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SOME PERTURBED TRAPEZOID INEQUALITIES FOR m- AND (α, m)-CONVEX FUNCTIONS AND APPLICATIONS

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Abstract – In this paper, the Authors establish some new inequalities related to perturbed trapezoid inequality for the classes of functions whose second derivatives of absolute values are m and (α, m)-convex. After, applications to special means have also been presented.

Keywords – *Hermite-Hadamard inequalities, m- and (α, m)-convex functions, perturbed trapezoid inequality, means.*

1 Introduction

Definition 1.1. [11] A function $f : I \to \mathbb{R}$ is said to be convex on I if inequality

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) \tag{1}$$

holds for all $u, v \in I$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below the chord PR.

In [14], G. Toader defined m-convexity: another intermediate between the usual convexity and starshaped convexity.

Definition 1.2. [14] A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \tag{2}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$ .We say that f is m-concave if $-f$ is m-convex. Denote by $K_m(b)$ the class of all m-convex functions on $[0, b]$ for which $f(0) \leq 0$.

Remark 1.3. For $m = 1$ in (2), we recapture the concept of convex functions defined on $[0, b]$ and, for $m = 0$, the concept of star-shaped functions defined on $[0, b]$ is obtained.

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Definition 1.4. [1] The function $f:[0,b]\to\mathbb{R}$ is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$; if for every $u,v\in[0,b]$ and $t\in[0,1]$, we have

$$f\left(tu+(1-t)v\right)\leq t^{\alpha}f\left(u\right)+m\left(1-t^{\alpha}\right)f\left(v\right). \tag{3}$$

Remark 1.5. Note that for $(\alpha,m)\in\{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex.

Theorem 1.6. (The Hermite-Hadamard inequality) Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and $u,v\in I$ with $u<v$. The following double inequality:

$$f\left(\frac{u+v}{2}\right)\leq\frac{1}{v-u}\int_u^v f\left(x\right)dx\leq\frac{f\left(u\right)+f\left(v\right)}{2} \tag{4}$$

is known in the literature as Hadamard's inequality (or Hermite-Hadamard inequality) for convex functions. If f is a positive concave function, then the inequality is reversed.

In the literature [2]-[7] on numerical integration, the following estimation is well known as the trapezoid inequality:

$$\left|\int_u^v f\left(x\right)dx-\frac{1}{2}\left(v-u\right)\left(f\left(u\right)+f\left(v\right)\right)\right|\leq\frac{1}{12}M_2\left(v-u\right)^3, \tag{5}$$

where $f:[u,v]\to\mathbb{R}$ is supposed to be twice differentiable on the interval (u,v), with the second derivative bounded on (u,v) by $M_2=\sup_{x\in(u,v)}\left|f''\left(x\right)\right|<+\infty$.

For the perturbed trapezoid inequality, Dragomir et al. [4] obtained the following inequality by an application of the Grüss inequality:

$$\begin{aligned}&\left|\int_u^v f\left(x\right)dx-\frac{1}{2}\left(v-u\right)\left(f\left(u\right)+f\left(v\right)\right)+\frac{1}{12}\left(v-u\right)^2\left(f'\left(v\right)-f'\left(u\right)\right)\right|\\ \leq\ &\frac{1}{32}\left(\Gamma_2-\gamma_2\right)\left(v-u\right)^3,\end{aligned} \tag{6}$$

where f is supposed to be twice differentiable on the interval (u,v), with the second derivative bounded on (u,v) by $\Gamma_2=\sup_{x\in(u,v)}f''\left(x\right)<+\infty$ and $\gamma_2=\inf_{x\in(u,v)}f''\left(x\right)>-\infty$.

For recent results and generalizations concerning Hadamard's inequality, concepts of convexity, m-, (α,m)-convexity and trapezoid inequality see [1]-[19] and the references therein.

Throughout this paper we will use the following notations and conventions. Let $J=[0,\infty)\subset\mathbb{R}=(-\infty,+\infty)$, and $u,v\in J$ with $0<u<v$ and $f'\in L[u,v]$ and

$$A\left(u,v\right)=\frac{u+v}{2},\ G\left(u,v\right)=\sqrt{uv},\ I\left(u,v\right)=\frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}}\ \text{(for } u\neq v\text{)},$$

be the arithmetic mean, geometric mean, identric mean, for $u,v>0$ respectively.

The aim of this paper is to establish some results connected with the perturbed trapezoid inequality for m and (α,m)-convex functions as well as to apply them for some elementary inequalities for real numbers and in numerical integration.

