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r-τ_{12}-θ-GENERALIZED FUZZY CLOSED SETS IN SMOOTH BITOPOLOGICAL SPACES

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Abstract – In [34] we introduced the notion of r-(τ_i, τ_j)-θ-generalized fuzzy closed sets in smooth bitopological spaces by using $(\tau_i, \tau_j)\theta$-fuzzy closure $T^{\tau_i}_{\tau_j}$ defined in [19]. Recently, [33] we defined a new θ-fuzzy closure, denoted C^{θ}_{12} on smooth bitopological spaces by using smooth supra topological space (X, τ_{12}) which is generated from smooth bitopological space (X, τ_1, τ_2) [1], such that $C^{\theta}_{12} \leq T^{\tau_i}_{\tau_j}$. In this paper, we introduce a new class of r-θ-generalized fuzzy closed sets, namely, r-τ_{12}-θ-gfc in smooth bitopological spaces via C^{θ}_{12}-fuzzy closure operator. The basic properties of these sets are studied. Furthermore, the relationship with other notions of r-generalized fuzzy closed sets in [31, 32, 33, 34] are investigated and we give many examples for reverse. In addition, by using r-τ_{12}-θ-gfc sets, we define a new fuzzy closure operator which generates a new smooth topology. Finally, generalized fuzzy θ-continuous (resp. irresolute) and fuzzy strongly θ-continuous mappings are introduced and some of their properties are studied.

Keywords – *Smooth topology, θ-generalized fuzzy closed, generalized fuzzy closure operator, generalized fuzzy θ-continuous mapping, generalized fuzzy θ-irresolute mapping, fuzzy strongly θ- continuous mapping.*

1 Introduction

Kubiak [20] and Šostak [29] independently in (1985) introduced the fundamental concept of a fuzzy topology as an extension of both crisp topology and Chang's fuzzy topology [5]. Šostak presented some rules and showed how such an extension can be realized. Subsequently, Badard [3], introduced the concept of 'smooth topological space'. Chattopadhyay et al. [6] and Chattopadhyay and Samanta [7] have re-introduced the same concept, calling it 'gradation of openess'. Ramadan [26] and his colleagues have introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [6, 7, 11, 22, 28, 30]). Lee et al. [21] introduced the concept of smooth bitopological space as a generalization of smooth topological space and Kandil's defined fuzzy bitopological space [14].

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The so-called supra topology was established, by Mashhour et al. [24] (recall that a supra topology on a set X is a collection of subsets of X, which is closed under arbitrary unions). Abd El-Monsef and Ramadan [2] introduced the concept supra fuzzy topology, followed by Ghanim et al. [13] who introduced the supra fuzzy topology in Šostak sense. Abbas [1] generated the supra fuzzy topology (X, τ_{12}) from fuzzy bitopological space (X, τ_1, τ_2) in Šostak sense as an extension of supra fuzzy topology due to Kandil et al. [15].

The first attempt of generalizing closed sets was done by Levine [23]. Subsequently, Fukutake [12], generalized this concept in bitopological space. Balasubramanian and Sundaram [4], introduced the concept of generalized fuzzy closed sets within Chang's fuzzy topology. Kim and Ko [18] defined r-generalized fuzzy closed sets in smooth topological spaces. Recently, in [31], we introduced the concept of generalized fuzzy closed sets in smooth bitopological spaces. Noiri [25] and Dontchev and Maki [8] introduced another new generalization of Livine generalized closed set by utilizing the θ-closure operator. The concept of θ-generalized closed sets was applied to the digital line [9]. Khedr and Al-Saadi [16] generalized the notion of θ-generalized sets to bitopological space. El-Shafei and Zakari [10] introduced the concept of θ-generalized fuzzy closed sets in Chang's fuzzy topology. Recently, in [34], we introduced the notion of θ-generalized fuzzy closed sets in smooth bitopological spaces by utilizing the $(\tau_i, \tau_j)\theta$-fuzzy closure $T^{\tau_i}_{\tau_j}$ defined in [19]. In this paper we define another type of r-θ-generalized fuzzy closed sets in smooth bitopological spaces via C^{θ}_{12}-fuzzy closure which was established by us [33], and study its relationship with other types of r-generalized fuzzy closed sets which introduced in ([31, 32, 33, 34]). By using this new class of generalized fuzzy closed sets we define a new fuzzy closure operator which generates a new smooth topology. Finally, we define and study generalized fuzzy θ-continuous (resp. irresolute) and fuzzy strongly θ-continuous mappings.

2 Preliminary

Throughout this paper, let X be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$. A fuzzy set μ of X is a mapping $\mu : X \longrightarrow I$, and I^X be the family of all fuzzy sets on X. For any $\mu_1, \mu_2 \in I^X$, then $(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x) : x \in X\}$, $(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x) : x \in X\}$. The complement of a fuzzy set λ is denoted by $\bar{1} - \lambda$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha \ \forall x \in X$. By $\bar{0}$ and $\bar{1}$, we denote constant maps on X with value 0 and 1, respectively. For $x \in X$ and $t \in I_0$, the fuzzy set x_t of X whose value t at x and 0 otherwise is called the fuzzy point in X. Let $Pt(X)$ be a family of all fuzzy points in X. For $\lambda \in I^X$, $x_t \in \lambda$ if and only if $\lambda(x) \geq t$ and x_t is said to be quasi-coincident (q-coincident, for short) with λ, denoted by $x_t \ q \ \lambda$ if and only if $1 - \lambda(x) < t$. For $\mu, \lambda \in I^X$, μ is called q-coincident with λ, denoted by $\mu \ q \ \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu \ \bar{q} \ \lambda$. Also, for two fuzzy sets λ_1 and $\lambda_2 \in I^X$, $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1 \ \bar{q} \ \bar{1} - \lambda_2$. FP (resp. FP^*) stand for fuzzy pairwise (resp. fuzzyP^*). The indices $i, j \in \{1, 2\}$ and $i \neq j$.

Definition 2.1. [3, 6, 26, 29] A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, $\forall \ \mu_1, \mu_2 \in I^X$,

(3) $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is an r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is an r-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geq r$. Note, Šostak [29] used the term 'fuzzy topology' and Chattopadhyay et al. [6], the term 'gradation of openness' for a smooth topology τ.

If τ satisfies conditions (1) and (3), then τ is said to be supra smooth topology and (X, τ) is said to be a supra smooth topological space [13].

Definition 2.2. [21, 29] A triple (X, τ_1, τ_2) consisting of the set X endowed with smooth topologies τ_1 and τ_2 on X is called a smooth bitopological space (smooth bts, for short). For $\lambda \in I^X$ and $r \in I_0$, r-τ_i-open (resp. closed) fuzzy set denotes the r-open (resp. closed) fuzzy set in (X, τ_i), for $i = 1, 2$.

