http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 26.02.2015
Accepted: 10.07.2015

Year: **2015**, *Number*: **6** , *Pages*: **01-19**
Original Article[**]

FUZZY SOFT SET PREFERENCE RELATION

Jayanta Ghosh[1,*] <ghoshjay_04@yahoo.com>
Tapas Kumar Samanta[2] <mumpu_tapas5@yahoo.co.in>

[1]*Department of Mathematics, Manickpur Adarsha Vidyapith, Howrah-711309, West Bengal, India*
[2]*Department of Mathematics, Uluberia College, Uluberia, Howrah-711315, West Bengal, India*

Abstract – In this paper, at first we define fuzzy strict preference relation in our way motivated from strict preference relation in crisp concept and then define fuzzy weak preference relation, fuzzy indifference relation. Hence we discuss some properties like fuzzy semi-symmetric, fuzzy negatively transitive, fuzzy connectedness and give supporting examples. Thereafter we introduce the notion of fuzzy soft set strict preference relation and define fuzzy soft set weak preference relation, fuzzy soft set indifference relation. Also we verify some properties with suitable examples.

Keywords – *Fuzzy strict preference relation, Fuzzy weak preference relation, Fuzzy soft set strict preference relation, Fuzzy soft set weak preference relation.*

1 Introduction

In real life situation, almost all objects have an ambiguous status with respect to belongingness in a particular class. To reduce this ambiguity, in 1965, Zadeh[13] introduced fuzzy set with a continuum of grades of membership. Fuzzy sets and relations

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

have applications in diverse types of areas, for example in data bases, pattern recognition, neural networks, fuzzy modelling, economics, medicine, multicriteria decision making etc. But fuzzy set theory, probability theory etc. have inherent difficulties to deal with uncertainties in above mentioned areas. To deal with uncertainties free from some difficulties, in 1999, Molodtsov[12] proposed a new parameterized mathematical tool named as soft set. Thereafter in 2001, Maji[11] et al. introduced the notion of fuzzy soft set as hybrid structure of fuzzy set and soft set. Then gradually so many contributions comes from several authors [2, 5, 7, 8, 10] in the area of soft set and fuzzy soft set. On the other hand, Preference modelling is an inevitable step in a variety of field like economics, sociology, psychology, mathematical programming, medicine, decision analysis etc. In decision making problem, procedures are usually based on pair comparisons in the sense that process are linked to some degree of credibility of preference. But preference value can not be express accurately. Hence the use of fuzzy preference is needed. Some papers concerning preference relation and fuzzy preference relation have been published; see, e.g., [1, 6, 9, 15, 16]. Here we have been motivated to introduce fuzzy preference relation in our way following the notion of strict preference relation in crisp method [14]. Also, to introduce fuzzy soft set preference relation, we have considered soft set relation which was recently introduced by Babitha et al. [4] in 2010, as a soft subset of cartesian product of the soft sets. The rest of this paper is organized as follows. In section 2, we recollect basic definitions and notations for later section. In section 3, we redefine fuzzy strict preference relation by extending the concept of preference relation[14] in crisp method and hence define fuzzy weak preference relation, fuzzy indifference relation and study some of their properties. In section 4, we define fuzzy soft set strict preference, fuzzy soft set weak preference, fuzzy soft set indifference relation and examine their properties with supporting examples.

2 Preliminary

Throughout this paper, let U be the initial universe, E be the set of parameters and A, B, C are subsets of E. We denote $\max\{x, y\}$ by $x \vee y$ and $\min\{x, y\}$ by $x \wedge y$. Let $P(U)$ be the collection of all subsets of U and I^U, $I^{U\times U}$ denote the collection of all fuzzy subsets of U, $U \times U$ respectively.

Definition 2.1. Let μ, ν be two fuzzy subsets of U. Then μ is called a fuzzy subset of ν if $\mu(x) \leq \nu(x)$, $\forall\, x \in U$. We write $\mu \subseteq \nu$.

Definition 2.2. [13] A fuzzy binary relation μ on U is a fuzzy subset of $U \times U$ i.e. $\mu : U \times U \to [0, 1]$.

Definition 2.3. A fuzzy subset of $U \times U$ is said to be a null fuzzy set, denoted by $\widetilde{0}_{U\times U}$ and defined by $\widetilde{0}_{U\times U}(x,y) = 0$ for all $(x,y) \in U \times U$. A fuzzy subset of $U \times U$ is said to be a absolute fuzzy set, denoted by $\widetilde{1}_{U\times U}$ and defined by $\widetilde{1}_{U\times U}(x,y) = 1$ for all $(x,y) \in U \times U$.

Definition 2.4. [3] The Cartesian product of two fuzzy subsets μ, ν of U is denoted by $\mu \times \nu$ and defined by

$$(\mu \times \nu)(x,y) = \mu(x) \wedge \nu(y), \ \ \forall\, x,\, y \in U.$$

Definition 2.5. [13] Let μ, ν be two fuzzy relation on U. Then for all $(x,y) \in U \times U$,
(i) union of μ, ν is denoted by $\mu \cup \nu$ and defined by

$$(\mu \cup \nu)(x,y) = \mu(x,y) \ \vee \ \nu(x,y);$$

(ii) intersection of μ, ν is denoted by $\mu \cap \nu$ and defined by

$$(\mu \cap \nu)(x,y) = \mu(x,y) \ \wedge \ \nu(x,y);$$

(iii) complement of μ is denoted by μ^c and defined by

$$\mu^c(x,y) = 1 - \mu(x,y);$$

(iv) algebraic product of μ, ν is denoted by $\mu.\nu$ and defined by

$$(\mu.\nu)(x,y) = \mu(x,y).\nu(x,y);$$

(v) algebraic sum of μ, ν is denoted by $\mu \oplus \nu$ and defined by

$$(\mu \oplus \nu)(x,y) = \mu(x,y) + \nu(x,y) - (\mu.\nu)(x,y).$$

Definition 2.6. [12] Let $A \subseteq E$. A pair (F,A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.7. [4] Let (F,A) and (G,B) be two soft sets over U, then the cartesian product of (F,A) and (G,B) is defined as, $(F,A) \times (G,B) = (H, A \times B)$, where $H : A \times B \to P(U \times U)$ and $H(a,b) = F(a) \times G(b)$, for all $(a,b) \in A \times B$, i.e.