2 The New Results for m- and (α,m)-convex Functions

To prove perturbed trapezoid inequalities for m-convex and (α,m)-convex functions, we use following Lemma which was used by Tunç et al. (see [16])

Lemma 2.1. [16] Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f'' \in L[a, b]$, then the following equality holds:

$$\begin{aligned} & \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a)) \\ = & \frac{(b-a)^3}{4}\int_0^1 (t+1)^2\left[f''(ta+(1-t)b) + f''(tb+(1-t)a)\right]dt \end{aligned} \tag{7}$$

Theorem 2.2. [16] Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f''|$ is convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} & \left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \le & \frac{7}{12}(b-a)^3\left(|f''(a)| + |f''(b)|\right). \end{aligned} \tag{8}$$

Theorem 2.3. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and $m \in [0, 1]$. If $|f''|$ is m-convex on I, then the following inequality holds:

$$\begin{aligned} & \left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ & \frac{(b-a)^3}{4}\left\{\frac{17\left[|f''(a)| + |f''(b)|\right]}{12} + m\frac{11\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]}{12}\right\}. \end{aligned} \tag{9}$$

Proof. Using Lemma 2.1 and Definition 1.2, it follows that

$$\begin{aligned} & \left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \le & \frac{(b-a)^3}{4}\int_0^1 (t+1)^2\left(|f''(ta+(1-t)b)| + |f''(tb+(1-t)a)|\right)dt \\ \le & \frac{(b-a)^3}{4}\int_0^1 (t+1)^2\left\{t|f''(a)| + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right. \\ & \left. +t|f''(b)| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|dt\right\} \\ \le & \frac{(b-a)^3}{4}\left\{\left(\left[|f''(a)| + |f''(b)|\right]\int_0^1 t(t+1)^2\,dt\right)\right. \\ & \left. +m\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]\int_0^1 (t+1)^2(1-t)\,dt\right\} \\ \le & \frac{(b-a)^3}{4}\left\{\frac{17\left[|f''(a)| + |f''(b)|\right]}{12} + m\frac{11\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]}{12}\right\}. \end{aligned}$$

□

Theorem 2.4. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and $(\alpha, m) \in [0, 1]^2$. If $|f''|$ is (α, m)-convex on I, then the following inequality holds:

$$\begin{aligned} & \left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \le & \frac{(b-a)^3}{4}\left\{\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}\left[|f''(a)| + |f''(b)|\right]\right. \\ & \left. +\left(\frac{7}{3} - \frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}\right)m\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]\right\}. \end{aligned} \tag{10}$$

Proof. Using Lemma 2.1 and Definition 1.4, it follows that

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{(b-a)^3}{4}\int_0^1 (t+1)^2 \left(\left|f''(ta+(1-t)b)\right| + \left|f''(tb+(1-t)a)\right|\right)dt \\
\leq\ & \frac{(b-a)^3}{4}\int_0^1 (t+1)^2 \left\{t^\alpha \left|f''(a)\right| + m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|\right. \\
& \left. + t^\alpha \left|f''(b)\right| + m(1-t^\alpha)\left|f''\left(\frac{a}{m}\right)\right| dt\right\} \\
\leq\ & \frac{(b-a)^3}{4}\left\{\left(\left[\left|f''(a)\right| + \left|f''(b)\right|\right]\int_0^1 t^\alpha (t+1)^2\,dt\right)\right. \\
& \left. + m\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]\int_0^1 (t+1)^2(1-t^\alpha)\,dt\right\} \\
\leq\ & \frac{(b-a)^3}{4}\left\{\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}\left[\left|f''(a)\right| + \left|f''(b)\right|\right]\right. \\
& \left. + m\left(\frac{7}{3} - \frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}\right)\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]\right\}.
\end{aligned}$$

□

Remark 2.5. i) In inequality (10), if we choose $\alpha = 1$, inequality (10) reduces to inequality (9).
ii) In inequality (10), if we take $\alpha = 1$, $m = 1$, inequality (10) reduces to inequality (8).

