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$\ddot{g}$-LOCALLY CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract − The aim of this paper is to introduce and study the classes of $\ddot{g}$-locally closed sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets which are weaker forms of the class of locally closed sets. Furthermore the relations with other notions connected with the forms of locally closed sets are investigated.

Keywords − *Topological space, g-closed set, $\ddot{g}$-closed set, $\ddot{g}$-lc* set and $\ddot{g}$-lc** set.*

1 Introduction

The first step of locally closedness was done by Bourbaki [4]. He defined a set A to be locally closed if it is the intersection of an open set and a closed set. In literature many general topologists introduced the studies of locally closed sets. Extensive research on locally closedness and generalizing locally closedness were done in recent years. Stone [20] used the term FG for a locally closed set. Ganster and Reilly used locally closed sets in [7] to define LC-continuity and LC-irresoluteness. Balachandran et al [2] introduced the concept of generalized locally closed sets. Veera Kumar [23] (Sheik John [19]) introduced $\hat{g}$-locally closed sets (=ω-locally closed sets) respectively.

In this paper, we introduce three forms of locally closed sets called $\ddot{g}$-locally closed sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets. Properties of these new concepts are studied as well as their relations to the other classes of locally closed sets will be investigated.

2 Preliminaries

Throughout this paper (X, τ) (or X) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A, respectively.

We recall the following definitions, Corollary and Remarks which are useful in the sequel.

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Definition 2.1. *A subset A of a space (X, τ) is called:*

1. *semi-open set [10] if A$\subseteq$cl(int(A));*
2. *α-open set [11] if A$\subseteq$int(cl(int(A)));*
3. *regular open set [21] if A=int(cl(A)).*

The complements of the above mentioned open sets are called their respective closed sets.

The semi-closure [5] of a subset A of X, denoted by scl(A), is defined to be the intersection of all semi-closed sets of (X, τ) containing A. It is known that scl(A) is a semi-closed set.

Definition 2.2. *A subset A of a space (X, τ) is called*

1. *a generalized closed (briefly g-closed) set [9] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is open in (X, τ). The complement of g-closed set is called g-open set;*
2. *a semi-generalized closed (briefly sg-closed) set [3] if scl(A)$\subseteq$U whenever A$\subseteq$U and U is semi-open in (X, τ). The complement of sg-closed set is called sg-open set;*
3. *a regular generalized closed (briefly rg-closed) set [12] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is regular open in (X, τ). The complement of rg-closed set is called rg-open set;*
4. *a $\hat{g}$-closed set [22] (=ω-closed set [19]) if cl(A)$\subseteq$U whenever A$\subseteq$U and U is semi-open in (X, τ). The complement of $\hat{g}$-closed set is called $\hat{g}$-open set;*
5. *a $\ddot{g}$-closed set [15] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is sg-open in (X, τ). The complement of $\ddot{g}$-closed set is called $\ddot{g}$-open set.*

Remark 2.3. *The collection of all $\ddot{g}$-closed (resp. ω-closed, g-closed, rg-closed, sg-closed) sets in X is denoted by $\ddot{G}C(X)$ (resp. $\omega C(X)$, GC(X), RGC(X), SGC(X)).*

The collection of all $\ddot{g}$-open (resp. ω-open, g-open, rg-open, sg-open) sets in X is denoted by $\ddot{G}O(X)$ (resp. $\omega O(X)$, GO(X), RGO(X), SGO(X)).

We denote the power set of X by P(X).

Definition 2.4. *A subset S of a space (X, τ) is called:*

1. *locally closed (briefly lc) [7] if S=U$\cap$F, where U is open and F is closed in (X, τ).*
2. *generalized locally closed (briefly glc) [2] if S=U$\cap$F, where U is g-open and F is g-closed in (X, τ).*
3. *semi-generalized locally closed (briefly sglc) [13] if S=U$\cap$F, where U is sg-open and F is sg-closed in (X, τ).*
4. *regular-generalized locally closed (briefly rg-lc) [1] if S=U$\cap$F, where U is rg-open and F is rg-closed in (X, τ).*
5. *generalized locally semi-closed (briefly glsc) [8] if S=U$\cap$F, where U is g-open and F is semi-closed in (X, τ).*
6. *locally semi-closed (briefly lsc) [8] if S=U$\cap$F, where U is open and F is semi-closed in (X, τ).*
7. *α-locally closed (briefly α-lc) [8] if S=U$\cap$F, where U is α-open and F is α-closed in (X, τ).*
8. *ω-locally closed (briefly ω-lc) [19] if S=U$\cap$F, where U is ω-open and F is ω-closed in (X, τ).*
9. *$sglc^{\star}$ [13] if S=U$\cap$F, where U is sg-open and F is closed in (X, τ).*

The class of all locally closed (resp. generalized locally closed, generalized locally semi-closed, locally semi-closed, ω-locally closed) sets in X is denoted by LC(X) (resp. GLC(X), GLSC(X), LSC(X), ω-LC(X)).

