THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR $p$-CONVEX FUNCTIONS IN HILBERT SPACE

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Abstract — In this paper, we introduce operator $p$-convex functions and establish some Hermite-Hadamard type inequalities in which some operator $p$-convex functions of positive operators in Hilbert spaces are involved.

Keywords — The Hermite-Hadamard inequality, $p$-convex functions, operator $p$-convex functions, selfadjoint operator, inner product space, Hilbert space.

1 Introduction

The following inequality holds for any convex function $f$ define on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{0}^{1} f(x)dx \leq \frac{f(a) + f(b)}{2} $$ (1)

both inequalities hold in the reversed direction if $f$ is concave.

The inequality (1) is known in the literature as the Hermite-Hadamard’s inequality. The Hermite-Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

In this paper, firstly we defined for bounded positive selfadjoint operator $p$-convex functions in Hilbert space, secondly established some new theorems for them and finally Hermite-Hadamard type inequalities for product two bounded positive selfadjoint operators $p$-convex set up in Hilbert space.

In the paper [1] Dragomir et al. consider $P(I)$. This class is defined in the following way.

**Definition 1.1.** [1] We say that $f : I \to \mathbb{R}$ is a $P$-function, or that $f$ belongs to the class $P(I)$, if $f$ is a non-negative function and for all $x, y \in I, \alpha \in [0, 1]$, we have

$$ f(\alpha x + (1-\alpha)y) \leq f(x) + f(y). $$

For some results about the class $P(I)$ see, e.g., [2] and [3].

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2 Preliminary

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write, for every $x \in H$

$$A \leq B \text{ (or } B \geq A) \text{ if } \langle Ax, x \rangle \leq \langle Bx, x \rangle \text{ (or } \langle Bx, x \rangle \geq \langle Ax, x \rangle)$$

we call it the operator order.

Let $A$ be a selfadjoint linear operator on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $C(Sp(A))$ the $C^*$-algebra of all continuous complex-valued functions on the spectrum $A$. The Gelfand map establishes a *-isometrically isomorphism $\Phi$ between $C(Sp(A))$ and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows [6].

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

i. $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$ ;

ii. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

iii. $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)}|f(t)|$ ;

iv. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$

If $f$ is a continuous complex-valued functions on $C(Sp(A))$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator $A$.

If $A$ is bounded selfadjoint operator and $f$ is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on $H$. Moreover, if both $f$ and $g$ are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq g(A)$ in the operator order $B(H)$.

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\alpha \in [0,1]$ and for every bounded self-adjoint operator $A$ and $B$ in $B(H)$ whose spectra are contained in $I$.

3 Operator $p$-convex Functions in Hilbert Space

The following definition and function class are firstly defined by Seren Salaş.

**Definition 3.1.** Let $I$ be interval in $\mathbb{R}$ and $K$ be a convex subset of $B(H)^+$. A continuous function $f : I \to \mathbb{R}$ is said to be operator $p$-convex on $I$, operators in $K$ if

$$f(\alpha A + (1-\alpha)B) \leq f(A) + f(B)$$

(2)

in the operator order in $B(H)$, for all $\alpha \in [0,1]$ and for every positive operators $A$ and $B$ in $K$ whose spectra are contained in $I$.

In the other words, if $f$ is an operator $p$-convex on $I$, we denote by $f \in SpO$.

**Lemma 3.2.** If $f$ belongs to $SpO$ for operators in $K$, then $f(A)$ is positive for every $A \in K$.

**Proof.** For $A \in K$, we have

$$f(A) = f\left(\frac{A}{2} + \frac{A}{2}\right) \leq f(A) + f(A) = 2f(A).$$

This implies that $f(A) \geq 0$.

Moslehian and Najafi [4] proved the following theorem for positive operators as follows :

**Theorem 3.3.** [4] Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(A + B) \leq f(A) + f(B)$ for all non-negative operator functions $f$ on $[0, \infty)$. 
Dragomir in [5] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

**Theorem 3.4.** [5] Let \( f : I \rightarrow \mathbb{R} \) be an operator convex function on the interval \( I \). Then for all selfadjoint operators \( A, B \) with spectra in \( I \) we have the inequality

\[
\left( f\left( \frac{A + B}{2} \right) \right) \leq \frac{1}{2} \left[ f\left( \frac{3A + B}{4} \right) + f\left( \frac{A + 3B}{4} \right) \right] \leq \int_0^1 f\left( \left( 1 - t \right) A + tB \right) dt \leq \frac{1}{2} \left[ f\left( \frac{A + B}{2} \right) + \frac{f(A) + f(B)}{2} \right] \leq \left( \frac{f(A) + f(B)}{2} \right).
\]

Let \( X \) be a vector space, \( x, y \in X, x \neq y \). Define the segment

\[
[x, y] := (1 - t)x + ty; t \in [0, 1].
\]

We consider the function \( g(x, y) : [0, 1] \rightarrow \mathbb{R} \) and the associated function

\[
g(x, y)(t) := f((1 - t)x + ty), t \in [0, 1].
\]

Note that \( f \) is convex on \([x, y]\) if and only if \( g(x, y) \) is convex on \([0, 1]\). For any convex function defined on a segment \([x, y] \in X\), we have the Hermite-Hadamard integral inequality

\[
f\left( \frac{x + y}{2} \right) \leq \int_0^1 f((1 - t)x + ty) dt \leq \frac{f(x) + f(y)}{2}
\]

which can be derived from the classical Hermite-Hadamard inequality for the convex \( g(x, y) : [0, 1] \rightarrow \mathbb{R} \).