$$H(a,b) = \{(h_i, h_j);\ \text{where}\, h_i \in F(a)\, \text{and}\, h_j \in G(b)\}.$$

Definition 2.8. [4] Let (F,A) and (G,B) be two soft sets over U, then a soft set relation from (F,A) to (G,B) is a soft subset of $(F,A) \times (G,B)$, i.e., a soft set relation from (F,A) to (G,B) is of the form (H_1, C) where $C \subseteq A \times B$ and

$H_1(a,b) = H(a,b),\ \forall\,(a,b) \in C,$ where $(H, A \times B) = (F,A) \times (G,B)$ as defined in Definition 2.7. Any soft subset of $(F,A) \times (F,A)$ is called a soft set relation on (F,A). In an equivalent way, the soft set relation R on the soft set (F,A) in the parameterized form are as follows:

If $(F,A) = \{F(a),\ F(b), ...\}$, then $F(a)RF(b)$ iff $F(a) \times F(b) \in R$.

Definition 2.9. [11] Let $A \subseteq E$. A pair $(\mathcal{F},A)$ is called a fuzzy soft set over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : A \to I^U$.

Definition 2.10. [11] Let $(\mathcal{F},A)$, $(\mathcal{G},B)$ be two fuzzy soft set over U. Then we say that $(\mathcal{F},A)$ is a fuzzy soft subset of $(\mathcal{G},B)$ if

(i) $A \subseteq B$,
(ii) $\forall\, a \in A,\ \mathcal{F}(a) \subseteq \mathcal{G}(a)$.

We write $(\mathcal{F},A) \,\widetilde{\subseteq}\, (\mathcal{G},B)$, if $(\mathcal{F},A)$ is fuzzy soft subset of $(\mathcal{G},B)$.

Definition 2.11. [11] The intersection of two fuzzy soft set $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over common universe U, denoted by $(\mathcal{F},A) \,\widetilde{\cap}\, (\mathcal{G},B)$, is defined as the fuzzy soft set $(\mathcal{H},C)$, where $C = A \cap B$ and for all $e \in C,\ \mathcal{H}(e) = \mathcal{F}(e) \cap \mathcal{G}(e)$.

Definition 2.12. [11] The union of two fuzzy soft set $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over common universe U, denoted by $(\mathcal{F},A) \,\widetilde{\cup}\, (\mathcal{G},B)$, is defined as the fuzzy soft set $(\mathcal{H},C)$, where $C = A \cap B$ and for all $e \in C,\ \mathcal{H}(e) = \mathcal{F}(e) \cup \mathcal{G}(e)$.

Definition 2.13. The complement of a fuzzy soft set $(\mathcal{F},A)$ over U is denoted by $(\mathcal{F},A)^c$ and defined by $(\mathcal{F},A)^c = (\mathcal{F}^c,A)$, where $\mathcal{F}^c : A \to I^U$ is given by $\mathcal{F}^c(e) = [\mathcal{F}(e)]^c$ for all $e \in A$.

Definition 2.14. The cartesian product of two fuzzy soft set $(\mathcal{F},A)$, $(\mathcal{G},B)$ over U is defined as $(\mathcal{F},A) \times (\mathcal{G},B) = (\mathcal{H}, A \times B)$, where $\mathcal{H} : A \times B \to I^{U \times U}$ and $\mathcal{H}(a,b) = \mathcal{F}(a) \times \mathcal{G}(b),\ \forall\,(a,b) \in A \times B$.

Definition 2.15. Let $(\mathcal{F},A)$, $(\mathcal{G},B)$ be two fuzzy soft set over U. Then a fuzzy soft set relation from $(\mathcal{F},A)$ to $(\mathcal{G},B)$ is a fuzzy soft subset of $(\mathcal{F},A) \times (\mathcal{G},B)$, i.e., a fuzzy soft set relation from $(\mathcal{F},A)$ to $(\mathcal{G},B)$ is of the form $(\mathcal{R},C)$, where $C \subseteq A \times B$ and $\mathcal{R}(a,b) \subseteq \mathcal{H}(a,b)$, $\forall\,(a,b) \in C$, where $(\mathcal{H}, A \times B) = (\mathcal{F},A) \times (\mathcal{G},B)$ as defined in Definition 2.14.

If $(\mathcal{R},C)$ is a fuzzy soft subset of $(\mathcal{F},A) \times (\mathcal{F},A)$, then $(\mathcal{R},C)$ is called a fuzzy soft set relation on $(\mathcal{F},A)$. Fuzzy soft set relation $(\mathcal{R},C)$ may be denoted simply by $\mathcal{R}$.

Definition 2.16. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set relation on $(\mathcal{F}, A)$. Then for all $a, b \in A$,

(i) $\mathcal{R}$ is called reflexive if $\mathcal{R}(a, a) = \widetilde{1}_{U\times U}$;
(ii) $\mathcal{R}$ is called irreflexive if $\mathcal{R}(a, a) = \widetilde{0}_{U\times U}$;
(iii) $\mathcal{R}$ is called symmetric if $\mathcal{R}(a, b) = \mathcal{R}(b, a)$;
(iv) $\mathcal{R}$ is called asymmetric if $\mathcal{R}(a, b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b, a) = \widetilde{0}_{U\times U}$.

3 Fuzzy Strict Preference Relation

Here we give the definition of fuzzy strict preference relation and then define fuzzy weak preference relation and fuzzy indifference relation with the help of fuzzy strict preference relation, motivated from the notion of strict preference relation in crisp method[14].