Theorem 2.6. [16] Let $f : I \subseteq R \to R$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and let $p > 1$ with $1/p + 1/q = 1$. If the mapping $|f''|^q$ is convex on $[a, b]$ then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \qquad (11)\\
\leq\ & \frac{(b-a)^3}{2}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.7. Let $f : I \subseteq R \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $m \in [0, 1]$, and let $p > 1$ with $1/p + 1/q = 1$. If the mapping $|f''|^q$ is m-convex on I, then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \qquad (12)\\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}} \\
& \times\left\{\left[\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}} + \left[\frac{\left|f''(b)\right|^q + m\left|f''\left(\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Using Lemma 2.1, Definition 1.2 and Hölder's integral inequality, we get

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{(b-a)^3}{4}\left[\int_0^1 |t+1|^2 |f''(ta+(1-t)b)|\,dt \right.\\
&\left. + \int_0^1 |t+1|^2 |f''(tb+(1-t)a)|\,dt\right] \\
\leq\ & \frac{(b-a)^3}{4}\left[\left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left. + \left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\int_0^1 \left(t|f''(a)|^q + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left. + \left(\int_0^1 \left(t|f''(b)|^q + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\frac{|f''(a)|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}} + \left[\frac{|f''(b)|^q + m\left|f''\left(\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

□

Theorem 2.8. Let $f : I \subseteq R \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$, $(\alpha, m) \in [0,1]^2$, and let $p > 1$ with $1/p + 1/q = 1$. If the mapping $|f''|^q$ is (α, m)-convex on I, then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \qquad (13)\\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\frac{|f''(a)|^q}{\alpha+1} + \frac{m\alpha\left|f''\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}} + \left[\frac{|f''(b)|^q}{\alpha+1} + \frac{m\alpha\left|f''\left(\frac{a}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Using Lemma 2.1, Definition 1.4 and Hölder's integral inequality, we get

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{(b-a)^3}{4}\left[\int_0^1 |t+1|^2 |f''(ta+(1-t)b)|\,dt\right. \\
&\left. + \int_0^1 |t+1|^2 |f''(tb+(1-t)a)|\,dt\right] \\
\leq\ & \frac{(b-a)^3}{4}\left[\left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 |t+1|^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\int_0^1 \left(t^\alpha |f''(a)|^q + m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 \left(t^\alpha |f''(b)|^q + m(1-t^\alpha)\left|f''\left(\frac{a}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^3}{4}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\frac{|f''(a)|^q}{\alpha+1} + \frac{m\alpha\left|f''\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}} + \left[\frac{|f''(b)|^q}{\alpha+1} + \frac{m\alpha\left|f''\left(\frac{a}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

□

Remark 2.9. i) In (13), if we choose $\alpha = 1$, we have the inequality in (12).

ii) In Theorem 2.8, if we choose $\alpha = m = 1$, we obtain the inequality in (11).

Corollary 2.10. i) Under the assumptions of Theorem 2.7, if we choose $p = m = 1$, we obtain the inequality;

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{7(b-a)^3}{6}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

ii) Under the assumptions of Theorem 2.8, if we choose $p = m = 1$, we obtain the inequality;

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\leq\ & \frac{7(b-a)^3}{6}\left\{\left[\frac{|f''(a)|^q}{\alpha+1} + \frac{\alpha|f''(b)|^q}{\alpha+1}\right]^{\frac{1}{q}} + \left[\frac{|f''(b)|^q}{\alpha+1} + \frac{\alpha|f''(a)|^q}{\alpha+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 2.11. [16] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and let $p > 1$with $1/p + 1/q = 1$. If the mapping $|f''|^p$ convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned}&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}}\left\{\left(\frac{17|f''(a)|^p + 11|f''(b)|^p}{12}\right)^{\frac{1}{p}} \right. \\ &\left. + \left(\frac{17|f''(b)|^p + 11|f''(a)|^p}{12}\right)^{\frac{1}{p}}\right\}.\end{aligned} \tag{14}$$

Theorem 2.12. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $m \in [0, 1]$, and let $p > 1$ with $1/p+1/q = 1$. If the mapping $|f''|^q$ is m-convex on I, then the following inequality holds:

$$\begin{aligned}&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}}\left\{\left(\frac{17|f''(a)|^p + m11\left|f''\left(\frac{b}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}} \right. \\ &\left. + \left(\frac{17|f''(b)|^p + m11\left|f''\left(\frac{a}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}}\right\}.\end{aligned} \tag{15}$$