Definition 2.5. *[16] For any A$\subseteq$X, $\ddot{g}$-int(A) is defined as the union of all $\ddot{g}$-open sets contained in A. i.e., $\ddot{g}$-int(A)=$\cup\{G : G\subseteq A$ and G is $\ddot{g}$-open$\}$.*

Definition 2.6. *[16] For every set $A\subseteq X$, we define the $\ddot{g}$-closure of A to be the intersection of all $\ddot{g}$-closed sets containing A. ie., $\ddot{g}$-cl(A)$=\cap\{F : A\subseteq F\in \ddot{G}C(X)\}$.*

Definition 2.7. *[17] A space (X, τ) is called a $T_{\ddot{g}}$-space if every $\ddot{g}$-closed set in it is closed.*

Recall that a subset A of a space (X, τ) is called dense if cl(A)=X.

Definition 2.8. *A topological space (X, τ) is called:*

1. *submaximal [6, 23] if every dense subset is open.*
2. *$\hat{g}$ (or ω)-submaximal [19, 23] if every dense subset is ω-open.*
3. *g-submaximal [2] if every dense subset is g-open.*
4. *rg-submaximal [12] if every dense subset is rg-open.*

Remark 2.9. *For a topological space X, the following statements hold:*

1. *Every closed set is $\ddot{g}$-closed but not conversely [15].*
2. *Every $\ddot{g}$-closed set is ω-closed but not conversely [15].*
3. *Every $\ddot{g}$-closed set is g-closed but not conversely [15].*
4. *Every $\ddot{g}$-closed set is sg-closed but not conversely [15].*
5. *Every $\ddot{g}$-open set is ω-open but not conversely [18].*
6. *A subset A of X is $\ddot{g}$-closed if and only if $\ddot{g}$-cl(A)=A [16].*
7. *A subset A of X is $\ddot{g}$-open if and only if $\ddot{g}$-int(A)=A [16].*

Corollary 2.10. *[15] If A is a $\ddot{g}$-closed set and F is a closed set, then $A\cap F$ is a $\ddot{g}$-closed set.*

Theorem 2.11. *[23] Let (X, τ) be a topological space.*

1. *If X is submaximal, then X is $\hat{g}$-submaximal.*
2. *If X is $\hat{g}$-submaximal, then X is g-submaximal.*
3. *If X is g-submaximal, then X is rg-submaximal.*
4. *The respective converses of the above need not be true in general.*

3 $\ddot{g}$-locally Closed Sets

We introduce the following definition.

Definition 3.1. *A subset A of (X, τ) is called $\ddot{g}$-locally closed (briefly $\ddot{g}$-lc) if $A=S\cap G$, where S is $\ddot{g}$-open and G is $\ddot{g}$-closed in (X, τ).*

The class of all $\ddot{g}$-locally closed sets in X is denoted by $\ddot{G}$LC(X).

Proposition 3.2. *Every $\ddot{g}$-closed (resp. $\ddot{g}$-open) set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Definition 3.1.

Example 3.3. *Let $X=\{a, b, c\}$ with $\tau=\{\emptyset, \{b\}, X\}$. Then the set $\{b\}$ is $\ddot{g}$-lc set but it is not $\ddot{g}$-closed and the set $\{a, c\}$ is $\ddot{g}$-lc set but it is not $\ddot{g}$-open in (X, τ).*

Proposition 3.4. *Every lc set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Remark 2.9 (1).

Example 3.5. *Let $X=\{a, b, c\}$ with $\tau=\{\emptyset, \{b, c\}, X\}$. Then the set $\{b\}$ is $\ddot{g}$-lc set but it is not lc set in (X, τ).*

Proposition 3.6. *Every $\ddot{g}$-lc set is a (1) ω-lc set, (2) glc set and (3) sglc set. However the separate converses are not true.*

Proof. It follows from Remark 2.9 (2), (3) and (4).

Example 3.7. *Let X={a, b, c} with τ={$\emptyset$, {a}, X}. Then the set {b} is g-lc set but it is not $\ddot{g}$-lc set in (X, τ). Moreover, the set {c} is sg-lc set but it is not $\ddot{g}$-lc set in (X, τ).*

Example 3.8. *Let X={a, b, c} with τ={$\emptyset$, {b}, {a, c}, X}. Then the set {a} is ω-lc set but it is not $\ddot{g}$-lc set in (X, τ).*

Remark 3.9. *The concepts of α-lc set and $\ddot{g}$-lc set are independent of each other.*

Example 3.10. *The set {b, c} in Example 3.3 is α-lc set but it is not a $\ddot{g}$-lc set in (X, τ) and the set {a, b} in Example 3.5 is $\ddot{g}$-lc set but it is not an α-lc set in (X, τ).*