**Lemma 3.5.** Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function on the interval \( I \). Then for every two positive operators \( A, B \in K \subseteq B(H)^+ \) with spectra in \( I \) the function \( f \in S_pO \) for operators in

\[
[A, B] := (1 - t)A + tB; t \in [0, 1]
\]

if and only if the function \( \varphi_{x,A,B} : [0, 1] \rightarrow \mathbb{R} \) defined by

\[
\varphi_{x,A,B} := \langle f((1 - t)A + tB)x, x \rangle
\]

is operator \( p \)-convex on \([0, 1]\) for every \( x \in H \) with \( \|x\| = 1 \).

**Proof.** Since \( f \in S_pO \) operator in \([A, B]\), then for any \( t_1, t_2 \in [0, 1] \) and \( \alpha \in [0, 1] \) we have

\[
\varphi_{x,A,B}(\alpha t_1 + (1 - \alpha)t_2) = \langle f((1 - (\alpha t_1 + (1 - \alpha)t_2)A + (\alpha t_1 + (1 - \alpha)t_2)B)x, x \rangle
\]

\[
= \langle f((\alpha[(1 - t_1)A + t_1B] + (1 - \alpha)][(1 - t_2)A + t_2B])x, x \rangle
\]

\[
\leq \langle f((1 - t_1)A + t_1B)x, x \rangle + f((1 - t_2)A + t_2B)x, x \rangle
\]

\[
\leq \varphi_{x,A,B}(t_1) + \varphi_{x,A,B}(t_2)
\]

**Theorem 3.6.** Let \( f \in S_pO \) on the interval \( I \subseteq [0, \infty) \) for operators \( K \subseteq B(H)^+ \). Then for all positive operators \( A \) and \( B \) in \( K \) with spectra in \( I \), we have the inequality

\[
\frac{1}{2} f\left( \frac{A + B}{2} \right) \leq \int_0^1 f((1 - t)A + tB) dt \leq [f(A) + (B)]
\]
Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have
\[
\langle ((1 - t)A + tB)x, x \rangle = (1 - t)\langle Ax, x \rangle + t\langle Bx, x \rangle \in I,
\]
(4)
Since $\langle Ax, x \rangle \in \text{Sp}(A) \subseteq I$ and $\langle Bx, x \rangle \in \text{Sp}(B) \subseteq I$.
Continuity of $f$ and 4 imply that the operator-valued integral $\int_0^1 f(tA + (1 - t)B)dt$ exists.
Since $f$ is operator $p$-convex, therefore for $t$ in $[0, 1]$, and $A, B \in K$ we have
\[
f(tA + (1 - t)B)dt \leq f(A) + f(B)
\]
(5)
Integrating both sides of 5 over $[0, 1]$ we get the following inequality
\[
\int_0^1 f(tA + (1 - t)B)dt \leq f(A) + f(B)
\]
To prove the first inequality of 3, we observe that
\[
f\left(\frac{A + B}{2}\right) \leq f(tA + (1 - t)B) + f((1 - t)A + tB)
\]
(6)
Integrating the inequality 6 over $t \in [0, 1]$ and taking into account that
\[
\int_0^1 f(tA + (1 - t)B)dt = \int_0^1 f((1 - t)A + tB)dt
\]
then we deduce the first part of 3.

4 The Hermite-Hadamard Type Inequality for the Product Two Operators $p$-convex Functions

Let $f, g \in S_pO$ on the interval in $I$. Then for all positive operators $A$ and $B$ on a Hilbert space $H$
with spectra in $I$, we define real functions $M(A, B)$ and $N(A, B)$ on $H$ by
\[
M(A, B)(x) = \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle \quad (x \in H),
\]
\[
N(A, B)(x) = \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle \quad (x \in H).
\]

Theorem 4.1. Let $f, g \in S_pO$ be on the interval $I$ for operators in $K \subseteq B(H)^+$. Then for all positive operators $A$ and $B$ in $K$ with spectra in $I$, we have the inequality
\[
\int_0^1 \langle f(tA + (1 - t)B)x, x \rangle \langle g(tA + (1 - t)B)x, x \rangle dt
\]
\[
\leq M(A, B) + N(A, B)
\]
hold for any $x \in H$ with $\|x\| = 1$.