Proof. Using Lemma 2.1, Definition 1.2 and power mean integral inequality, we establish

$$\begin{aligned}&\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\ \leq\ & \frac{(b-a)^3}{4}\int_0^1 |t+1|^2\,|f''(ta+(1-t)b) + f''(tb+(1-t)a)|\,dt \\ \leq\ & \frac{(b-a)^3}{4}\left(\int_0^1 |t+1|^2\,dt\right)^{1-\frac{1}{p}} \\ & \left\{\left(\int_0^1 (t+1)^2\left(t|f''(a)|^p + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}} \right. \\ & \left. + \left(\int_0^1 (t+1)^2\left(t|f''(b)|^p + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}}\right\} \\ \leq\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}} \\ & \times\left\{\left(\frac{17|f''(a)|^p + m11\left|f''\left(\frac{b}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}} + \left(\frac{17|f''(b)|^p + m11\left|f''\left(\frac{a}{m}\right)\right|^p}{12}\right)^{\frac{1}{p}}\right\}.\end{aligned}$$

□

Theorem 2.13. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $(\alpha, m) \in [0, 1]^2$, and let $p > 1$ with $1/p + 1/q = 1$. If the mapping $|f''|^q$ is (α, m)-convex on I then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\le\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}} \\
&\times\left\{\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(a)|^p+\frac{m\alpha\left(7\alpha^2+30\alpha+29\right)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{b}{m}\right)\right|^p\right]^{\frac{1}{p}}\right. \\
&\left.+\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(b)|^p+\frac{m\alpha\left(7\alpha^2+30\alpha+29\right)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{a}{m}\right)\right|^p\right]^{\frac{1}{p}}\right\}.
\end{aligned} \tag{16}$$

Proof. Using Lemma 2.1, Definition 1.4 and power mean integral inequality, we obtain

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\le\ & \frac{(b-a)^3}{4}\int_0^1 |t+1|^2\,|f''(ta+(1-t)b)+f''(tb+(1-t)a)|\,dt \\
\le\ & \frac{(b-a)^3}{4}\left(\int_0^1 |t+1|^2\,dt\right)^{1-\frac{1}{p}} \\
&\left\{\left(\int_0^1 (t+1)^2\left(t^\alpha|f''(a)|^p+m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}}\right. \\
&\left.+\left(\int_0^1 (t+1)^2\left(t^\alpha|f''(b)|^p+m(1-t^\alpha)\left|f''\left(\frac{a}{m}\right)\right|^p\right)dt\right)^{\frac{1}{p}}\right\} \\
\le\ & \frac{(b-a)^3}{4}\left(\frac{7}{3}\right)^{1-\frac{1}{p}} \\
&\times\left\{\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(a)|^p+\frac{m\alpha\left(7\alpha^2+30\alpha+29\right)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{b}{m}\right)\right|^p\right]^{\frac{1}{p}}\right. \\
&\left.+\left[\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}|f''(b)|^p+\frac{m\alpha\left(7\alpha^2+30\alpha+29\right)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|f''\left(\frac{a}{m}\right)\right|^p\right]^{\frac{1}{p}}\right\}.
\end{aligned}$$

□

Remark 2.14. i) In (16), if we choose $\alpha=1$, we have the inequality in (15).
ii) In (16), if we choose $\alpha=m=1$, we obtain the inequality in (14).

Corollary 2.15. i) Under the assumptions of Theorem 2.12, if we choose $p=m=1$, we obtain the inequality;

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{5}{4}(b-a)^2(f'(b)-f'(a))\right| \\
\le\ & \frac{(b-a)^3}{2}\left(\frac{17|f''(a)|+11|f''(b)|}{12}\right).
\end{aligned}$$

ii) Under the assumptions of Theorem 2.13, if we choose $p = m = 1$, we obtain the inequality;

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{5}{4}(b-a)^2(f'(b)-f'(a))\right|$$
$$\leq \frac{(b-a)^3}{4}\frac{7\alpha^3+34\alpha^2+45\alpha+14}{3(\alpha^3+6\alpha^2+11\alpha+6)}\left[|f''(a)|+|f''(b)|\right].$$

3 Applications to Special Means

Now we shall use the results of Section 2 to prove the following new inequalities connecting the above means for arbitrary real numbers.