Remark 3.11. *The concepts of lsc set and $\ddot{g}$-lc set are independent of each other.*

Example 3.12. *The set {a} in Example 3.3 is lsc set but it is not a $\ddot{g}$-lc set in (X, τ) and the set {a, b} in Example 3.5 is $\ddot{g}$-lc set but it is not a lsc set in (X, τ).*

Remark 3.13. *The concepts of $\ddot{g}$-lc set and glsc set are independent of each other.*

Example 3.14. *The set {b, c} in Example 3.3 is glsc set but it is not a $\ddot{g}$-lc set in (X, τ) and the set {a, b} in Example 3.5 is $\ddot{g}$-lc set but it is not a glsc set in (X, τ).*

Remark 3.15. *The concepts of $\ddot{g}$-lc set and sglc* set are independent of each other.*

Example 3.16. *The set {b, c} in Example 3.3 is sglc* set but it is not a $\ddot{g}$-lc set in (X, τ) and the set {a, b} in Example 3.5 is $\ddot{g}$-lc set but it is not a sglc* set in (X, τ).*

Theorem 3.17. *For a $T_{\ddot{g}}$-space (X, τ), the following properties hold:*

1. *$\ddot{G}$LC(X)=LC(X).*
2. *$\ddot{G}$LC(X)$\subseteq$GLC(X).*
3. *$\ddot{G}$LC(X)$\subseteq$GLSC(X).*
4. *$\ddot{G}$LC(X)$\subseteq\omega$-LC(X).*

Proof. (1) Since every $\ddot{g}$-open set is open and every $\ddot{g}$-closed set is closed in (X, τ), $\ddot{G}$LC(X)$\subseteq$LC(X) and hence $\ddot{G}$LC(X)=LC(X).

(2), (3) and (4) follows from (1), since for any space (X, τ), LC(X)$\subseteq$GLC(X), LC(X)$\subseteq$GLSC(X) and LC(X)$\subseteq\omega$-LC(X).

Corollary 3.18. *If GO(X)=τ, then $\ddot{G}$LC(X)$\subseteq$GLSC(X)$\subseteq$LSC(X).*

Proof. GO(X)=τ implies that (X, τ) is a $T_{\ddot{g}}$-space and hence by Theorem 3.17, $\ddot{G}$LC(X) $\subseteq$GLSC(X). Let A$\in$GLSC(X). Then A=U$\cap$F, where U is g-open and F is semi-closed. By hypothesis, U is open and hence A is a lsc-set and so A$\in$LSC(X).

Definition 3.19. *A subset A of a space (X, τ) is called*

1. *$\ddot{g}$-lc* set if A=S$\cap$G, where S is $\ddot{g}$-open in (X, τ) and G is closed in (X, τ).*
2. *$\ddot{g}$-lc** set if A=S$\cap$G, where S is open in (X, τ) and G is $\ddot{g}$-closed in (X, τ).*

The class of all $\ddot{g}$-lc* (resp. $\ddot{g}$-lc**) sets in a topological space (X, τ) is denoted by $\ddot{G}$LC*(X) (resp. $\ddot{G}$LC**(X)).

Proposition 3.20. *Every lc set is $\ddot{g}$-lc* set but not conversely.*

Proof. It follows from Definitions 2.4 (1) and 3.19 (1).

Example 3.21. *The set {b} in Example 3.5 is $\ddot{g}$-lc* set but it is not a lc set in (X, τ).*

Proposition 3.22. *Every lc set is $\ddot{g}$-lc** set but not conversely.*

Proof. It follows from Definitions 2.4 (1) and 3.19 (2).

Example 3.23. *The set {a, c} in Example 3.5 is $\ddot{g}$-lc** set but it is not a lc set in (X, τ).*

Proposition 3.24. *Every $\ddot{g}$-lc* set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Definitions 3.1 and 3.19 (1).

Example 3.25. *The set {a, b} in Example 3.5 is $\ddot{g}$-lc set but it is not a $\ddot{g}$-lc* set in (X, τ).*

Proposition 3.26. *Every $\ddot{g}$-lc** set is $\ddot{g}$-lc set but not conversely.*

Proof. It follows from Definitions 3.1 and 3.19 (2).

Question 1. Give an example for a set which is $\ddot{g}$-lc set but not $\ddot{g}$-lc** set.