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have
\[
\langle (A + B)x, x \rangle = \langle Ax, x \rangle + \langle Bx, x \rangle \in I,
\]
(7)
since $\langle Ax, x \rangle \in \text{Sp}(A) \subseteq I$ and $\langle Bx, x \rangle \in \text{Sp}(B) \subseteq I$.
Continuity of $f, g$ and 7 imply that the operator-valued integrals
\[
\int_0^1 f(tA + (1 - t)B)dt, \int_0^1 g(tA + (1 - t)B)dt \text{ and } \int_0^1 (fg)(tA + (1 - t)B)dt
\]
exist.
Since \( f, g \in S_pO \), therefore for \( t \) in \([0, 1]\) and \( x \in H \) we have

\[
\langle f(tA + (1 - t)B)x, x \rangle \leq \langle f(A) + f(B)x, x \rangle \tag{8}
\]

\[
\langle g(tA + (1 - t)B)x, x \rangle \leq \langle g(A) + g(B)x, x \rangle. \tag{9}
\]

From 8 and 9, we obtain

\[
\langle f(tA + (1 - t)B)x, x \rangle \langle g(tA + (1 - t)B)x, x \rangle \leq \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle. \tag{10}
\]

Integrating both sides of 10 over \([0, 1]\), we get the required inequality 7.

**Theorem 4.2.** Let \( f, g \) belong to \( S_pO \) on the interval \( I \) for operators in \( K \subseteq B(H)^+ \). Then for all positive operators \( A \) and \( B \) in \( K \) with spectra in \( I \), we have the inequality

\[
\frac{1}{2} \left\langle f \left( \frac{A + B}{2} \right)x, x \right\rangle \left\langle g \left( \frac{A + B}{2} \right)x, x \right\rangle \tag{11}
\]

\[
\leq \int_0^1 \left\langle f \left( tA + (1 - t)B \right)x, x \right\rangle \left\langle g \left( tA + (1 - t)B \right)x, x \right\rangle dt + M(A, B) + N(A, B) \tag{12}
\]

hold for any \( x \in H \) with \( \|x\| = 1 \).

**Proof.** Since \( f, g \in S_pO \), therefore for any \( t \) in \( I \) and any \( x \in H \) with \( \|x\| = 1 \), we observe that

\[
\bigg\langle f \left( \frac{A + B}{2} \right)x, x \right\rangle \left\langle g \left( \frac{A + B}{2} \right)x, x \right\rangle \leq \bigg\langle f \left( \frac{tA + (1 - t)B}{2} + \frac{(1 - t)A + tB}{2} \right)x, x \right\rangle \times \left\langle g \left( \frac{tA + (1 - t)B}{2} + \frac{(1 - t)A + tB}{2} \right)x, x \right\rangle \]

\[
\leq \left\{ \langle f(tA + (1 - t)B)x, x \rangle + \langle f((1 - t)A + tB)x, x \rangle \right\} \times \left\{ \langle g(tA + (1 - t)B)x, x \rangle + \langle g((1 - t)A + tB)x, x \rangle \right\} \]

\[
\leq \left\{ \langle f(tA + (1 - t)B)x, x \rangle \langle g(tA + (1 - t)B)x, x \rangle \right\} \]

\[+ \left\{ \langle f((1 - t)A + tB)x, x \rangle \langle g((1 - t)A + tB)x, x \rangle \right\} + \left\{ \langle f(A)x, x \rangle \langle f(B)x, x \rangle \times \left[ \langle g(A)x, x \rangle + \langle g(B)x, x \rangle \right] \right\} \]

\[+ \left\{ \langle f(A)x, x \rangle + \langle f(B)x, x \rangle \times \left[ \langle g(A)x, x \rangle + \langle g(B)x, x \rangle \right] \right\} \]
\[
\begin{align*}
&= \left\{ \left[ \langle f(tA + (1-t)B)x, x \rangle g(tA + (1-t)B)x, x \rangle \right] \\
&\quad + \left[ \langle f((1-t)A+tB)x, x \rangle g((1-t)A+tB)x, x \rangle \right] \\
&\quad + 2 \left[ \langle f(A)x, x \rangle \langle g(A)x, x \rangle \right] + 2 \left[ \langle f(B)x, x \rangle \langle g(B)x, x \rangle \right] \\
&\quad + 2 \left[ \langle f(A)x, x \rangle \langle g(B)x, x \rangle \right] + 2 \left[ \langle f(B)x, x \rangle \langle g(A)x, x \rangle \right] \right\}
\end{align*}
\]

By integration over \([0, 1]\), we obtain
\[
\langle f\left(\frac{A+B}{2}\right)x, x \rangle \langle g\left(\frac{A+B}{2}\right)x, x \rangle 
\leq \int_0^1 \left[ \langle f((1-t)A+tB)x, x \rangle g((1-t)A+tB)x, x \rangle \right] dt \\
+ \langle f(tA + (1-t)B)x, x \rangle \langle g((1-t)A+tB)x, x \rangle \right] dt
\]
+ 2M(A, B) + 2N(A, B)
\]

This implies the inequality 11.

References