Definition 3.1. [14] A binary relation P on U, i.e. $P \subseteq U \times U$ is said to be strict preference relation if (i) P is irreflexive i.e. $(x, x) \notin P$, $\forall\, x \in U$,

(ii) P is asymmetric i.e. $(x, y) \in P \Rightarrow (y, x) \notin P$, where $x, y \in U$.

Given a strict preference relation P on U, two new relations on U, called indifference relation (denoted by I) and weak preference relation (denoted by W) are as follows: For all $x, y \in U$,

(i) $(x, y) \in I \Leftrightarrow (x, y) \notin P$ and $(y, x) \notin P$,
(ii) $(x, y) \in W \Leftrightarrow$ either $(x, y) \in P$ or, $(x, y) \in I$.

Definition 3.2. A fuzzy binary relation μ on U, i.e. $\mu : U \times U \to [0, 1]$ is said to be fuzzy strict preference relation if

(i) μ is fuzzy irreflexive i.e. $\mu(x, x) = 0$, $\forall\, x \in U$,
(ii) μ is fuzzy asymmetric i.e. $\mu(x, y) > 0 \Rightarrow \mu(y, x) = 0$, $\forall\, x, y \in U$.

Given a fuzzy strict preference relation μ on U, we can define two new fuzzy relations called fuzzy indifference relation (denoted by μ_I) and fuzzy weak preference relation (denoted by μ_W) as follows:

(i) $\mu_I(x, y) > 0 \Leftrightarrow \mu(x, y) = 0$ and $\mu(y, x) = 0$,
(ii) $\mu_W(x, y) > 0 \Leftrightarrow$ either $\mu(x, y) > 0$ or, $\mu_I(x, y) > 0$, $\forall\, x, y \in U$.

Note 3.3. Let μ, ν be two fuzzy strict preference relation on U. Then $\mu \cup \nu$ may or may not be fuzzy strict preference relation on U.

Example 3.4. Let $U = \{1, 2\}$. Then $U \times U = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Also let $\mu : U \times U \to [0, 1]$ is defined by $\mu(1, 1) = 0$, $\mu(1, 2) = 0.1$, $\mu(2, 1) = 0$, $\mu(2, 2) = 0$ and $\nu : U \times U \to [0, 1]$ is defined by $\nu(1, 1) = 0$, $\nu(1, 2) = 0$, $\nu(2, 1) = 0.2$, $\nu(2, 2) = 0$. Then by Definition 3.2, μ, ν are obviously fuzzy strict preference relation on U. By definition 2.5, $(\mu \cup \nu)(1, 1) = 0$, $(\mu \cup \nu)(1, 2) = 0.1$, $(\mu \cup \nu)(2, 1) = 0.2$, $(\mu \cup \nu)(2, 2) = 0$. Obviously, $(\mu \cup \nu)$ is fuzzy irreflexive. Now $(\mu \cup \nu)(1, 2) = 0.1 > 0$ but $(\mu \cup \nu)(2, 1) = 0.2 \neq 0$. Hence, $(\mu \cup \nu)$ is not fuzzy asymmetric. Therefore $(\mu \cup \nu)$ is not fuzzy strict preference relation on U.

Example 3.5. Let $U = \{1, 2\}$. Define $\mu : U \times U \to [0, 1]$ by $\mu(1, 1) = 0$, $\mu(1, 2) = 0.1$, $\mu(2, 1) = 0$, $\mu(2, 2) = 0$ and define $\nu : U \times U \to [0, 1]$ by $\nu(1, 1,) = 0$, $\nu(1, 2) = 0.2$, $\nu(2, 1) = 0$, $\nu(2, 2) = 0$. Then by definition 3.2, μ, ν, $\mu \cup \nu$ are fuzzy strict preference relation on U.

Theorem 3.6. Let μ, ν be two fuzzy strict preference relation on U. Then $\mu \cap \nu$ is also a fuzzy strict preference relation on U.

Proof. Let μ, ν be two fuzzy strict preference relation on U. Then

$$\forall\, (x, x) \in U \times U,\ \mu(x, x) = 0 = \nu(x, x)$$

and $\mu(x, y) > 0 \Rightarrow \mu(y, x) = 0$, $\nu(u, v) > 0 \Rightarrow \nu(v, u) = 0$ for all (x, y), $(u, v) \in U \times U$.

Then by Definition 2.5, $(\mu \cap \nu)(x, x) = \mu(x, x) \wedge \nu(x, x) = 0$.

Now let, for some $(x, y) \in U \times U$, $(\mu \cap \nu)(x, y) > 0$. This implies $\mu(x, y) \wedge \nu(x, y) > 0$.

(i) If $\mu(x, y) > 0$ then $\mu(y, x) = 0$.
(ii) If $\nu(x, y) > 0$ then $\nu(y, x) = 0$.

In both cases $(\mu \cap \nu)(y, x) = \mu(y, x) \wedge \nu(y, x) = 0$. So, $\mu \cap \nu$ is a fuzzy strict preference on U. □

Note 3.7. If μ is a fuzzy strict preference relation on U then μ is fuzzy irreflexive i.e. $\mu(x, x) = 0$, $\forall\, (x, x) \in U \times U$. Hence $\mu^c(x, x) = 1 - \mu(x, x) = 1$. So, μ^c is a fuzzy reflexive relation. Therefore μ^c is not a fuzzy strict preference relation on U.

Theorem 3.8. If μ, ν are fuzzy strict preference relation on U then $\mu.\nu$ is also a fuzzy strict preference relation on U.

Proof. Proof is straightforward. □

Note 3.9. If μ, ν are fuzzy strict preference relation on U then $\mu \oplus \nu$ may or may not be fuzzy strict preference relation on U.