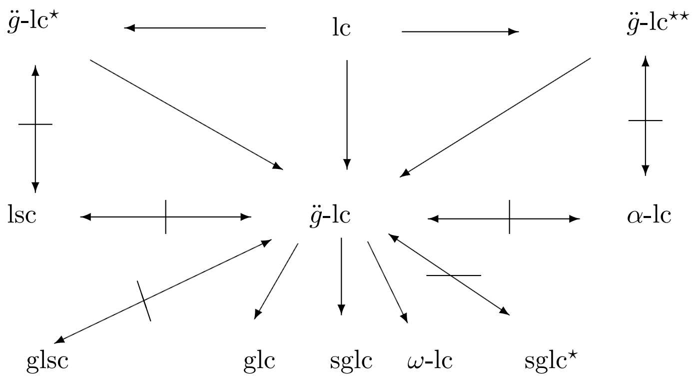
Remark 3.27. *The concepts of $\ddot{g}$-lc* set and lsc set are independent of each other.*

Example 3.28. *The set {c} in Example 3.5 is $\ddot{g}$-lc* set but it is not a lsc set in (X, τ) and the set {a} in Example 3.3 is lsc set but it is not a $\ddot{g}$-lc* set in (X, τ).*

Remark 3.29. *The concepts of $\ddot{g}$-lc** set and α-lc set are independent of each other.*

Example 3.30. *The set {a, b} in Example 3.5 is $\ddot{g}$-lc** set but it is not an α-lc set in (X, τ) and the set {a, b} in Example 3.3 is α-lc set but it is not a $\ddot{g}$-lc** set in (X, τ).*

Remark 3.31. *From the above discussions we have the following implications where A $\rightarrow$ B (resp. A $\nleftrightarrow$ B) represents A implies B but not conversely (resp. A and B are independent of each other).*



Proposition 3.32. *If GO(X)=τ, then $\ddot{G}LC$(X)=$\ddot{G}LC^{\star}$(X)=$\ddot{G}LC^{\star\star}$(X).*

Proof. For any space (X, τ), $\tau\subseteq\ddot{G}O(X)\subseteq$GO(X). Therefore by hypothesis, $\ddot{G}O(X)=\tau$. i.e., (X, τ) is a T$_{\ddot{g}}$-space and hence $\ddot{G}$LC(X)=$\ddot{G}$LC*(X)=$\ddot{G}$LC**(X).

Remark 3.33. *The converse of Propositions 3.32 need not be true.*

For the topological space (X, τ) in Example 3.3. $\ddot{G}$LC(X)=$\ddot{G}$LC*(X)=$\ddot{G}$LC**(X). However GO(X)={$\emptyset$, {a}, {b}, {c}, {a, b}, {b, c}, X}$\neq\tau$.

Proposition 3.34. *Let (X, τ) be a topological space. If GO(X)$\subseteq$LC(X), then $\ddot{G}LC$(X) =$\ddot{G}LC^{\star\star}$(X).*

Proof. Let A$\in\ddot{G}$LC(X). Then A=S$\cap$G where S is $\ddot{g}$-open and G is $\ddot{g}$-closed. Since $\ddot{G}O(X)\subseteq$GO(X) and by hypothesis GO(X)$\subseteq$LC(X), S is locally closed. Then S=P$\cap$Q, where P is open and Q is closed. Therefore, A=P$\cap$(Q$\cap$G). By Corollary 2.10, Q$\cap$G is $\ddot{g}$-closed and hence A$\in\ddot{G}$LC**(X). i.e., $\ddot{G}$LC(X)$\subseteq\ddot{G}$LC**(X). For any topological space, $\ddot{G}$LC**(X)$\subseteq\ddot{G}$LC(X) and so $\ddot{G}$LC(X)=$\ddot{G}$LC**(X).

Remark 3.35. *The converse of Proposition 3.34 need not be true in general.*

For the topological space (X, τ) in Example 3.3, then $\ddot{G}$LC(X)=$\ddot{G}$LC**(X)={$\emptyset$, {b}, {a, c}, X}. But GO(X)={$\emptyset$, {a}, {b}, {c}, {a, b}, {b, c}, X}$\nsubseteq$LC(X)={$\emptyset$, {b}, {a, c}, X}.

Corollary 3.36. *Let (X, τ) be a topological space. If ωO(X)$\subseteq$LC(X), then $\ddot{G}$LC(X) =$\ddot{G}$LC**(X).*

Proof. It follows from the fact that ωO(X)$\subseteq$GO(X) and Proposition 3.34.

Remark 3.37. *The converse of Corollary 3.36 need not be true in general.*

For the topological space (X, τ) in Example 3.8, then $\ddot{G}$LC(X)=$\ddot{G}$LC**(X)={$\emptyset$, {b}, {a,c}, X}. But ωO(X)=P(X)$\nsubseteq$LC(X)={$\emptyset$, {b}, {a, c}, X}.

The following results are characterizations of $\ddot{g}$-lc sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets.