Subsequently, the fuzzy closure (resp. interior) for any fuzzy set in smooth topological space is given as follows:

Definition 2.3. [7] Let (X, τ) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, a fuzzy closure is a mapping $C_\tau : I^X \times I_0 \to I^X$ such that

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \ \mu \geq \lambda, \ \tau(\bar{1} - \mu) \geq r\}. \tag{1}$$

And, a fuzzy interior of λ is a mapping $I_\tau : I^X \times I_0 \to I^X$ defined as

$$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \ \mu \leq \lambda, \ \tau(\mu) \geq r\}, \tag{2}$$

satisfies

$$I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r). \tag{3}$$

Remark 2.4. If (X, τ) is a supra smooth topological space. Then the definition of fuzzy closure (resp. interior) for any fuzzy set is defined as (1) and (2) in Definition 2.3 respectively.

Definition 2.5. [7] A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping C satisfies the following conditions:

(C1) $C(\bar{0}, r) = \bar{0}$,

(C2) $\lambda \leq C(\lambda, r)$,

(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,

(C4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$,

(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \longrightarrow I$ given by

$$\tau_C(\lambda) = \bigvee\{r \in I | \ C(\bar{1} - \lambda, r) = \bar{1} - \lambda\} \tag{4}$$

If C satisfies conditions (C1),(C2),(C4),(C5) and the following inequality:
$(C3)^* \quad C(\lambda, r) \vee C(\mu, r) \leq C(\lambda \vee \mu, r)$,
then C is called supra fuzzy closure operator on X [1]. and it generates a supra smooth topology $\tau_C : I^X \longrightarrow I$ as in (4)

By using (3), the definitions of fuzzy interior operator and supra fuzzy interior operator are obtained. In analogs of Definition 2.5, a fuzzy interior operator was defined.
The following theorem shows how to generate a supra fuzzy closure operator from smooth bts (X, τ_1, τ_2).

Theorem 2.6. [1] Let (X, τ_1, τ_2) be a smooth bts, for each $\lambda \in I^X$ and $r \in I_0$. Then:

(1) The mapping $C_{12} : I^X \times I_0 \to I^X$ such that $C_{12}(\lambda, r) = C_{\tau_1}(\lambda, r) \wedge C_{\tau_2}(\lambda, r)$ is a supra fuzzy closure operator, and (X, C_{12}) is a supra fuzzy closure space.

(2) The mapping $I_{12} : I^X \times I_0 \to I^X$ defined by $I_{12}(\lambda, r) = I_{\tau_1}(\lambda, r) \vee I_{\tau_2}(\lambda, r)$ is a supra fuzzy interior operator, satisfies $I_{12}(\bar{1} - \lambda, r) = \bar{1} - C_{12}(\lambda, r)$.

Theorem 2.7. [1] Let (X, τ_1, τ_2) be a smooth bts, let (X, C_{12}) be a supra fuzzy closure space. Define the mapping $\tau_S : I^X \to I$ on X by

$$\tau_S(\lambda) = \bigvee\{\tau_1(\lambda_1) \wedge \tau_2(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2, \ \ \lambda_1, \lambda_2 \in I^X\}$$

where $\bigvee$ is taken over all families $\{\lambda_1, \lambda_2 \in I^X : \lambda = \lambda_1 \vee \lambda_2\}$. Then:
(1) $\tau_S = \tau_{C_{12}}$ is the coarsest smooth supra topology on X which is finer than τ_1 and τ_2.
(2) $C_{12} = C_{\tau_s} = C_{\tau_{C_{12}}}$.

Remark 2.8. In this paper we will denote to $\tau_{C_{12}}$ by τ_{12}.

Definition 2.9. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then, a fuzzy set λ is called:

(1) an r-(τ_i, τ_j)-generalized fuzzy closed (r-(τ_i, τ_j)-gfc, for short), if $C_{\tau_j}(\lambda, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s \ \forall\ 0 < s \leq r$. The complement of r-$(\tau_i, \tau_j)$-gfc is an r-$(\tau_i, \tau_j)$-generalized fuzzy open (r-(τ_i, τ_j)-gfo, for short) [31].

(2) an r-τ_{12}-generalized fuzzy closed (r-τ_{12}-gfc, for short) if $C_{12}(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau_{12}(\mu) \geq s \quad \forall\ 0 < s \leq r$. The complement of r-τ_{12}-gfc is an r-τ_{12}-generalized fuzzy open (r-τ_{12}-gfo, for short) [32].

The concepts of r-τ_{12}-gfc and r-(i, j)-gfc sets are independent.

Recall next the definitions of open Q-nbd, θ-cluster point and θ-fuzzy closure operator in smooth bts (X, τ_1, τ_2).

Definition 2.10. [19] Let (X, τ_1, τ_2) be a smooth bts, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. Then, μ is called an r-open Q_{τ_i}-nighborhood of x_t if $x_t\ q\ \mu$ with $\tau_i(\mu) \geq r$, we denote

$$Q_{\tau_i}(x_t, r) = \{\mu \in I^X |\ x_t\ q\ \mu, \tau_i(\mu) \geq r\}.$$

Definition 2.11. [19] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) A fuzzy point $x_t \in Pt(X)$ is called an r-$(\tau_i, \tau_j)\theta$-cluster point of λ if for every $\mu \in Q_{\tau_i}(x_t, r)$, $C_{\tau_j}(\mu, r)\ q\ \lambda$.

(2) An $(\tau_i, \tau_j)\theta$-closure is a mapping $T_{\tau_j}^{\tau_i} : I^X \times I_0 \longrightarrow I^X$ defined as follows:

$$T_{\tau_j}^{\tau_i}(\lambda, r) = \bigvee\{x_t \in Pt(X)|\ x_t \textit{ is } r\text{-}(\tau_i, \tau_j)\theta\textit{-cluster point of } \lambda\}.$$

(3) λ is called an r-(τ_i, τ_j) fuzzy θ-closed iff $\lambda = T_{\tau_j}^{\tau_i}(\lambda, r)$. The complement of an r-(τ_i, τ_j) fuzzy θ-closed is called r-(τ_i, τ_j) fuzzy θ-open.

Theorem 2.12. [19] Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. Then:

(1) $T_{\tau_j}^{\tau_i}(\lambda, r) = \bigwedge\{\mu \in I^X |\ I_{\tau_j}(\mu, r) \geq \lambda,\ \tau_i(\bar{1} - \mu) \geq r\}$, i.e., $T_{\tau_j}^{\tau_i}(\lambda, r)$ is an r-τ_i-closed fuzzy set.

(2) x_t is an r-$(\tau_i, \tau_j)\theta$-cluster point of λ iff $x_t \in T_{\tau_j}^{\tau_i}(\lambda, r)$.