Example 3.10. Let $U = \{1, 2\}$ and define two fuzzy strict preference relation μ, ν as in Example 3.4. Then by Definition 2.5, we have $(\mu \oplus \nu)(1, 1) = 0 = (\mu \oplus \nu)(2, 2)$, $(\mu \oplus \nu)(1, 2) = 0.1$, $(\mu \oplus \nu)(2, 1) = 0.2$.

This implies that $\mu \oplus \nu$ is fuzzy irreflexive relation but it is not fuzzy asymmetric. Hence $\mu \oplus \nu$ is not fuzzy strict preference relation on U.

Example 3.11. Let $U = \{1, 2\}$ and define two fuzzy strict preference relation μ, ν as in Example 3.5. Then by Definition 2.5, we have $(\mu \oplus \nu)(1, 1) = 0 = (\mu \oplus \nu)(2, 2)$, $(\mu \oplus \nu)(1, 2) = 0.28$, $(\mu \oplus \nu)(2, 1) = 0$. This implies that $\mu \oplus \nu$ is fuzzy strict preference relation on U.

Theorem 3.12. Let μ be a fuzzy strict preference relation on U. Then $\mu^{-1}(r) = \{(x, y) \in U \times U \ : \mu(x, y) = r\}$, where $r \in (0, 1]$, is a strict preference relation on U.

Proof. Take an element $r \in (0, 1]$ and fixed. As $\mu^{-1}(r) \subseteq U \times U$, $\mu^{-1}(r)$ is a binary relation on U. Since μ is a fuzzy strict preference relation on U, then $\mu(x, x) = 0$, $\forall\, x \in U$. Therefore $(x, x) \notin \mu^{-1}(r)$, $\forall\, x \in U$. So, $\mu^{-1}(r)$ is irreflexive.

Again, let $(x, y) \in \mu^{-1}(r)$, where $x, y \in U$. Then $\mu(x, y) = r > 0$. This implies $\mu(y, x) = 0$. Therefore $(y, x) \notin \mu^{-1}(r)$. Hence, $\mu^{-1}(r)$ is asymmetric. So, $\mu^{-1}(r)$ is strict preference relation on U for each $r \in (0, 1]$. □

Now we define fuzzy semi-reflexive relation, fuzzy semi-symmetric relation, fuzzy connected, negatively fuzzy transitive and fuzzy transitive relation as follows:

Definition 3.13. Let μ be a fuzzy binary relation on U. Then for all $x, y, z \in U$,

(i) μ is called fuzzy semi-reflexive if $\mu(x, x) > 0$;
(ii) μ is called fuzzy semi-symmetric if $\mu(x, y) > 0 \Rightarrow \mu(y, x) > 0$;
(iii) μ is called fuzzy connected if either $\mu(x, y) > 0$ or $\mu(y, x) > 0$;
(iv) μ is called negatively fuzzy transitive if
$\mu(x, y) = 0 = \mu(y, z) \Rightarrow \mu(x, z) = 0$;
(v) μ is called fuzzy transitive if
$\mu(x, y) > 0$ and $\mu(y, z) > 0 \Rightarrow \mu(x, z) > 0$.

Theorem 3.14. If μ is a fuzzy strict preference relation on U then fuzzy indifference relation μ_I on U is a fuzzy semi-reflexive and fuzzy semi-symmetric on U.

Proof. Since μ is a fuzzy strict preference relation then $\mu(x,x) = 0 \ \ \forall\, x \in U$. Hence by definition of μ_I we have $\mu_I(x,x) > 0$. This implies μ_I is fuzzy semi-reflexive.
Again $\mu_I(x,y) > 0 \Leftrightarrow \mu(x,y) = 0$ and $\mu(y,x) = 0 \Leftrightarrow \mu_I(y,x) > 0$. This implies μ_I is fuzzy semi-symmetric on U. □

Theorem 3.15. If μ is a fuzzy strict preference relation on U, then fuzzy weak preference relation μ_W on U is a fuzzy semi-reflexive and fuzzy connected on U.

Proof. $\mu_W(x,y) > 0 \Leftrightarrow \mu(x,y) > 0$ or $\mu_I(x,y) > 0 \ \ \forall\, x,\, y \in U$. Since $\mu_I(x,x) > 0 \ \ \forall\, x \in U$, as μ_I is fuzzy semi-reflexive.

Hence $\mu_W(x,x) > 0 \ \ \forall\, x \in U$. This implies μ_W is fuzzy semi-reflexive.
Now we are going to prove μ_W is fuzzy connected.

Since, μ is fuzzy strict preference relation, then for all $x, y \in U$, either $\mu(x,y) > 0$ or $\mu(y,x) > 0$ or $\mu(x,y) = \mu(y,x) = 0$.

Case 1: Let $\mu(x,y) > 0$. Then $\mu(y,x) = 0$, since μ is fuzzy asymmetric relation on X. Hence $\mu_I(y,x) \not> 0$. This implies $\mu_W(x,y) > 0$ but $\mu_W(y,x) \not> 0$.

Case 2: Let $\mu(y,x) > 0$. Then similarly as in case 1, we can prove that $\mu_W(y,x) > 0$ but $\mu_W(x,y) \not> 0$.

Case 3: Let $\mu(x,y) = \mu(y,x) = 0$. Then by Definition 3.2, we have $\mu_I(x,y) > 0$. This implies $\mu_W(x,y) > 0$.
Hence for all $x, y \in U$, either $\mu_W(x,y) > 0$ or $\mu_W(y,x) > 0$. So, by Definition 3.13, μ_W is fuzzy connected on U. □

Theorem 3.16. Let μ is a fuzzy strict preference relation on U. Then for $x, y \in U$, $\mu_W(x,y) > 0$ and $\mu_W(y,x) > 0 \ \Leftrightarrow \mu_I(x,y) > 0$.