Theorem 3.38. *For a subset A of (X, τ) the following statements are equivalent:*

1. *A$\in\ddot{G}$LC(X),*
2. *A=S$\cap\ddot{g}$-cl(A) for some $\ddot{g}$-open set S,*
3. *$\ddot{g}$-cl(A)$-$A is $\ddot{g}$-closed,*
4. *A$\cup$($\ddot{g}$-cl(A))^c is $\ddot{g}$-open,*
5. *A$\subseteq\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c).*

Proof. (1) $\Rightarrow$ (2). Let A$\in\ddot{G}$LC(X). Then A=S$\cap$G where S is $\ddot{g}$-open and G is $\ddot{g}$-closed. Since A$\subseteq$G, $\ddot{g}$-cl(A)$\subseteq$G and so S$\cap\ddot{g}$-cl(A)$\subseteq$A. Also A$\subseteq$S and A$\subseteq\ddot{g}$-cl(A) implies A$\subseteq$S$\cap\ddot{g}$-cl(A) and therefore A=S$\cap\ddot{g}$-cl(A).

(2) $\Rightarrow$ (3). A=S$\cap\ddot{g}$-cl(A) implies $\ddot{g}$-cl(A)$-$A=$\ddot{g}$-cl(A)$\cap$S^c which is $\ddot{g}$-closed since S^c is $\ddot{g}$-closed and $\ddot{g}$-cl(A) is $\ddot{g}$-closed.

(3) $\Rightarrow$ (4). A$\cup$($\ddot{g}$-cl(A))^c=($\ddot{g}$-cl(A)$-$A)^c and by assumption, ($\ddot{g}$-cl(A)$-$A)^c is $\ddot{g}$-open and so is A$\cup$($\ddot{g}$-cl(A))^c.

(4) $\Rightarrow$ (5). By assumption, A$\cup$($\ddot{g}$-cl(A))^c=$\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c) and hence A$\subseteq\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c).

(5) $\Rightarrow$ (1). By assumption and since A$\subseteq\ddot{g}$-cl(A), A=$\ddot{g}$-int(A$\cup$($\ddot{g}$-cl(A))^c)$\cap\ddot{g}$-cl(A). Therefore, A$\in\ddot{G}$LC(X).

Theorem 3.39. *For a subset A of (X, τ), the following statements are equivalent:*

1. *A$\in\ddot{G}$LC*(X),*
2. *A=S$\cap$cl(A) for some $\ddot{g}$-open set S,*
3. *cl(A)$-$A is $\ddot{g}$-closed,*
4. *A$\cup$(cl(A))^c is $\ddot{g}$-open.*

Proof. (1) $\Rightarrow$ (2). Let A$\in\ddot{G}$LC*(X). There exist an $\ddot{g}$-open set S and a closed set G such that A=S$\cap$G. Since A$\subseteq$S and A$\subseteq$cl(A), A$\subseteq$S$\cap$cl(A). Also since cl(A)$\subseteq$G, S$\cap$cl(A)$\subseteq$S$\cap$G=A. Therefore A=S$\cap$cl(A).

(2) $\Rightarrow$ (1). Since S is $\ddot{g}$-open and cl(A) is a closed set, A=S$\cap$cl(A)$\in\ddot{G}$LC*(X).

(2) $\Rightarrow$ (3). Since cl(A)$-$A=cl(A)$\cap$S^c, cl(A)$-$A is $\ddot{g}$-closed by Corollary 2.10.

(3) $\Rightarrow$ (2). Let S=(cl(A)$-$A)^c. Then by assumption S is $\ddot{g}$-open in (X, τ) and A=S$\cap$cl(A).

(3) $\Rightarrow$ (4). Let G=cl(A)$-$A. Then G^c=A$\cup$(cl(A))^c and A$\cup$(cl(A))^c is $\ddot{g}$-open.

(4) $\Rightarrow$ (3). Let S=A$\cup$(cl(A))^c. Then S^c is $\ddot{g}$-closed and S^c=cl(A)$-$A and so cl(A)$-$A is $\ddot{g}$-closed.

Theorem 3.40. *Let A be a subset of (X, τ). Then A$\in\ddot{G}$LC**(X) if and only if A=S$\cap\ddot{g}$-cl(A) for some open set S.*

Proof. Let A$\in\ddot{G}$LC**(X). Then A=S$\cap$G where S is open and G is $\ddot{g}$-closed. Since A$\subseteq$G, $\ddot{g}$-cl(A)$\subseteq$G. We obtain A=A$\cap\ddot{g}$-cl(A)=S$\cap$G$\cap\ddot{g}$-cl(A)=S$\cap\ddot{g}$-cl(A).

Converse part is trivial.

Corollary 3.41. *Let A be a subset of (X, τ). If A$\in\ddot{G}$LC**(X), then $\ddot{g}$-cl(A)$-$A is $\ddot{g}$-closed and A$\cup$($\ddot{g}$-cl(A))^c is $\ddot{g}$-open.*

Proof. Let A$\in\ddot{G}$LC**(X). Then by Theorem 3.40, A=S$\cap\ddot{g}$-cl(A) for some open set S and $\ddot{g}$-cl(A)$-$A=$\ddot{g}$-cl(A)$\cap$S^c is $\ddot{g}$-closed in (X, τ). If G=$\ddot{g}$-cl(A)$-$A, then G^c=A$\cup$($\ddot{g}$-cl(A))^c and G^c is $\ddot{g}$-open and so is A$\cup$($\ddot{g}$-cl(A))^c.