Definition 2.13. [34] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. A fuzzy set λ is an r-(τ_i, τ_j)-θ-generalized fuzzy closed (r-(τ_i, τ_j)-θ-gfc, for short) if $T_{\tau_i}^{\tau_j}(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s \ \forall\ 0 < s \leq r$. The complement of r-(τ_i, τ_j)-θ-gfc is an r-(τ_i, τ_j)-θ-generalized fuzzy open (r-(τ_i, τ_j)-θ-gfo, for short).

Definition 2.14. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$, $r \in I_0$ and $x_t \in Pt(X)$. Then:

(1) A fuzzy point x_t is said to be an r-τ_{12}-θ-cluster point of λ if and only if $C_{12}(\mu, r)$ q λ, for each $\mu \in Q_{\tau_{12}}(x_t, r)$, where $Q_{\tau_{12}}(x_t, r) = \{\mu \in I^X |\ x_t\ q\ \mu, \tau_{12}(\mu) \geq r\}$. The set of all r-τ_{12}-θ-cluster points of λ is called C_{12}^{θ}-fuzzy closure of λ, i.e. $C_{12}^{\theta} : I^X \times I_0 \longrightarrow I^X$ defined as

$$C_{12}^{\theta}(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid x_t \textit{ is } r\text{-}\tau_{12}\text{-}\theta\textit{-cluster point of}\lambda\}.$$

(2) λ is said to be an r-τ_{12}-θ-closed fuzzy set iff $C_{12}^{\theta}(\lambda, r) = \lambda$. The complement of r-τ_{12}-θ-closed fuzzy set is an r-τ_{12}-θ-open fuzzy set.

Theorem 2.15. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) $C_{12}(\lambda, r) \leq C_{12}^{\theta}(\lambda, r) \leq T_{\tau_j}^{\tau_i}(\lambda, r)$.

(2) If λ is an r-τ_{12}-open fuzzy set in X, then $C_{12}(\lambda, r) = C_{12}^{\theta}(\lambda, r)$.

Some properties of C_{12}^{θ} are given in the following proposition:

Proposition 2.16. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. Then:

(1) $C_{12}^{\theta}(\lambda, r) = \bigwedge\{C_{12}(\rho, r) :\ \rho \geq \lambda\ , \tau_{12}(\rho) \geq r\}$.

(2) If $\lambda_1 \leq \lambda_2$, then $C^{\theta}_{12}(\lambda_1, r) \leq C^{\theta}_{12}(\lambda_2, r)$.

(3) $C^{\theta}_{12}(\lambda_1, r) \vee C^{\theta}_{12}(\lambda_2, r) = C^{\theta}_{12}(\lambda_1 \vee \lambda_2, r)$.

(4) $C^{\theta}_{12}(\lambda, r) \leq C^{\theta}_{12}(\lambda, s)$, if $r \leq s$.

(5) $C^{\theta}_{12}(\lambda_1 \wedge \lambda_2, r) \leq C^{\theta}_{12}(\lambda_1, r) \wedge C^{\theta}_{12}(\lambda_2, r)$.

(6) $C^{\theta}_{12}(\lambda, r) \leq C^{\theta}_{12}(C^{\theta}_{12}(\lambda, r), r)$.

Next we introduce the concept of I^{θ}_{12}-fuzzy interior in smooth bts (X, τ_1, τ_2).

Definition 2.17. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. A fuzzy point x_t is said to be an r-τ_{12}-θ-interior point of λ if there exists $\mu \in Q_{\tau_{12}}(x_t, r)$ such that $C_{12}(\mu, r)\ \bar{q}\ \bar{1} - \lambda$. The set of all r-τ_{12}-θ-interior points of λ is called I^{θ}_{12}-fuzzy interior of λ. i.e. $I^{\theta}_{12} : I^X \times I_0 \longrightarrow I^X$ defined as

$$I^{\theta}_{12}(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid x_t \textit{ is } r\text{-}\tau_{12}\text{-}\theta\textit{-interior point of } \lambda\}.$$

Equivalently, I^{θ}_{12}-fuzzy interior can be stated as folllows.

Proposition 2.18. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

$$I^{\theta}_{12}(\lambda, r) = \bigvee\{\mu \in I^X \mid C_{12}(\mu, r) \leq \lambda, \tau_{12}(\mu) \geq r\}.$$

Throughout this paper (X, τ_{12}) and (Y, τ^*_{12}) denote the supra smooth topological spaces which are induced from smooth bitopological spaces (X, τ_1, τ_2) and (Y, τ^*_1, τ^*_2) respectively.

Definition 2.19. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau^*_1, \tau^*_2)$ from a smooth bts (X, τ_1, τ_2) to another one (Y, τ^*_1, τ^*_2) is said to be:

(1) FP-continuous if and only if $\tau_i(f^{-1}(\mu)) \geq \tau^*_i(\mu)$ for each $\mu \in I^Y$ and $i = 1, 2$ [17].

(2) FP^*-continuous if and only if $f : (X, \tau_{12}) \longrightarrow (Y, \tau^*_{12})$ is F-continuous [27]. That is, $\tau_{12}(f^{-1}(\mu)) \geq \tau^*_{12}$ for each $\mu \in I^Y$.

(3) FP^*-open if and only if $f : (X, \tau_{12}) \longrightarrow (Y, \tau^*_{12})$ is F-open [17]. That is, $\tau^*_{12}(f(\lambda)) \geq \tau_{12}(\lambda)$ for each $\lambda \in I^X$.

(4) generalized FP^*-continuous (GFP^*-continuous, for short) if and only if $f^{-1}(\mu)$ is an r-τ_{12}-gfc for all $\mu \in I^Y$ with $\tau^*_{12}(\bar{1} - \mu) \geq r$ [32].

(5) generalized FP^*-irresolute closed (GFP^*-irresolute closed, for short) if and only if $f(\mu)$ is an r-τ^*_{12}-gfc in Y for each r-τ_{12}-gfc μ in X [32].

3 r-τ_{12}-θ-generalized Fuzzy Closed Sets

In this section we introduce a new class of generalized fuzzy closed sets via a fuzzy closure C^{θ}_{12} defined in [33], and we study its relationship with other types of generalized fuzzy closed sets which introduced in ([31, 32, 33, 34]).

Definition 3.1. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) A fuzzy set λ is called an r-τ_{12}-θ-generalized fuzzy closed (r-τ_{12}-θ-gfc, for short) if $C^{\theta}_{12}(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau_{12}(\mu) \geq s$ for all $0 < s \leq r$.

(2) A fuzzy set λ is called an r-τ_{12}-θ-generalized fuzzy open (r-τ_{12}-θ-gfo, for short) if $\bar{1} - \lambda$ is an r-τ_{12}-θ-gfc.

Proposition 3.2. Let (X, τ_1, τ_2) be a smooth bts, $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. Then:

(1) If λ_1, λ_2 are r-τ_{12}-θ-gfc sets, then $\lambda_1 \vee \lambda_2$ is an r-τ_{12}-θ-gfc set.

