac{1}{r_n}\sum_{j=0}^{n} s_{n-j}t_j a_{jk} \quad \text{for all } n,k \in \mathbb{N}.$$

Let μ be any given sequence space. Then, $A = (a_{nk}) \in (\mu : \bar{f}(r,s,t))$ if and only if $C \in (\mu : f)$.

Proof. Let $z = (z_k) \in \mu$ and consider the following equality

$$\sum_{k=0}^{m} c_{nk}z_k = \frac{1}{r_n}\sum_{j=0}^{n} s_{n-j}t_j\left(\sum_{k=0}^{m} a_{jk}z_k\right) \quad \text{for all } m,n \in \mathbb{N},$$

which yields as $m \to \infty$ that $(Cz)_n = \{\bar{A}(r,s,t)(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in \bar{f}(r,s,t)$ whenever $z \in \mu$ if and only if $Cz \in f$ whenever $z \in \mu$. This completes the proof. □

Of course, Theorems 4.1 and 4.2 have several consequences depending on the choice of the sequence space μ. Define $a(n,k)$, $a(n,k,m)$ and Δa_{nk} for all $k,m,n \in \mathbb{N}$ as follows;

$$a(n,k) = \sum_{j=0}^{n} a_{jk}, \quad a(n,k,m) = \frac{1}{m+1}\sum_{j=0}^{m} a_{n+j,k} \text{ and } \Delta a_{nk} = a_{nk} - a_{n,k+1}.$$

Prior to giving some results as an application of this idea, we give the following basic lemma, which is the collection of the characterizations of matrix transformations related to almost convergence:

Lemma 4.3. Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:

(i) [33, J. P. Duran]$A = (a_{nk}) \in (\ell_\infty : f)$ if and only if (8) holds and

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \quad \text{for all } k \in \mathbb{N}, \tag{15}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{m\to\infty}\sum_k |a(n,k,m) - \alpha_k| = 0 \quad \text{uniformly in } n \tag{16}$$

also hold .

(ii) [33, J. P. Duran]$A = (a_{nk}) \in (f : f)$ if and only if (8) and (15) hold, and

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \alpha, \tag{17}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{m\to\infty} \sum_k \left| \Delta\big[a(n,k,m) - \alpha_k\big] \right| = 0 \quad \text{uniformly in } n \tag{18}$$

also hold .

(iii) [34, J. P. King]$A = (a_{nk}) \in (f : f)$ if and only if (8), (15) and (17) hold .

(iv) [35, Başar and Çolak]$A = (a_{nk}) \in (cs : f)$ if and only if (15) holds, and

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a_{nk}| < \infty \tag{19}$$

also holds .

(v) [36, Başar and Solak]$A = (a_{nk}) \in (bs : f)$ if and only if (15) and (19) hold, and

$$\lim_{k\to\infty} a_{nk} = 0 \quad \text{for all } n \in \mathbb{N}, \tag{20}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{q\to\infty} \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \Delta\big[a(n+i,k) - \alpha_k\big] \right| = 0 \quad \text{uniformly in } n \tag{21}$$

also hold .

(vi) [37, Başar]$A = (a_{nk}) \in (f : cs)$ if and only if the following conditions hold:

$$\sup_{n\in\mathbb{N}} \sum_k |a(n,k)| < \infty, \tag{22}$$

$$\exists \alpha_k \in \mathbb{C} \ni \sum_n a_{nk} = \alpha_k \quad \text{for all } k \in \mathbb{N}, \tag{23}$$

$$\exists \alpha \in \mathbb{C} \ni \sum_n \sum_k a_{nk} = \alpha, \tag{24}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} \sum_k \left| \Delta\big[a(n,k) - \alpha_k\big] \right| = 0. \tag{25}$$

Now, we can give the following two corollaries as a direct consequence of Theorems 4.1 and 4.2 and Lemma 4.3:

Corollary 4.4. The following statements hold:

(i) $A = (a_{nk}) \in (\bar{f}(r,s,t) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (8) holds with e_{nk} instead of a_{nk}.

(ii) $A = (a_{nk}) \in (\bar{f}(r,s,t) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (8), (9), (10) and (11) hold with e_{nk} instead of a_{nk}.

(iii) $A = (a_{nk}) \in (\bar{f}(r,s,t) : bs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (22) holds with e_{nk} instead of a_{nk}.

(iv) $A = (a_{nk}) \in (\bar{f}(r,s,t) : cs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (22), (23), (24) and (25) hold with e_{nk} instead of a_{nk}.

(v) $A = (a_{nk}) \in (\bar{f}(r,s,t) : f)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\bar{f}(r,s,t)\}^\beta$ and (8), (15), (17) and (18) hold with e_{nk} instead of a_{nk}.

Corollary 4.5. The following statements hold:

(i) $A = (a_{nk}) \in (\ell_\infty : \bar{f}(r,s,t))$ if and only if (8), (15) and (17) hold with e_{nk} instead of a_{nk}.

(ii) $A = (a_{nk}) \in (f : \bar{f}(r,s,t))$ if and only if (8), (15), (17) and (18) hold with e_{nk} instead of a_{nk}.

(iii) $A = (a_{nk}) \in (c : \bar{f}(r,s,t))$ if and only if (8), (15) and (17) hold with e_{nk} instead of a_{nk}.

(iv) $A = (a_{nk}) \in (bs : \bar{f}(r,s,t))$ if and only if (15), (19), (20) and (21) hold with e_{nk} instead of a_{nk}.

(v) $A = (a_{nk}) \in (cs : \bar{f}(r,s,t))$ if and only if (15) and (19) hold with e_{nk} instead of a_{nk}.

Acknowledgement

We wish to thank the referees for their valuable suggestions and comments which improved the paper considerably.