4 $\ddot{g}$-dense Sets and $\ddot{g}$-submaximal Spaces

We introduce the following definition.

Definition 4.1. *A subset A of a space (X, τ) is called $\ddot{g}$-dense if $\ddot{g}$-cl(A)=X.*

Example 4.2. *Consider the topological space (X, τ) in Example 3.5. Then the set A={b, c} is $\ddot{g}$-dense in (X, τ).*

Proposition 4.3. *Every $\ddot{g}$-dense set is dense.*

Proof. Let A be an $\ddot{g}$-dense set in (X, τ). Then $\ddot{g}$-cl(A)=X. Since $\ddot{g}$-cl(A)⊆cl(A), we have cl(A)=X and so A is dense.

The converse of Proposition 4.3 need not be true as can be seen from the following example.

Example 4.4. *The set {a, c} in Example 3.5 is a dense in (X, τ) but it is not $\ddot{g}$-dense in (X, τ).*

Definition 4.5. *A topological space (X, τ) is called $\ddot{g}$-submaximal if every dense subset in it is $\ddot{g}$-open in (X, τ).*

Proposition 4.6. *Every submaximal space is $\ddot{g}$-submaximal.*

Proof. Let (X, τ) be a submximal space and A be a dense subset of (X, τ). Then A is open. But every open set is $\ddot{g}$-open and so A is $\ddot{g}$-open. Therefore (X, τ) is $\ddot{g}$-submaximal.

The converse of Proposition 4.6 need not be true as can be seen from the following example.

Example 4.7. *For the topological space (X, τ) of Example 3.5, every dense subset is $\ddot{g}$-open and hence (X, τ) is $\ddot{g}$-submaximal. However, the set A={a, b} is dense in (X, τ), but it is not open in (X, τ). Therefore (X, τ) is not submaximal.*

Proposition 4.8. *Every $\ddot{g}$-submaximal space is ω-submaximal.*

Proof. Let (X, τ) be an $\ddot{g}$-submaximal space and A be a dense subset of (X, τ). Then A is $\ddot{g}$-open. But every $\ddot{g}$-open set is ω-open [Remark 2.9 (5)] and so A is ω-open. Therefore is (X, τ) is ω-submaximal.

The converse of Proposition 4.8 need not be true as can be seen from the following example.

Example 4.9. *Consider the topological space (X, τ) in Example 3.8. Then (X, τ) is ω-submaximal but it is not $\ddot{g}$-submaximal, because the set A={b, c} is a dense set in (X, τ) but it is not $\ddot{g}$-open in (X, τ).*

Remark 4.10. *From Propositions 4.6, 4.8 and Theorem 2.11, we have the following diagram:*

submaximal ⟶ $\ddot{g}$-submaximal ⟶ ω-submaximal ⟶ g-submaximal
↓
rg-submaximal

Theorem 4.11. *A space (X, τ) is $\ddot{g}$-submaximal if and only if P(X)=$\ddot{G}$LC*(X).*

Proof. Necessity. Let A∈P(X) and let V=A∪(cl(A))^c. This implies that cl(V)=cl(A)∪ (cl(A))^c=X. Hence cl(V)=X. Therefore V is a dense subset of X. Since (X, τ) is $\ddot{g}$-submaximal, V is $\ddot{g}$-open. Thus A∪(cl(A))^c is $\ddot{g}$-open and by Theorem 3.39, we have A∈$\ddot{G}$LC*(X).

Sufficiency. Let A be a dense subset of (X, τ). This implies A∪(cl(A))^c=A∪X^c=A∪∅ =A. Now A∈$\ddot{G}$LC*(X) implies that A=A∪(cl(A))^c is $\ddot{g}$-open by Theorem 3.39. Hence (X, τ) is $\ddot{g}$-submaximal.