(2) If λ_1, λ_2 are r-τ_{12}-θ-gfo sets, then $\lambda_1 \wedge \lambda_2$ is an r-τ_{12}-θ-gfo set.

Proof. To prove part (1), let $\lambda_1 \vee \lambda_2 \leq \mu$ such that $\tau_{12}(\mu) \geq s$ for $0 < s \leq r$. This implies $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$. Since λ_1 and λ_2 are r-τ_{12}-θ-gfc sets, then in view of Proposition 2.16(3) and Definition 3.1(1), we have, $C^{\theta}_{12}(\lambda_1 \vee \lambda_2, s) = C^{\theta}_{12}(\lambda_1, s) \vee C^{\theta}_{12}(\lambda_2, s) \leq \mu \vee \mu = \mu$. Hence, $\lambda_1 \vee \lambda_2$ is an r-τ_{12}-θ-gfc. The prove of part (2), follows from the duality of (1). □

Remark 3.3. The finite intersection (resp. union) of r-τ_{12}-θ-gfc (resp. gfo) sets in a smooth bts (X, τ_1, τ_2) need not to be an r-τ_{12}-θ-gfc (resp. gfo), as the following example shows.

Example 3.4. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.2} \vee b_{0.5}, \qquad \lambda_2 = a_{0.4} \vee b_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}. \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{3}{4} & \text{if } \lambda = \lambda_2 \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1, \\ \frac{3}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then, for $r = \frac{1}{4}$ the fuzzy sets $\eta_1 = a_{0.2} \vee b_{0.6}$ and $\eta_2 = a_{0.6} \vee b_{0.2}$ are $\frac{1}{4}$-τ_{12}-θ-gfc sets but $\eta_1 \wedge \eta_2$ is not a $\frac{1}{4}$-τ_{12}-θ-gfc. By taking the complement of η_1 and η_2 we obtain the finite union of r-τ_{12}-θ-gfo sets. This union need not to be r-τ_{12}-θ-gfo.

In the following Propositions 3.5, 3.7, 3.9, 3.10 and 3.11 with the examples following them show that the class of r-τ_{12}-θ-gfc sets is properly placed between the classes of r-τ_{12}-gfc sets and r-τ_{12}-θ-closed fuzzy sets.

Proposition 3.5. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-closed fuzzy set, then λ is an r-τ_{12}-θ-gfc set.

Proof. Let $\lambda \leq \mu$ such that $\tau_{12}(\mu) \geq s$ for $0 < s \leq r$. Since λ is an r-τ_{12}-θ-closed fuzzy set, then $C^{\theta}_{12}(\lambda, r) = \lambda$ and from Proposition 2.16(4) , for $s \leq r$ we have $C^{\theta}_{12}(\lambda, s) \leq C^{\theta}_{12}(\lambda, r) = \lambda \leq \mu$. Hence, λ is an r-τ_{12}-θ-gfc set. □

The converse of Proposition 3.5 is not true as we show in the next example.

Example 3.6. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.2} \vee b_{0.5}, \quad \lambda_2 = a_{0.5} \vee b_{0.3}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \ \lambda_2, \ \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then, for $r = \frac{1}{2}$, the fuzzy set $\lambda = a_{0.4} \vee b_{0.4}$ is a $\frac{1}{2}$-τ_{12}-θ-gfc but is not a $\frac{1}{2}$-τ_{12}-θ-closed fuzzy set.

Proposition 3.7. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-gfc set, then λ is an r-τ_{12}-gfc set.

Proof. The proof follows directly from Theorem 2.15(1). □

The following example shows the converse of the previous proposition is not true.

Example 3.8. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.7} \vee b_{0.5}, \quad \lambda_2 = a_{0.2} \vee b_{0.9}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = \frac{1}{2}$, the fuzzy set $\lambda = a_{0.3} \vee b_{0.5}$ is a $\frac{1}{2}$-τ_{12}-gfc but is not a $\frac{1}{2}$-τ_{12}-θ-gfc.

Proposition 3.9. [32] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-closed fuzzy set, then λ is an r-τ_{12}-gfc set.

The converse of Proposition 3.9 is not true (see [32]).

Proposition 3.10. [33] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-closed fuzzy set, then λ is an r-τ_{12}-closed fuzzy set.

The converse of Proposition 3.10 is not true (see [33]).

Proposition 3.11. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is an r-(τ_j, τ_i) fuzzy θ-closed set, then λ is an r-τ_{12}-θ-closed fuzzy set.

Proof. To prove λ is an r-τ_{12}-θ-closed fuzzy set, we must prove $C^{\theta}_{12}(\lambda, r) = \lambda$. Clearly $\lambda \leq C^{\theta}_{12}(\lambda, r)$. On the other hand, from Theorem 2.15(1), $C^{\theta}_{12}(\lambda, r) \leq T^{\tau_j}_{\tau_i}(\lambda, r)$. Since λ is an r-(τ_j, τ_i) fuzzy θ-closed set, then $T^{\tau_j}_{\tau_i}(\lambda, r) = \lambda$. Consequently, $C^{\theta}_{12}(\lambda, r) \leq \lambda$. Hence, λ is an r-τ_{12}-θ-closed fuzzy set. □

The next example shows the converse of Proposition 3.11 is not true in general.

Example 3.12. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.4} \vee b_{0.5}, \quad \lambda_2 = a_{0.5} \vee b_{0.4}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then, for $r = \frac{1}{4}$, the fuzzy set $\lambda = a_{0.5} \vee b_{0.5}$ is a $\frac{1}{4}$-τ_{12}-θ-closed fuzzy set but is not a $\frac{1}{4}$-(τ_1, τ_2)fuzzy θ-closed set.

From the above discussion we have the following diagram which is an enlargement of a Diagram from [33].

$$\begin{array}{ccccccc} \mathbf{r}\text{-}(\tau_\mathbf{i}, \tau_\mathbf{j})\text{-}\theta\text{-}\mathbf{gfc} & \Leftarrow & \mathbf{r}\text{-}(\tau_\mathbf{j}, \tau_\mathbf{i})\mathbf{fuzzy}\ \theta\text{-}\mathbf{closed} & \Rightarrow & \mathbf{r}\text{-}\tau_{\mathbf{12}}\text{-}\theta\text{-}\mathbf{closed\ fuzzy\ set} & \Rightarrow & \mathbf{r}\text{-}\tau_{\mathbf{12}}\text{-}\mathbf{closed\ fuzzy\ set} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \mathbf{r}\text{-}(\tau_\mathbf{i}, \tau_\mathbf{j})\text{-}\mathbf{gfc} & \Longleftarrow & \mathbf{r}\text{-}\tau_\mathbf{j}\text{-}\mathbf{closed\ fuzzy\ set} & & \mathbf{r}\text{-}\tau_{\mathbf{12}}\text{-}\theta\text{-}\mathbf{gfc} & \Longrightarrow & \mathbf{r}\text{-}\tau_{\mathbf{12}}\text{-}\mathbf{gfc} \end{array}$$

From the above diagram one can notice that the concepts of r-(τ_i, τ_j)-θ-gfc and r-τ_{12}-θ-gfc sets are independent as the following two examples show.