References

[1] G.G. Lorentz, A contribution to the theory of divergent sequences, Acta Math. **80**(1948), 167-190.

[2] F. Başar, B. Altay, On the space of sequences of $p-$ bounded variation and related matrix mappings, Ukrainian Math. J. **55**(1)(2003), 136-147.

[3] R. Çolak, M. Et, E. Malkowsky, Some topics of sequence spaces, in: Lecture Notes in Mathematics, Fırat Univ. Press, Turkey, 2004, pp.1-63. Fırat Univ. Elazığ, 2004, ISBN:975-394-038-6.

[4] C. S. Wang, On Nörlund sequence spaces, Tamkang J. Math. **9**(1978), 269-274.

[5] P. N. Ng, P.Y. Lee, Cesàro sequence spaces of non-absolute type, Comment. Math. Prace Mat. **20**(2)(1978), 429-433.

[6] C. Aydın, F. Başar, On the new sequence spaces which include the spaces c_0 and c, Hokkaido Math. J. **33**(2) (2004), 383-398.

[7] B. Altay, F. Başar, Some Euler sequence spaces of non-absolute type, Ukrainian Math. J. **57**(1)(2005), 1-17.

[8] B. Altay, F. Başar, M. Mursaleen, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I, Inform. Sci. **176**(10)(2006), 1450-1462.

[9] F. Başar, M. Kirişci, Almost convergence and generalized difference matrix, Comput. Math. Appl. **61**(2011), 602-611.

[10] M. Mursaleen, F. Başar, B. Altay, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II, Nonlinear Anal. TMA, **65**(3)(2006), 707-717.

[11] J. A. Sıddıqi, Infinite matrices summing every almost periodic sequences, Pasific. J. Math. **33**(1)(1971), 235-251.

[12] A. M. Jarrah, E. Malkowsky, BK-spaces, bases and linear operators, Ren. Circ. Mat. Palermo II **52**(1990), 177-191.

[13] F. Başar, Summability theory and its applications, Bentham Science Publishers, Istanbul, 2012, e-books, Monographs.

[14] K. Kayaduman and M. Şengönül, The spaces of Cesàro almost convergent sequences and core theorems, Acta Math. Sci. **32B**(6) (2012), 2265-2278.

[15] M. Şengönül and K. Kayaduman, On the Riesz almost convergent sequences space, Abstr. Appl. Anal. (2012), Article ID 691694, 18pp.

[16] A. Karaisa and Ümit Karabıyık, Almost sequence spaces by the domain of the matrix A^r, Abstr. Appl. Anal., (2013), Article ID 783731, 8pp.

[17] M. Kirişçi, Almost convergence and generalized weighted mean II, Journal of Inequalities and Applications (2014), 2014:93.

[18] M. Kirişçi, The spaces of Euler almost null and Euler almost convergent sequences, Commun. Fac. Sci. Univ. Ank. Series A1, **62**(2) (2013), 85-100.

[19] A. Sönmez, Almost convergence and triple band matrix, Mathematical and Computer Modelling, **57**(2013), 2393-2402.

[20] M. Yeşilkayagil, F. Başar, Spaces of $A_\lambda-$almost null and $A_\lambda-$almost convergence sequences, Journal of Egyptian Mathematical Society (2014), http://dx.doi.org/10.1016/j.joems.2014.01.013.

[21] E. E. Kara, K. Elmaağaç, On the $u-$difference almost sequence space and matrix transformations, Int. J. Mod. Math. Sci., **8**(1) (2013), 57-76.

[22] Y. Okuyama, On absolute generalized Nörlund summability of orthogonal series, Tamkang J. Math., **33**(2) (2002), 161-165.

[23] U. Stadtmüller, A. Tali, A note on families of generalized Nörlund matrices as bounded operators on ℓ_p, Proc. Estonian Acad. Sci. **58**(3) (2009), 137-145.

[24] J. Boos, Classical and Modern Methods in Summability, Oxford University Press, New York, 2000.

[25] H. Bor, On a summability factor theorem, Commun. Math. Anal., **1**(1) (2006), 46-51.

[26] M. Şengönül, F. Başar, Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c, Soochow J. Math., **31**(1) (2005), 107-119.

[27] E. Malkowsky, V. Rakočević, An introduction into the theory of sequence spaces and measures of noncompactness, Zb. Rad. (Beogr.), **9**(17) (2000), 143-234.

[28] C. Aydın, F. Başar, Some new sequence spaces which iclude the spaces ℓ_p and ℓ_∞, Demonstratio Math., **38**(3) (2005), 641-656.

[29] B. de Malafosse, The Banach algebra $\mathfrak{B}(X)$, where X is a $BK-$space and applications, Mat. Vesnik, **57** (2005), 41-60.

[30] F. M. Mears, The inverse Nörlund mean, Ann. Math., **44**(3) (1943), 401-410.

[31] M. Mursaleen, A. K. Noman, On generalized means and some related sequence spaces, Comput. Math. Appl. **61**(4) (2011), 988-999.

[32] A. Wilansky, Summability Through Functional Analysis, Mathematical Studies, North Holland, Amsterdam, 1984.

[33] J. P. Duran, Infinite matrices and almost convergence, Math. Z. **128** (1972), 75-83.

[34] J. P. King, Almost summable sequences, Proc. Am. Math. Soc., **17** (1966), 1219-1225.

[35] F. Başar, R. Çolak, Almost conservative matrix transformations, Turk. J. Math. **13**(3) (1989), 91-100.

[36] F. Başar, I. Solak, Almost-coercive matrix transformations, Rend. Math. Appl. **11**(2) (1991), 249-256.

[37] F. Başar, Strongly-conservative sequence-to-series matrix transformations, Erc. Uni. Fen. Bil. Derg. **5**(12) (1989), 888-893.