Proof. Suppose for $x, y \in U$, $\mu_W(x,y) > 0$ and $\mu_W(y,x) > 0$. Then $\mu_W(x,y) > 0 \Rightarrow \mu(x,y) > 0$ or $\mu_I(x,y) > 0$. Let $\mu(x,y) > 0$. Then $\mu(y,x) = 0$, since μ is fuzzy asymmetric relation.

Again $\mu_W(y,x) > 0 \ \Rightarrow \ \mu(y,x) > 0$ or $\mu_I(y,x) > 0$. But $\mu(y,x) = 0$, as proved earlier. Hence $\mu_I(y,x) > 0$. This implies $\mu_I(x,y) > 0$, since μ_I is fuzzy semi-symmetric relation, by Theorem 3.14.

Conversely, let $\mu_I(x,y) > 0$. Then by Definition 3.2, $\mu_W(x,y) > 0$. Since μ_I is fuzzy semi-symmetric relation, then $\mu_I(x,y) > 0 \Rightarrow \mu_I(y,x) > 0 \Rightarrow \mu_W(y,x) > 0$. $\square$

Note 3.17. Fuzzy weak preference relation μ_W on U is fuzzy semi-symmetric if and only if $\mu_I(x,y) > 0$ for all $x, y \in U$.

Theorem 3.18. If fuzzy strict preference relation μ on U is a negatively fuzzy transitive then μ, μ_W, μ_I are all fuzzy transitive relation on U.

Proof. Let μ be a negatively fuzzy transitive relation on U.

1. To prove μ_W is fuzzy transitive relation on U, suppose that there exist $x, y, z \in U$ such that $\mu_W(x,y) > 0$ and $\mu_W(y,z) > 0$. By Definition 3.2,

$\mu_W(x,y) > 0 \Rightarrow \mu(x,y) > 0$ or $\mu_I(x,y) > 0$;

$\mu_W(y,z) > 0 \Rightarrow \mu(y,z) > 0$ or $\mu_I(y,z) > 0$.

Case 1: Let $\mu(x,y) > 0$ and $\mu(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition 3.2; $\Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive.

Now if $\mu(x,z) > 0$ then $\mu_W(x,z) > 0$. Hence μ_W is fuzzy transitive.

If $\mu(x,z) = 0$ then $\mu(x,z) = 0 = \mu(z,x) \Rightarrow \mu_I(x,z) > 0$. This implies $\mu_W(x,z) > 0$. Hence μ_W is fuzzy transitive on U.

Case 2: Let $\mu(x,y) > 0$ and $\mu_I(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition of μ, μ_I $\Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive.
Hence, as in case 1, μ_W is fuzzy transitive.

Case 3: Let $\mu_I(x,y) > 0$ and $\mu(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition of μ, μ_I $\Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive.
Then we can prove similarly as in Case 1 that μ_W is fuzzy transitive.

Case 4: Let $\mu_I(x,y) > 0$ and $\mu_I(y,z) > 0$. This implies $\mu(y,x) = 0$ and $\mu(z,y) = 0$, by Definition of μ_I $\Rightarrow \mu(z,x) = 0$, since μ is negatively fuzzy transitive, which implies that μ_W is fuzzy transitive.

2. To prove μ is fuzzy transitive on U, suppose that there exist $x, y, z \in U$ such that $\mu(x, y) > 0$ and $\mu(y, z) > 0$ but $\mu(x, z) = 0$.

Now $\mu(x, y) > 0$ and $\mu(y, z) > 0$

$\Rightarrow \mu(y, x) = 0$ and $\mu(z, y) = 0$, by Definition of 3.2

$\Rightarrow \mu(z, x) = 0$, since μ is negatively fuzzy transitive.

Now $\mu(z, x) = 0$ and $\mu(x, z) = 0$

$\Rightarrow \mu_I(z, x) > 0 \Rightarrow \mu_W(z, x) > 0$.

Again given that $\mu(y, z) > 0$. This implies $\mu_W(y, z) > 0$.

Hence $\mu_W(y, z) > 0$ and $\mu_W(z, x) > 0$

$\Rightarrow \mu_W(y, x) > 0$, since μ_W is fuzzy transitive

$\Rightarrow \mu(y, x) > 0$ or $\mu_I(y, x) > 0$.

If $\mu(y, x) > 0$ then it implies $\mu(x, y) = 0$, which contradicts our assumption.

If $\mu_I(y, x) > 0$ then it implies $\mu(x, y) = 0$ and $\mu(y, x) = 0$, which also contradicts our assumption. So, $\mu(x, z) > 0$. Hence μ is fuzzy transitive on U.

3. To prove μ_I is fuzzy transitive on U, suppose that there exist $x, y, z \in U$ such that $\mu_I(x, y) > 0$ and $\mu_I(y, z) > 0$.

Now $\mu_I(x, y) > 0$ and $\mu_I(y, z) > 0$

$\Rightarrow \mu(x, y) = 0, \mu(y, x) = 0$ and $\mu(y, z) = 0, \mu(z, y) = 0$

$\Rightarrow \mu(x, y) = 0, \mu(y, z) = 0$ and $\mu(y, x) = 0, \mu(z, y) = 0$

$\Rightarrow \mu(x, z) = 0$ and $\mu(z, x) = 0$, by transitivity of μ

$\Rightarrow \mu_I(x, z) > 0$, by Definition of μ_I.

So, μ_I is a fuzzy transitive relation on U. □

Theorem 3.19. If for $x, y, z \in U$, $\mu_W(x, y) > 0$ and $\mu(y, z) > 0$, then $\mu(x, z) > 0$.

Proof. Suppose there exist $x, y, z \in U$ such that $\mu_W(x, y) > 0$ and $\mu(y, z) > 0$ but $\mu(x, z) = 0$.
Now $\mu_W(x, y) > 0 \Rightarrow \mu(x, y) > 0$ or $\mu_I(x, y) > 0$.