Proposition 4.12. *For subsets A and B in (X, τ), the following are true:*

1. *If A, B∈$\ddot{G}$LC(X), then A∩B∈$\ddot{G}$LC(X).*
2. *If A, B∈$\ddot{G}$LC*(X), then A∩B∈$\ddot{G}$LC*(X).*
3. *If A, B∈$\ddot{G}$LC**(X), then A∩B∈$\ddot{G}$LC**(X).*
4. *If A∈$\ddot{G}$LC(X) and B is $\ddot{g}$-open (resp. $\ddot{g}$-closed), then A∩B∈$\ddot{G}$LC(X).*

5. *If $A \in \ddot{G}LC^{\star}(X)$ and B is $\ddot{g}$-open (resp. closed), then $A \cap B \in \ddot{G}LC^{\star}(X)$.*
6. *If $A \in \ddot{G}LC^{\star\star}(X)$ and B is $\ddot{g}$-closed (resp. open), then $A \cap B \in \ddot{G}LC^{\star\star}(X)$.*
7. *If $A \in \ddot{G}LC^{\star}(X)$ and B is $\ddot{g}$-closed, then $A \cap B \in \ddot{G}LC(X)$.*
8. *If $A \in \ddot{G}LC^{\star\star}(X)$ and B is $\ddot{g}$-open, then $A \cap B \in \ddot{G}LC(X)$.*
9. *If $A \in \ddot{G}LC^{\star\star}(X)$ and $B \in \ddot{G}LC^{\star}(X)$, then $A \cap B \in \ddot{G}LC(X)$.*

Proof. By Remark 2.9 and Corollary 2.10., (1) to (8) hold.

(9). Let A=S∩G where S is open and G is $\ddot{g}$-closed and B=P∩Q where P is $\ddot{g}$-open and Q is closed. Then A∩B=(S∩P)∩(G∩Q) where S∩P is $\ddot{g}$-open and G∩Q is $\ddot{g}$-closed, by Corollary 2.10. Therefore A∩B∈$\ddot{G}$LC(X).

Remark 4.13. *Union of two $\ddot{g}$-lc sets (resp. $\ddot{g}$-lc* sets, $\ddot{g}$-lc** sets) need not be an $\ddot{g}$-lc set (resp. $\ddot{g}$-lc* set, $\ddot{g}$-lc** set) as can be seen from the following examples.*

Example 4.14. *Let X={a, b, c} with τ={∅, {a}, {a, b}, X}. Then $\ddot{G}LC(X)$={∅, {a}, {b}, {c}, {a, b}, {b, c}, X}. Then the sets {a} and {c} are $\ddot{g}$-lc sets, but their union {a, c}∉$\ddot{G}LC(X)$.*

Example 4.15. *Let X={a, b, c} with τ ={∅, {b}, {a, b}, X}. Then $\ddot{G}LC^{\star}(X)$={∅, {a}, {b}, {c}, {a, b}, {a, c}, X}. Then the sets {b} and {c} are $\ddot{g}$-lc* sets, but their union {b, c}∉$\ddot{G}LC^{\star}(X)$.*

Example 4.16. *Let X={a, b, c} with τ={∅, {b}, {b, c}, X}. Then $\ddot{G}LC^{\star\star}(X)$={∅, {a}, {b}, {c}, {a, c}, {b, c}, X}. Then the sets {a} and {b} are $\ddot{g}$-lc** sets, but their union {a, b}∉$\ddot{G}LC^{\star\star}(X)$.*

We introduce the following definition.

Definition 4.17. *Let A and B be subsets of (X, τ). Then A and B are said to be $\ddot{g}$-separated if A∩$\ddot{g}$-cl(B)=∅ and $\ddot{g}$-cl(A)∩B=∅.*

Example 4.18. *For the topological space (X, τ) of Example 3.5. Let A={b} and let B={c}. Then $\ddot{g}$-cl(A)={a, b} and $\ddot{g}$-cl(B)={a, c} and so the sets A and B are $\ddot{g}$-separated.*

Proposition 4.19. *For a topological space (X, τ), the followings are true:*

1. *Let A, B∈$\ddot{G}LC(X)$. If A and B are $\ddot{g}$-separated then A∪B∈$\ddot{G}LC(X)$.*
2. *Let A, B∈$\ddot{G}LC^{\star}(X)$. If A and B are separated (i.e., A∩cl(B)=∅ and cl(A)∩B=∅), then A∪B∈$\ddot{G}LC^{\star}(X)$.*
3. *Let A, B∈$\ddot{G}LC^{\star\star}(X)$. If A and B are $\ddot{g}$-separated then A∪B∈$\ddot{G}LC^{\star\star}(X)$.*

Proof. (1) Since A, B∈$\ddot{G}$LC(X), by Theorem 3.38, there exist $\ddot{g}$-open sets U and V of (X, τ) such that A=U∩$\ddot{g}$-cl(A) and B=V∩$\ddot{g}$-cl(B). Now G=U∩(X−$\ddot{g}$-cl(B)) and H=V∩(X−$\ddot{g}$-cl(A)) are $\ddot{g}$-open subsets of (X, τ). Since A∩$\ddot{g}$-cl(B)=∅, A⊆$(\ddot{g}\text{-cl(B)})^c$. Now A=U∩$\ddot{g}$-cl(A) becomes A∩$(\ddot{g}\text{-cl(B)})^c$=G∩$\ddot{g}$-cl(A). Then A=G∩$\ddot{g}$-cl(A). Similarly B=H∩$\ddot{g}$-cl(B). Moreover G∩$\ddot{g}$-cl(B)=∅ and H∩$\ddot{g}$-cl(A)=∅. Since G and H are $\ddot{g}$-open sets of (X, τ), G∪H is $\ddot{g}$-open. Therefore A∪B=(G∪H)∩$\ddot{g}$-cl(A∪B) and hence A∪B∈$\ddot{G}$LC(X).