Proposition 3.1. Let $a, b \in (0, x)$ and $x > 0$, $m \in [0, 1]$ with $a < b$. Then, the following inequality holds:

$$\left|-\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|$$
$$\leq \frac{(b-a)^2}{2}\frac{\left(17+11m^3\right)}{12}\frac{A\left(a^2,b^2\right)}{G^4(a,b)}.$$

Proof. The proof is immediate from Theorem 2.3 applied for $f(x) = -\ln x$, $x \in R$. □

Proposition 3.2. Let $(0, x)$, $a, b \in (0, x)$ and $x > 0$, $(\alpha, m) \in [0, 1]^2$ with $a < b$. Then, the following inequality holds:

$$\left|-\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|$$
$$\leq \frac{(b-a)^2}{2}\left(\frac{4\alpha^2+16\alpha+14}{\alpha^3+6\alpha^2+11\alpha+6}\left(1-m^3\right)+\frac{7m^3}{3}\right)\frac{A\left(a^2,b^2\right)}{G^4(a,b)}.$$

Proof. The proof is immediate from Theorem 2.4 applied for $f(x) = -\ln x$, $x \in R$. □

Proposition 3.3. Let $(0, x)$, $a, b \in (0, x)$ and $x > 0$, $m \in [0, 1]$, $p > 1$ with $a < b$. Then, the following inequality holds:

$$\left|-\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|$$
$$\leq \frac{(b-a)^2}{2^{2+\frac{1}{q}}G^4(a,b)}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{1/p}\left\{\left[b^{2q}+a^{2q}m^{1+q}\right]^{\frac{1}{q}}+\left[a^{2q}+b^{2q}m^{1+q}\right]^{\frac{1}{q}}\right\}.$$

Proof. The proof is immediate from Theorem 2.7 applied for $f(x) = -\ln x$, $x \in R$. □

Proposition 3.4. Let $(0, x)$, $a, b \in (0, x)$ and $x > 0$, $(\alpha, m) \in [0, 1]^2$, $p > 1$ with $a < b$. Then, the following inequality holds:

$$\left|-\ln I(a,b) + A(\ln a, \ln b) + \frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|$$
$$\leq \frac{(b-a)^2}{4G^4(a,b)}\left(\frac{2^{2p+1}-1}{2p+1}\right)^{1/p}\frac{1}{(\alpha+1)^{\frac{1}{q}}}\left\{\left[b^{2q}+a^{2q}m^{1+q}\alpha\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[a^{2q}+b^{2q}m^{1+q}\alpha\right]^{\frac{1}{q}}\right\}.$$

Proof. The proof is immediate from Theorem 2.8 applied for $f(x) = -\ln x$, $x \in R$.. □

Proposition 3.5. Let $(0,x)$, $a,b\in(0,x)$ and $x>0$, $m\in[0,1]$, $p>1$ with $a<b$. Then, the following inequality holds:

$$\begin{aligned}&\left|-\ln I(a,b)+A(\ln a,\ln b)+\frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|\\ \leq\ &\frac{7(b-a)^2}{12G^4(a,b)}\left(\frac{3}{84}\right)^{\frac{1}{p}}\left\{\left(17b^{2p}+11m^{p+1}a^{2p}\right)^{1/p}\right.\\ &\left.+\left(17a^{2p}+11m^{p+1}b^{2p}\right)^{1/p}\right\}\end{aligned}$$

Proof. The proof is immediate from Theorem 2.12 applied for $f(x)=-\ln x$, $x\in\mathbb{R}$. □

Proposition 3.6. Let $(0,x)$, $a,b\in(0,x)$ and $x>0$, $(\alpha,m)\in[0,1]^2$, $p>1$ with $a<b$. Then, the following inequality holds:

$$\begin{aligned}&\left|-\ln I(a,b)+A(\ln a,\ln b)+\frac{5}{4}\frac{(b-a)^2}{G^2(a,b)}\right|\\ \leq\ &\frac{(b-a)^2}{4G^4(a,b)}\left(\frac{7}{3}\right)^{1-\frac{1}{p}}\\ &\times\left\{\left(\frac{(4\alpha^2+16\alpha+14)\,b^{2p}}{(\alpha^3+6\alpha^2+11\alpha+6)}+\frac{m^{p+1}a^{2p}\left(7\alpha^3+30\alpha^2+29\alpha\right)}{3(\alpha^3+6\alpha^2+11\alpha+6)}\right)^{\frac{1}{p}}\right.\\ &\left.+\left(\frac{(4\alpha^2+16\alpha+14)\,a^{2p}}{(\alpha^3+6\alpha^2+11\alpha+6)}+\frac{m^{p+1}b^{2p}\left(7\alpha^3+30\alpha^2+29\alpha\right)}{3(\alpha^3+6\alpha^2+11\alpha+6)}\right)^{\frac{1}{p}}\right\}.\end{aligned}$$

Proof. The proof is immediate from Theorem 2.13 applied for $f(x)=-\ln x$, $x\in\mathbb{R}$. □

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