(i) If $\mu(x, y) > 0$ then by Definition 3.2, we have $\mu(y, x) = 0$.
Since μ is negatively fuzzy transitive then $\mu(y, x) = 0$ together with $\mu(x, z) = 0$ implies $\mu(y, z) = 0$, which contradicts our assumption $\mu(y, z) > 0$.

(ii) If $\mu_I(x, y) > 0$ then it implies $\mu(x, y) = 0$ and $\mu(y, x) = 0$.
Now $\mu(y, x) = 0$ together with $\mu(x, z) = 0$ implies that $\mu(y, z) = 0$, which contradicts our assumption $\mu(y, z) > 0$.

Hence our assumption $\mu(x, z) = 0$ is wrong. Therefore $\mu(x, z) > 0$. □

4 Fuzzy Soft Set Strict Preference Relation

In this section, at first we define fuzzy soft set strict preference relation, then we define fuzzy soft set weak preference relation and fuzzy soft set indifference relation with the help of fuzzy soft set strict preference relation.

Definition 4.1. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set relation on $(\mathcal{F}, A)$. Then $\mathcal{R}$ is said to be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$ if

(i) $\mathcal{R}$ is irreflexive, i.e. $\mathcal{R}(a, a) = \widetilde{0}_{U\times U}$, $\forall\, a \in A$.
(ii) $\mathcal{R}$ is asymmetric, i.e. $\mathcal{R}(a, b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b, a) = \widetilde{0}_{U\times U}$, $\forall\, a, b \in A$.

Given a fuzzy soft set strict preference relation $\mathcal{R}$ on $(\mathcal{F}, A)$, we can define two fuzzy soft set relation on $(\mathcal{F}, A)$ called fuzzy soft set indifference relation (denoted by $\mathcal{R}_I$) and fuzzy soft set weak preference relation (denoted by $\mathcal{R}_W$) as follows:
For all $a, b \in A$,

(i) $\mathcal{R}_I(a, b) \supset \widetilde{0}_{U\times U} \Leftrightarrow \mathcal{R}(a, b) = \widetilde{0}_{U\times U}$ and $\mathcal{R}(b, a) = \widetilde{0}_{U\times U}$,
(ii) $\mathcal{R}_W(a, b) \supset \widetilde{0}_{U\times U} \Leftrightarrow \mathcal{R}(a, b) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}_I(a, b) \supset \widetilde{0}_{U\times U}$.

Theorem 4.2. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$, $\mathcal{S}$ be two fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then $\mathcal{R}\,\widetilde{\cap}\,\mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Proof. Since $\mathcal{R}$, $\mathcal{S}$ are fuzzy soft set strict preference relation, then $\mathcal{R}(a,a) = \widetilde{0}_{U\times U}$, $\mathcal{S}(a,a) = \widetilde{0}_{U\times U}$, $\forall\,(a,a) \in A \times A$. Therefore $(\mathcal{R}\,\widetilde{\cap}\,\mathcal{S})(a,a) = \mathcal{R}(a,a) \cap \mathcal{S}(a,a) = \widetilde{0}_{U\times U}$, $\forall\,(a,a) \in A \times A$. So, $\mathcal{R}\,\widetilde{\cap}\,\mathcal{S}$ is irreflexive.

Now let, for some $(a,b) \in A \times A$, $(\mathcal{R}\,\widetilde{\cap}\,\mathcal{S})(a,b) \supset \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}(a,b) \cap \mathcal{S}(a,b) \supset \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$, $\mathcal{S}(a,b) \supset \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}(b,a) = \widetilde{0}_{U\times U}$, $\mathcal{S}(b,a) = \widetilde{0}_{U\times U}$, since $\mathcal{R}$, $\mathcal{S}$ is asymmetric

$\Rightarrow (\mathcal{R}\,\widetilde{\cap}\,\mathcal{S})(b,a) = \widetilde{0}_{U\times U}$.

Since $(a,b) \in A \times A$ is arbitrary, then $\mathcal{R}\,\widetilde{\cap}\,\mathcal{S}$ is asymmetric.

Hence $\mathcal{R}\,\widetilde{\cap}\,\mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. □

Note 4.3. Given two fuzzy soft set strict preference relation $\mathcal{R}$, $\mathcal{S}$ on $(\mathcal{F}, A)$. Then $\mathcal{R}\,\widetilde{\cup}\,\mathcal{S}$ may or may not be fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Example 4.4. Let U denotes the set of selected students in a school,

i.e. $U = \{s_1,\ s_2,\ s_3\}$.

Let A denotes different subjects.

Take $A = \{bengali,\ english,\ mathematics\}$,

i.e. $A = \{b, e, m\}$.

If a student get 95 marks out of 100 in a particular subject, then take the score of this student in that particular subject is 0.95.

Let a fuzzy soft set $(\mathcal{F}, A)$ over U describe students having different scores in different subjects in a particular examination and is given by

$F(b) = \{(s_1, 0.6),\ (s_2, 0.7),\ (s_3, 0.65)\}$;

$F(e) = \{(s_1, 0.59), (s_2, 0.75), (s_3, 0.6)\}$;

$F(m) = \{(s_1, 0.8), (s_2, 0.82), (s_3, 0.9)\}$;

Then

$A \times A = \{(b,b), (b,e), (b,m), (e,b), (e,e), (e,m), (m,b), (m,e), (m,m)\}$.