Example 3.13. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.3} \vee b_{0.5}, \quad \lambda_2 = a_{0.6} \vee b_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = \frac{1}{4}$, the fuzzy set $\lambda = a_{0.4} \vee b_{0.3}$ is a $\frac{1}{4}$-(τ_1, τ_2)-θ-gfc but is not a $\frac{1}{4}$-τ_{12}-θ-gfc set.

Example 3.14. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:

$$\lambda_1 = a_{0.4} \vee b_{0.5}, \quad \lambda_2 = a_{0.6} \vee b_{0.2}.$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \textit{otherwise}; \end{cases} \quad \textit{and} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{3} & \text{if } \lambda = \lambda_2, \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & \textit{otherwise}. \end{cases}$$

Then for $r = \frac{1}{3}$, the fuzzy set $\lambda = a_{0.1} \vee b_{0.3}$ is a $\frac{1}{3}$-τ_{12}-θ-gfc but is not a $\frac{1}{3}$-(τ_1, τ_2)-θ-gfc set.

4 Generalized C_{12}^{θ}-fuzzy Closure Operator

In this section we use the class of r-τ_{12}-θ-gfc (resp. gfo) sets to introduce a new fuzzy closure (resp. interior) operator on smooth bts (X, τ_1, τ_2). In fact this new fuzzy closure (resp. interior) operator represents a generalization of the fuzzy closure (resp. interior) operator C_{12}^{θ} (resp. I_{12}^{θ}) [32]. Some properties of these new fuzzy closure are given. We show that C_{12}^{θ} (resp. I_{12}^{θ}) generates a smooth fuzzy topology which is finer than τ_{12}^{θ}.

Definition 4.1. Let (X, τ_1, τ_2) be a smooth bts. For $\lambda \in I^X$ and $r \in I_0$, a generalized C_{12}^{θ}-fuzzy closure is a map $\mathcal{G}C_{12}^{\theta} : I^X \times I_0 \longrightarrow I^X$ define as

$$\mathcal{G}C_{12}^{\theta}(\lambda, r) = \bigwedge\{\rho \in I^X |\ \rho \geq \lambda \ and\ \rho\ is\ r\text{-}\tau_{12}\text{-}\theta\text{-}gfc\ set\}.$$

And a generalized I_{12}^{θ}-fuzzy interior of λ is a map $\mathcal{G}I_{12}^{\theta} : I^X \times I_0 \longrightarrow I^X$ define as

$$\mathcal{G}I_{12}^{\theta}(\lambda, r) = \bigvee\{\rho \in I^X |\ \rho \leq \lambda \ and\ \rho\ is\ r\text{-}\tau_{12}\text{-}\theta\text{-}gfo\ set\}.$$

Some properties of $\mathcal{G}C_{12}^{\theta}$ and $\mathcal{G}I_{12}^{\theta}$ are given next.

Proposition 4.2. Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \lambda_1, \lambda_2 \in I^X$ *and* $r \in I_0$. Then:

(1) $\mathcal{G}I_{12}^{\theta}(\bar{1} - \lambda, r) = \bar{1} - \mathcal{G}C_{12}^{\theta}(\lambda, r)$.

(2) If $\lambda_1 \leq \lambda_2$, then $\mathcal{G}C_{12}^{\theta}(\lambda_1, r) \leq \mathcal{G}C_{12}^{\theta}(\lambda_2, r)$.

(3) If $\lambda_1 \leq \lambda_2$, then $\mathcal{G}I_{12}^{\theta}(\lambda_1, r) \leq \mathcal{G}I_{12}^{\theta}(\lambda_2, r)$.

(4) If λ is an r-τ_{12}-θ-gfc, then $\mathcal{G}C_{12}^{\theta}(\lambda, r) = \lambda$.

(5) If λ is an r-τ_{12}-θ-gfo, then $\mathcal{G}I_{12}^{\theta}(\lambda, r) = \lambda$.

Proof. We prove (1), using Definition 4.1:

$$\begin{aligned}
\bar{1} - \mathcal{G}C_{12}^{\theta}(\lambda, r) &= \bar{1} - \bigwedge\{\rho \in I^X |\ \rho \geq \lambda,\ \rho\ is\ r\text{-}\tau_{12}\text{-}\theta\text{-}gfc\ set\} \\
&= \bigvee\{\bar{1} - \rho \in I^X |\ \bar{1} - \rho \leq \bar{1} - \lambda, \bar{1} - \rho\ is\ r\text{-}\tau_{12}\text{-}\theta\text{-}gfo\ set\} \\
&= \mathcal{G}I_{12}^{\theta}(\bar{1} - \lambda, r).
\end{aligned}$$

To prove (2), suppose there exist $x \in X$ and $t \in I_0$ such that

$$\mathcal{G}C_{12}^{\theta}(\lambda_1, r)(x) > t > \mathcal{G}C_{12}^{\theta}(\lambda_2, r)(x). \tag{5}$$

Since $\mathcal{G}C_{12}^{\theta}(\lambda_2, r)(x) < t$, then there exists an r-τ_{12}-θ-gfc ρ with $\rho \geq \lambda_2$ such that $\rho(x) < t$. Since $\lambda_1 \leq \lambda_2$, then $\mathcal{G}C_{12}^{\theta}(\lambda_1, r) \leq \rho$. It follows $\mathcal{G}C_{12}^{\theta}(\lambda_1, r)(x) < t$. This contradicts (5). Hence, $\mathcal{G}C_{12}^{\theta}(\lambda_1, r) \leq \mathcal{G}C_{12}^{\theta}(\lambda_2, r)$. The proof of (3), follows from taking the complement of (2) and then using (1). The proof of (4), follows from Definition 4.1. Finally, the proof of (5) is similar to the proof of (3). □

In Proposition 4.2 the converse of (4) and (5) are not true as the following example show. The example is inspired by the one introduced in $[18, p.333]$

Example 4.3. Let $X = \{a, b\}$. Define smooth topologies $\tau_1 = \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ 0.3 & \text{if } \lambda = a_{0.6}, \\ 0 & otherwise. \end{cases}$$

The induced supra smooth topological space of (X, τ_1, τ_2), is defined as $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ 0.3 & \text{if } \lambda = a_{0.6}, \\ 0 & otherwise. \end{cases}$$

The fuzzy set $a_{0.6}$ is not a 1-τ_{12}-θ-gfc, but $\mathcal{G}C_{12}^{\theta}(a_{0.6}, 1) = a_{0.6}$. Because, $a_{0.6} \vee b_s$ is a 1-τ_{12}-θ-gfc for $s \in I_0$. Therefore,

$\mathcal{G}C_{12}^{\theta}(a_{0.6}, 1) = \bigwedge_{s \in I_0}(a_{0.6} \vee b_s) = a_{0.6} \vee \bigwedge_{s \in I_0} b_s = a_{0.6}$.