(2) and (3) are similar to (1), using Theorems 3.39 and 3.40.

Remark 4.20. *The assumption that A and B are $\ddot{g}$-separated in (1) of Proposition 4.19 cannot be removed. In the topological space (X, τ) in Example 4.14, the sets {a} and {c} are not $\ddot{g}$-separated and their union {a, c}∉$\ddot{G}LC(X)$.*

5 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

Though the concept of topology has been identified as a difficult territory in Mathematics, we have taken it up as a challenge and cherishingly worked out this research study. It can also further up the understanding of basic structure of classical mathematics and offers new methods and results in obtaining significant results of classical mathematics. Moreover it also has applications in some important fields of Science and Technology.

In this paper we introduced and studied the classes of $\ddot{g}$-locally closed sets, $\ddot{g}$-lc* sets and $\ddot{g}$-lc** sets which are weaker forms of the class of locally closed sets. Furthermore the relations with other notions connected with the forms of locally closed sets are investigated.

References

[1] Arockiarani, I., Balachandran, K. and Ganster, M.: Regular-generalized locally closed sets and RGL-continuous functions, Indian J. Pure Appl. Math., 28 (1997), 661-669.

[2] Balachandran, K., Sundaram, P. and Maki, H.: Generalized locally closed sets and GLC-continuous functions, Indian J. Pure Appl. Math., 27 (3) (1996), 235-244.

[3] Bhattacharya, P. and Lahiri, B. K.: Semi-generalized closed sets in topology, Indian J. Math., 29 (3) (1987), 375-382.

[4] Bourbaki, N.: General topology, Part I, Addison-Wesley, Reading, Mass., 1966.

[5] Crossley, S. G. and Hildebrand, S. K.: Semi-closure, Texas J. Sci, 22 (1971), 99-112.

[6] Dontchev, J.: On submaximal spaces, Tamkang J. Math., 26 (1995), 253-260.

[7] Ganster, M. and Reilly, I. L.: Locally closed sets and LC-continuous functions, Internat J. Math. Math. Sci., 12 (3) (1989), 417-424.

[8] Gnanambal, Y.: Studies on generalized pre-regular closed sets and generalization of locally closed sets, Ph.D Thesis, Bharathiar University, Coimbatore 1998.

[9] Levine, N.: Generalized closed sets in topology, Rend. Circ. Math. Palermo, 19 (2) (1970), 89-96.

[10] Levine, N.: Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

[11] Njastad, O.: On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.

[12] Palaniappan, N. and Rao, K. C.: Regular generalized closed sets, Kyungpook Math. J., 33 (1993), 211-219.

[13] Park, J. H. and Park, J. K.: On semi-generalized locally closed sets and SGLC-continuous functions, Indian J. Pure Appl. Math., 31 (9) (2000), 1103-1112.

[14] Rajesh, N. and Ekici, E.: $\tilde{g}$-Locally closed sets in topological spaces, Kochi J. Math., 2 (2007), 1-9.

[15] Ravi, O. and Ganesan, S.: $\ddot{g}$-closed sets in topology, International Journal of Computer Science and Emerging Technologies, 2 (3) (2011), 330-337.

[16] Ravi, O. and Ganesan, S.: $\ddot{g}$-interior and $\ddot{g}$-closure in topological spaces, Bessel Journal of Mathematics (To appear).

[17] Ravi, O. and Ganesan, S.: On $T_{\ddot{g}}$-spaces, Journal of Advanced Studies in Topology, 2 (2) (2011), 1-6.

[18] Ravi, O. and Ganesan, S.: On $\ddot{g}$-continuous maps in topological spaces (submitted).

[19] Sheik John, M.: A study on generalizations of closed sets and continuous maps in topological and bitopological spaces, Ph.D Thesis, Bharathiar University, Coimbatore, September 2002.

[20] Stone, A. H.: Absolutely FG spaces, Proc. Amer. Math. Soc., 80 (1980), 515-520.

[21] Stone, M. H.: Application of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41 (1937), 374-481.

[22] Veera Kumar, M. K. R. S.: $\hat{g}$-closed sets in Topological spaces, Bull. Allahabad Math. Soc., 18 (2003), 99-112.

[23] Veera Kumar, M. K. R. S.: $\hat{g}$-locally closed sets and $\hat{G}$LC-functions, Indian J. Math., 43 (2) (2001), 231-247.