Then by Definition 2.14, the elements of the cartesian product

$(\mathcal{H}, A \times A) = (\mathcal{F}, A) \times (\mathcal{F}, A)$ are as follows:

$$\mathcal{H}(b,b) = \left\{ \frac{(s_1,s_1)}{0.6}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.6}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.65}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.65} \right\};$$

$$\mathcal{H}(b,e) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(b,m) = \left\{ \frac{(s_1,s_1)}{0.6}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.7}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.7}, \frac{(s_3,s_1)}{0.65}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.65} \right\};$$

$$\mathcal{H}(e,b) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.6}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.65}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(e,e) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.75}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(e,m) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.75}, \frac{(s_2,s_2)}{0.75}, \frac{(s_2,s_3)}{0.75}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(m,b) = \left\{ \frac{(s_1,s_1)}{0.6}, \frac{(s_1,s_2)}{0.7}, \frac{(s_1,s_3)}{0.65}, \frac{(s_2,s_1)}{0.6}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.65}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.7}, \frac{(s_3,s_3)}{0.65} \right\};$$

$$\mathcal{H}(m,e) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.75}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.75}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.75}, \frac{(s_3,s_3)}{0.6} \right\};$$

$$\mathcal{H}(m,m) = \left\{ \frac{(s_1,s_1)}{0.8}, \frac{(s_1,s_2)}{0.8}, \frac{(s_1,s_3)}{0.8}, \frac{(s_2,s_1)}{0.8}, \frac{(s_2,s_2)}{0.82}, \frac{(s_2,s_3)}{0.82}, \frac{(s_3,s_1)}{0.8}, \frac{(s_3,s_2)}{0.82}, \frac{(s_3,s_3)}{0.9} \right\}.$$

Define a fuzzy soft set relation $(\mathcal{R}, C)$ on $(\mathcal{F}, A)$ as follows:

Let $(x, y) \in C \subseteq A \times A$ if and only if either both x, y are art subjects or both are science subjects.

i.e. $C = \{(b, b),\ (b, e),\ (e, b),\ (e, e),\ (m, m)\}$, and take

$\mathcal{R}(b, b) = \mathcal{R}(e, e) = \mathcal{R}(m, m) = \mathcal{R}(e, b) = \widetilde{0}_{U \times U}$;

$$\mathcal{R}(b, e) = \left\{ \frac{(s_1, s_1)}{0.59}, \frac{(s_1, s_2)}{0.6}, \frac{(s_1, s_3)}{0.6}, \frac{(s_2, s_1)}{0.59}, \frac{(s_2, s_2)}{0.7}, \frac{(s_2, s_3)}{0.6}, \frac{(s_3, s_1)}{0.59}, \frac{(s_3, s_2)}{0.65}, \frac{(s_3, s_3)}{0.6} \right\}.$$

Define another fuzzy soft set relation $(\mathcal{S}, C)$ on $(\mathcal{F}, A)$ as follows:

$\mathcal{S}(b, b) = \mathcal{S}(e, e) = \mathcal{S}(m, m) = \mathcal{S}(b, e) = \widetilde{0}_{U \times U}$;

$$\mathcal{S}(e, b) = \left\{ \frac{(s_1, s_1)}{0.59}, \frac{(s_1, s_2)}{0.59}, \frac{(s_1, s_3)}{0.59}, \frac{(s_2, s_1)}{0.6}, \frac{(s_2, s_2)}{0.7}, \frac{(s_2, s_3)}{0.65}, \frac{(s_3, s_1)}{0.6}, \frac{(s_3, s_2)}{0.6}, \frac{(s_3, s_3)}{0.6} \right\}.$$

Obviously, $\mathcal{R}$, $\mathcal{S}$ are fuzzy soft set strict preference relations on $(\mathcal{F}, A)$. Then $(\mathcal{R} \widetilde{\cap} \mathcal{S}, C)$, $(\mathcal{R} \widetilde{\cup} \mathcal{S}, C)$ are as follows:

$(\mathcal{R} \widetilde{\cap} \mathcal{S})(b, b) = (\mathcal{R} \widetilde{\cap} \mathcal{S})(e, e) = (\mathcal{R} \widetilde{\cap} \mathcal{S})(m, m) = \widetilde{0}_{U \times U}$;

$(\mathcal{R} \widetilde{\cap} \mathcal{S})(b, e) = \widetilde{0}_{U \times U} = (\mathcal{R} \widetilde{\cap} \mathcal{S})(e, b)$ and

$(\mathcal{R} \widetilde{\cup} \mathcal{S})(b, b) = (\mathcal{R} \widetilde{\cup} \mathcal{S})(e, e) = (\mathcal{R} \widetilde{\cup} \mathcal{S})(m, m) = \widetilde{0}_{U \times U}$;

$(\mathcal{R} \widetilde{\cup} \mathcal{S})(b, e) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.6}, \frac{(s_1,s_3)}{0.6}, \frac{(s_2,s_1)}{0.59}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.6}, \frac{(s_3,s_1)}{0.59}, \frac{(s_3,s_2)}{0.65}, \frac{(s_3,s_3)}{0.6} \right\}$;

$(\mathcal{R} \widetilde{\cup} \mathcal{S})(e, b) = \left\{ \frac{(s_1,s_1)}{0.59}, \frac{(s_1,s_2)}{0.59}, \frac{(s_1,s_3)}{0.59}, \frac{(s_2,s_1)}{0.6}, \frac{(s_2,s_2)}{0.7}, \frac{(s_2,s_3)}{0.65}, \frac{(s_3,s_1)}{0.6}, \frac{(s_3,s_2)}{0.6}, \frac{(s_3,s_3)}{0.6} \right\}$.

This shows that $\mathcal{R} \widetilde{\cap} \mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$ but $\mathcal{R} \widetilde{\cup} \mathcal{S}$ is not a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Because $(\mathcal{R} \widetilde{\cup} \mathcal{S})(b, e) \supset \widetilde{0}_{U \times U}$, $(\mathcal{R} \widetilde{\cup} \mathcal{S})(e, b) \supset \widetilde{0}_{U \times U}$ implies $\mathcal{R} \widetilde{\cup} \mathcal{S}$ is not asymmetric on $(\mathcal{F}, A)$.