Next we show $\mathcal{G}C^{\theta}_{12}$ (resp. $\mathcal{G}I^{\theta}_{12}$) is fuzzy closure operator.

Theorem 4.4. Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:

(1) $\mathcal{G}C^{\theta}_{12}$ (resp. $\mathcal{G}I^{\theta}_{12}$) is a fuzzy closure (resp. interior) operator.

(2) The mapping $\tau^{\mathcal{G}\theta}_{12} : I^X \longrightarrow I$ defined as

$$\tau^{\mathcal{G}\theta}_{12}(\lambda) = \bigvee\{r \in I|\ \mathcal{G}C^{\theta}_{12}(\bar{1} - \lambda, r) = \bar{1} - \lambda\}.$$

is a smooth topology on X such that $\tau^{\theta}_{12} \leq \tau^{\mathcal{G}\theta}_{12}$.

Proof. We have shown that $\mathcal{G}C^{\theta}_{12}$ is a fuzzy closure operator and in a similar way can prove that $\mathcal{G}I^{\theta}_{12}$ is a fuzzy interior operator. To prove (1), we need to satisfy conditions $(C1)-(C5)$ in Definition 2.5.

(C1) Since $\bar{0}$ is an r-τ_{12}-θ-gfc set in X, then from Proposition 4.2(4), $\mathcal{G}C^{\theta}_{12}(\bar{0}, r) = \bar{0}$.

(C2) Follows immediately from the Definition of $\mathcal{G}C^{\theta}_{12}$.

(C3) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, then from Proposition 4.2(2),

$$\mathcal{G}C^{\theta}_{12}(\lambda, r) \leq \mathcal{G}C^{\theta}_{12}(\lambda \vee \mu, r) \text{ and } \mathcal{G}C^{\theta}_{12}(\mu, r) \leq \mathcal{G}C^{\theta}_{12}(\lambda \vee \mu, r).$$

This implies, $\mathcal{G}C^{\theta}_{12}(\lambda, r) \vee \mathcal{G}C^{\theta}_{12}(\mu, r) \leq \mathcal{G}C^{\theta}_{12}(\lambda \vee \mu, r)$.
Suppose $\mathcal{G}C^{\theta}_{12}(\lambda \vee \mu, r) \nleq \mathcal{G}C^{\theta}_{12}(\lambda, r) \vee \mathcal{G}C^{\theta}_{12}(\mu, r)$. Consequently, $x \in X$ and $t \in I_0$ exist such that

$$\mathcal{G}C^{\theta}_{12}(\lambda, r)(x) \vee \mathcal{G}C^{\theta}_{12}(\mu, r)(x) < t < \mathcal{G}C^{\theta}_{12}(\lambda \vee \mu, r)(x). \tag{6}$$

Since $\mathcal{G}C^{\theta}_{12}(\lambda, r)(x) < t$ and $\mathcal{G}C^{\theta}_{12}(\mu, r)(x) < t$, then there exist r-τ_{12}-θ-gfc sets ρ_1, ρ_2 with $\lambda \leq \rho_1$ and $\mu \leq \rho_2$ such that

$$\rho_1(x) < t, \rho_2(x) < t.$$

Since $\lambda \vee \mu \leq \rho_1 \vee \rho_2$ and $\rho_1 \vee \rho_2$ is an r-τ_{12}-θ-gfc from Proposition 3.2(1), we have $\mathcal{G}C^{\theta}_{12}(\lambda \vee \mu, r)(x) \leq (\rho_1 \vee \rho_2)(x) < t$. This, however, contradicts (6). Hence, $\mathcal{G}C^{\theta}_{12}(\lambda, r) \vee \mathcal{G}C^{\theta}_{12}(\mu, r) = \mathcal{G}C^{\theta}_{12}(\lambda \vee \mu, r)$.

(C4) Let $r \leq s$, $r, s \in I_0$. Suppose $\mathcal{G}C^{\theta}_{12}(\lambda, r) \nleq \mathcal{G}C^{\theta}_{12}(\lambda, s)$. Consequently, $x \in X$ and $t \in I_0$ exist such that

$$\mathcal{G}C^{\theta}_{12}(\lambda, s)(x) < t < \mathcal{G}C^{\theta}_{12}(\lambda, r)(x). \tag{7}$$

Since $\mathcal{G}C^{\theta}_{12}(\lambda, s)(x) < t$, then there is an s-τ_{12}-θ-gfc set ρ with $\lambda \leq \rho$ such that $\rho(x) < t$. This yields $C^{\theta}_{12}(\rho, s_1) \leq \mu$, whenever $\rho \leq \mu$ and $\tau_{12}(\mu) \geq s_1$, for $0 < s_1 \leq s$. Since $r \leq s$, then $C^{\theta}_{12}(\rho, r_1) \leq \mu$ whenever $\rho \leq \mu$ and $\tau_{12}(\mu) \geq r_1$, for $0 < r_1 \leq r \leq s_1 \leq s$. This implies ρ is an r-τ_{12}-θ-gfc. From Definition 4.1, we have $\mathcal{G}C^{\theta}_{12}(\lambda, r)(x) \leq \rho(x) < t$. This contradicts (7). Hence, $\mathcal{G}C^{\theta}_{12}(\lambda, r) \leq \mathcal{G}C^{\theta}_{12}(\lambda, s)$.

(C5) Let ρ be any r-τ_{12}-θ-gfc containing λ. Then, from Definition 4.1, we have $\mathcal{G}C^{\theta}_{12}(\lambda, r) \leq \rho$. From proposition 4.2(2), we obtain $\mathcal{G}C^{\theta}_{12}(\mathcal{G}C^{\theta}_{12}(\lambda, r), r) \leq \mathcal{G}C^{\theta}_{12}(\rho, r) = \rho$. This means that $\mathcal{G}C^{\theta}_{12}(\mathcal{G}C^{\theta}_{12}(\lambda, r), r)$ is contained in every r-τ_{12}-θ-gfc set containing λ. Hence, $\mathcal{G}C^{\theta}_{12}(\mathcal{G}C^{\theta}_{12}(\lambda, r), r) \leq \mathcal{G}C^{\theta}_{12}(\lambda, r)$. However, $\mathcal{G}C^{\theta}_{12}(\lambda, r) \leq \mathcal{G}C^{\theta}_{12}(\mathcal{G}C^{\theta}_{12}(\lambda, r), r)$. Therefore, $\mathcal{G}C^{\theta}_{12}(\mathcal{G}C^{\theta}_{12}(\lambda, r), r) = \mathcal{G}C^{\theta}_{12}(\lambda, r)$. Thus $\mathcal{G}C^{\theta}_{12}$ is a fuzzy closure operator.

To prove (2), we employ (1) and Definition 2.5, we get $\tau^{\mathcal{G}\theta}_{12}(\lambda)$ is a smooth topology on X. By Proposition 3.5, $C^{\theta}_{12}(\bar{1}-\lambda, r) = \bar{1}-\lambda$ which yields $\mathcal{G}C^{\theta}_{12}(\bar{1}-\lambda, r) = \bar{1}-\lambda$. Thus, $\tau^{\theta}_{12}(\lambda) \leq \tau^{\mathcal{G}\theta}_{12}(\lambda)$ for all $\lambda \in I^X$.

□

At the end of this section we state the following proposition which is describes each r-τ_{12}-θ-gfc set in smooth topological space $(X, \tau^{\mathcal{G}\theta}_{12})$.

Proposition 4.5. Let (X, τ_1, τ_2) be a smooth bts. $\lambda \in I^X$ and $r \in I_0$. If λ is an r-τ_{12}-θ-gfc, then λ is an r-$\tau^{\mathcal{G}\theta}_{12}$-closed fuzzy set.

Proof. The proof follows from Proposition 4.2(4) and Theorem 4.4(2). □

5 GFP^*-θ-continuous and GFP^*-θ-irresolute Mappings

In this section we use the smooth supra topological space (X, τ_{12}) which is generated from smooth bts (X, τ_1, τ_2) to introduce and study the concepts of generalized FP^*-θ- continuous (resp. irresolute) and FP^*-strongly-θ-continuous mappings for the smooth bts (X, τ_1, τ_2).

Definition 5.1. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is called:

(1) generalized FP^*-θ- continuous (GFP^*-θ-continuous , for short) if $f^{-1}(\mu)$ is an r-τ_{12}-θ-gfc in X for each r-τ_{12}^*-closed fuzzy set μ in Y.

(2) generalized-FP^*-θ-irresolute (GFP^*-θ-irresolute, for short) if $f^{-1}(\mu)$ is an r-τ_{12}-θ-gfc in X for each r-τ_{12}^*-θ-gfc μ in Y.

(3) FP^*-strongly-θ- continuous (FP^*-S-θ-continuous, for short) if for each $x_t \in Pt(X)$ and for each $\mu \in Q_{\tau_{12}^*}(f(x_t), r)$, there exists $\nu \in Q_{\tau_{12}}(x_t, r)$ such that $f(C_{12}(\nu, r)) \leq \mu$.

Next we study the relationships between GFP^*-θ-continuous, FP^*-S-θ-continuous, GFP^*-continuous and FP^*-continuous. Next proposition give the relationship between GFP^*-θ-continuous and GFP^*-continuous.

Proposition 5.2. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is GFP^*-θ-continuous, then f is GFP^*-continuous.

Proof. Let $\mu \in I^Y$ such that μ is an r-τ_{12}^*-fuzzy closed set. Since f is GFP^*-θ-continuous, then we have, $f^{-1}(\mu)$ is an r-τ_{12}-θ-gfc, and from Proposition 3.7, this yields $f^{-1}(\mu)$ is an r-τ_{12}-gfc. Hence, f is GFP^*-continuous. □

The converse of the above proposition in not true according to the following counterexample.

Example 5.3. Let $X = \{a, b\}$ and $Y = \{p, q, w\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:

$$\lambda_1 = a_{\frac{2}{3}} \vee b_{\frac{1}{2}}, \quad \lambda_2 = a_{\frac{3}{4}} \vee b_{\frac{1}{4}}, \quad \mu_1 = p_{\frac{3}{4}} \vee q_{\frac{2}{3}} \vee w_{\frac{1}{2}}, \quad \mu_2 = p_{\frac{2}{3}} \vee q_{\frac{3}{4}} \vee w_{\frac{1}{2}}.$$

We define the smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\tau_1^*, \tau_2^* : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & otherwise; \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ 0 & otherwise; \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & otherwise; \end{cases} \qquad and \qquad \tau_2^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & otherwise. \end{cases}$$

From the smooth bts's (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) we can induce the supra smooth topologies τ_{12} and τ_{12}^* as follows:

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ \frac{1}{4} & \text{if } \lambda = \lambda_2, \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ 0 & otherwise; \end{cases} \qquad and \qquad \tau_{12}^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ \frac{1}{3} & \text{if } \mu = \mu_1 \vee \mu_2, \\ 0 & otherwise. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ defined by $f(a) = q$ and $f(b) = w$. Then, f is GFP^*-continuous but is not GFP^*-θ-continuous because, there exists $\bar{1} - \mu_1$ is a $\frac{1}{2}$-τ_{12}^*-closed fuzzy set but $f^{-1}(\bar{1} - \mu_1)$ is not a $\frac{1}{2}$-τ_{12}-θ-gfc set.

Next we give the relationship between FP^*-S-θ-continuous and GFP^*-θ-continuous.

Proposition 5.4. If $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\tau_1^*,\tau_2^*)$ is FP^*-S-θ-continuous, then f is GFP^*-θ-continuous.

Proof. Let $\lambda\in$ be an r-τ_{12}^*-closed fuzzy set in Y. Let $f^{-1}(\lambda)\le\mu$ where $\tau_{12}(\mu)\ge s$ for $0<s\le r$. We must show $C_{12}^{\theta}(f^{-1}(\lambda),s)\le\mu$. Let $x_t\notin\mu$ this mean, $x_t\ q\ \bar{1}-\mu$. In fact that $f^{-1}(\lambda)\le\mu$, which implies $\bar{1}-\mu\le\bar{1}-f^{-1}(\lambda)$, and since $x_t\ q\ \bar{1}-\mu$ this yields, $x_t\ q\ \bar{1}-f^{-1}(\lambda)$. Thus, we have $f(x_t)\ q\ \bar{1}-\lambda$ such that $\bar{1}-\lambda$ is r-τ_{12}^*-open fuzzy set in Y. That is mean $\bar{1}-\lambda\in Q_{\tau_{12}^*}(f(x_t),r)$. Since f is FP^*-S-θ-continuous. Then, there exists $\eta\in Q_{\tau_{12}}(x_t,r)$ such that $f(C_{12}(\eta,r))\le\bar{1}-\lambda$. This implies, $f(C_{12}(\eta,r))\ \bar{q}\ \lambda$ and then $C_{12}(\eta,r)\ \bar{q}\ f^{-1}(\lambda)$. In view of Definition 2.14, we get $x_t\notin C_{12}^{\theta}(f^{-1}(\lambda),r)$. Since $s\le r$ then, from Proposition 2.16(4), we have $x_t\notin C_{12}^{\theta}(f^{-1}(\lambda),s)$. Hence, we obtain $C_{12}^{\theta}(f^{-1}(\lambda),s)\le\mu$. Thus, f is GFP^*-θ-continuous. □

The converse of the above Proposition not true as seen from the following example.

Example 5.5. Let $X=\{a,b\}$ and $Y=\{p,q\}$. Define $\lambda_1,\lambda_2\in I^X$ and $\mu_1,\mu_2\in I^Y$ as follows:

$$\lambda_1=a_{\frac{1}{2}}\vee b_{\frac{1}{3}},\quad \lambda_2=a_{\frac{1}{3}}\vee b_{\frac{1}{2}},\quad \mu_1=p_{\frac{1}{2}}\vee q_{\frac{1}{4}},\quad \mu_2=p_{\frac{1}{4}}\vee q_{\frac{1}{2}}.$$

We define the smooth topologies $\tau_1,\tau_2:I^X\longrightarrow I$ and $\tau_1^*,\tau_2^*:I^Y\longrightarrow I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \lambda=\bar{0},\bar{1},\\ \frac{1}{2} & \text{if } \lambda=\lambda_1,\\ 0 & otherwise;\end{cases}\qquad \tau_2(\lambda)=\begin{cases}1 & \text{if } \lambda=\bar{0},\bar{1},\\ \frac{1}{3} & \text{if } \lambda=\lambda_2,\\ 0 & otherwise;\end{cases}$$

$$\tau_1^*(\mu)=\begin{cases}1 & \text{if } \mu=\bar{0},\bar{1},\\ \frac{1}{2} & \text{if } \mu=\mu_1,\\ 0 & otherwise;\end{cases}\qquad and \qquad \tau_2^*(\mu)=\begin{cases}1 & \text{if } \mu=\bar{0},\bar{1},\\ \frac{1}{3} & \text{if } \mu=\mu_2,\\ 0 & otherwise.\end{cases}$$

From the smooth bts's (X,τ_1,τ_2) and (Y,τ_1^*,τ_2^*) we can induce the supra smooth topologies τ_{12} and τ_{12}^* as follows

$$\tau_{12}(\lambda)=\begin{cases}1 & \text{if } \lambda=\bar{0},\bar{1},\\ \frac{1}{2} & \text{if } \lambda=\lambda_1,\\ \frac{1}{3} & \text{if } \lambda=\lambda_2,\\ \frac{1}{3} & \text{if } \lambda=\lambda_1\vee\lambda_2,\\ 0 & otherwise;\end{cases}\qquad and \qquad \tau_{12}^*(\mu)=\begin{cases}1 & \text{if } \mu=\bar{0},\bar{1},\\ \frac{1}{2} & \text{if } \mu=\mu_1,\\ \frac{1}{3} & \text{if } \mu=\mu_2,\\ \frac{1}{3} & \text{if } \mu=\mu_1\vee\mu_2,\\ 0 & otherwise.\end{cases}$$

Consider the mapping $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\tau_1^*,\tau_2^*)$ defined by $f(a)=q$ and $f(b)=p$. Then, f is GFP^*-θ-continuous but is not FP^*-S-θ-continuous because, there exists $a_{0.7}\in Pt(X)$, $r=\frac{1}{3}$ and $\mu_1\in Q_{\tau_{12}^*}(f(a_{0.7}),\frac{1}{3})$ such that for any $\lambda\in Q_{\tau_{12}}(a_{0.7},\frac{1}{3})$, $f(C_{12}(\lambda,\frac{1}{3}))\not\le\mu_1$.

Proposition 5.6. [32] If $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\tau_1^*,\tau_2^*)$ is FP^*-continuous, then f is GFP^*-continuous.

The converse of the proceeded proposition is not true in general (see [32]).

To discuss the relation between FP^*-S-θ-continuous and FP^*-continuous, we need to give an equivalent definition to FP^*-continuous.

Theorem 5.7. A mapping $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\tau_1^*,\tau_2^*)$ is FP^*-continuous iff for each $x_t\in Pt(X)$ and for each $\mu\in Q_{\tau_{12}^*}(f(x_t),r)$, there exists $\eta\in Q_{\tau_{12}}(x_t,r)$ such that $f(\eta)\le\mu$.

Proof. The proof is similar to the one in [[34], Theorem 5.3]. □

Proposition 5.8. If $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\tau_1^*,\tau_2^*)$ is FP^*-S-θ-continuous, then f is FP^*-continuous.

Proof. Let $x_t\in Pt(X)$ and $\mu\in Q\tau_{12}^*(f(x_t),r)$. Since f is FP^*-S-θ-continuous. Then, there exists $\eta\in Q\tau_{12}(x_t,r)$ such that $f(C_{12}(\eta,r))\le\mu$. Since $\eta\le C_{12}(\eta,r)$, then $f(\eta)\le f(C_{12}(\eta,r))\le\mu$. Thus, in view of Theorem 5.7, f is FP^*-continuous. □

The converse of proposition 5.8 is not true as we have shown in Example 5.3. Note that Example 5.3 and Example 5.5 show that the FP^*-continuous and GFP^*-θ-continuous are independent. Therefore we have the following implications and none of them are reversible.

$$\begin{array}{ccc} \mathbf{GFP^*\text{-}\theta\text{-continuous}} & \Longrightarrow & \mathbf{GFP^*\text{-continuous}} \\ \Uparrow & & \Uparrow \\ \mathbf{FP^*\text{-}S\text{-}\theta\text{-continuous}} & \Longrightarrow & \mathbf{FP^*\text{-continuous}} \end{array}$$

The following theorem provides conditions to obtain GFP^*-θ-irresolute mapping from GFP^*-θ-continuous mapping.

Theorem 5.9. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is bijective, FP^*-open and GFP^*-θ-continuous, then f is GFP^*-θ-irresolute.

Proof. Let ν is an r-τ_{12}^*-θ-gfc set and $f^{-1}(\nu) \leq \mu$ such that $\tau_{12}(\mu) \geq s$ for $0 < s \leq r$. Since $f^{-1}(\nu) \leq \mu$, then $\nu \leq f(\mu)$. From the fact that f is FP^*-open, we obtain $f(\mu)$ is an s-τ_{12}^*-open fuzzy set. Now, we have ν is an r-τ_{12}^*-θ-gfc and $\nu \leq f(\mu)$. From Definition 3.1(1) we get, $C_{12}^{*\theta}(\nu, s) \leq f(\mu)$ and thus, $f^{-1}(C_{12}^{*\theta}(\nu, s)) \leq \mu$. Since $C_{12}^{*\theta}(\nu, s)$ is an r-τ_{12}^*-closed fuzzy set in Y and f is GFP^*-θ-continuous. Then, $f^{-1}(C_{12}^{*\theta}(\nu, s))$ is an r-τ_{12}-θ-gfc in X. Thus, from Definition 3.1(1), $C_{12}^{\theta}(f^{-1}(C_{12}^{*\theta})(\nu, s)), s) \leq \mu$ this yields $C_{12}^{\theta}(f^{-1}(\nu), s) \leq \mu$. Therefore, $f^{-1}(\nu)$ is an r-τ_{12}-θ-gfc. Hence, f is GFP^*-θ-irresolute. □

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