Example 4.5. As a continuation of Example 4.4, we define another fuzzy soft set relation $(\mathcal{T}, C)$ on $(\mathcal{F}, A)$ as follows:

$\mathcal{T}(b, b) = \mathcal{T}(e, e) = \mathcal{T}(m, m) = \mathcal{T}(e, b) = \widetilde{0}_{U \times U}$;

$$\mathcal{T}(b, e) = \left\{ \frac{(s_1, s_1)}{0.59}, \frac{(s_1, s_2)}{0.59}, \frac{(s_1, s_3)}{0.59}, \frac{(s_2, s_1)}{0.59}, \frac{(s_2, s_2)}{0.7}, \frac{(s_2, s_3)}{0.6}, \frac{(s_3, s_1)}{0.59}, \frac{(s_3, s_2)}{0.6}, \frac{(s_3, s_3)}{0.6} \right\}.$$

Then obviously, $\mathcal{T}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.
Now $(\mathcal{R}\,\widetilde{\cap}\,\mathcal{T}, C)$, $(\mathcal{R}\,\widetilde{\cup}\,\mathcal{T}, C)$ are as follows:

$$(\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(b,b) = (\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(e,e) = (\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(m,m) = (\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(e,b) = \widetilde{0}_{U\times U};$$

$$(\mathcal{R}\,\widetilde{\cap}\,\mathcal{T})(b,e) = \left\{\tfrac{(s_1,s_1)}{0.59}, \tfrac{(s_1,s_2)}{0.59}, \tfrac{(s_1,s_3)}{0.59}, \tfrac{(s_2,s_1)}{0.59}, \tfrac{(s_2,s_2)}{0.7}, \tfrac{(s_2,s_3)}{0.6}, \tfrac{(s_3,s_1)}{0.59}, \tfrac{(s_3,s_2)}{0.6}, \tfrac{(s_3,s_3)}{0.6}\right\};$$

and

$$(\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(b,b) = (\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(e,e) = (\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(m,m) = (\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(e,b) = \widetilde{0}_{U\times U};$$

$$(\mathcal{R}\,\widetilde{\cup}\,\mathcal{T})(b,e) = \left\{\tfrac{(s_1,s_1)}{0.59}, \tfrac{(s_1,s_2)}{0.6}, \tfrac{(s_1,s_3)}{0.6}, \tfrac{(s_2,s_1)}{0.59}, \tfrac{(s_2,s_2)}{0.7}, \tfrac{(s_2,s_3)}{0.6}, \tfrac{(s_3,s_1)}{0.59}, \tfrac{(s_3,s_2)}{0.65}, \tfrac{(s_3,s_3)}{0.6}\right\}.$$

This shows that $\mathcal{R}\,\widetilde{\cap}\,\mathcal{T}$ and $\mathcal{R}\,\widetilde{\cup}\,\mathcal{T}$ are both fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Note 4.6. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then $\mathcal{R}(a,a) = \widetilde{0}_{U\times U}$, $\forall\,(a,a) \in A \times A$. Hence $\mathcal{R}^c(a,a) = \widetilde{1}_{U\times U}$, $\forall\,(a,a) \in A \times A$. So, $\mathcal{R}^c$ is fuzzy soft set reflexive relation on $(\mathcal{F}, A)$. Therefore $\mathcal{R}^c$ is not a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Definition 4.7. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$, $\mathcal{S}$ be two fuzzy soft set relation on $(\mathcal{F}, A)$. The algebraic product of $\mathcal{R}$, $\mathcal{S}$ is denoted by $\mathcal{R}.\mathcal{S}$ and defined by

$$(\mathcal{R}.\mathcal{S})(a,b) = \mathcal{R}(a,b).\,\mathcal{S}(a,b),\ \forall\,(a,b) \in A \times B.$$

Theorem 4.8. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$, $\mathcal{S}$ be two fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then $\mathcal{R}.\mathcal{S}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$.

Proof. This theorem can be easily proved with the help of Definition 4.1 and Definition 4.7. $\square$

Definition 4.9. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set relation on $(\mathcal{F}, A)$. Then for all $a, b, c \in A$,

(i) $\mathcal{R}$ is called semi-reflexive if $\mathcal{R}(a,a) \supset \widetilde{0}_{U\times U}$;
(ii) $\mathcal{R}$ is called semi-symmetric if $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b,a) \supset \widetilde{0}_{U\times U}$;
(iii) $\mathcal{R}$ is called connected if either $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}(b,a) \supset \widetilde{0}_{U\times U}$;
(iv) $\mathcal{R}$ is called negatively transitive if $\mathcal{R}(a,b) = \widetilde{0}_{U\times U} = \mathcal{R}(b,c) \Rightarrow \mathcal{R}(a,c) = \widetilde{0}_{U\times U}$;
(v) $\mathcal{R}$ is called transitive if $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$ and $\mathcal{R}(b,c) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(a,c) \supset \widetilde{0}_{U\times U}$.

Theorem 4.10. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then fuzzy soft set indifference relation $\mathcal{R}_I$ on $(\mathcal{F}, A)$ is semi-reflexive and semi-symmetric on $(\mathcal{F}, A)$ and fuzzy soft set weak preference relation $\mathcal{R}_W$ is semi-reflexive on $(\mathcal{F}, A)$.

Proof. Since $\mathcal{R}$ is fuzzy soft set strict preference relation on $(\mathcal{F}, A)$, then $\mathcal{R}$ is irreflexive. Hence for all $a \in A$, $\mathcal{R}(a,a) = \widetilde{0}_{U\times U}$

$\Rightarrow \mathcal{R}_I(a,a) \supset \widetilde{0}_{U\times U}$, by Definition 4.1

$\Rightarrow \mathcal{R}_I$ is semi-reflexive, by Definition 4.9.

Now by Definition 4.1, for all $a, b \in A$,

$\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(a,b) = \widetilde{0}_{U\times U}$ and $\mathcal{R}(b,a) = \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$

Hence, by Definition 4.9, $\mathcal{R}_I$ is semi-symmetric.

Since $\mathcal{R}_I$ is semi-reflexive, $\mathcal{R}_I(a,a) \supset \widetilde{0}_{U\times U}$, $\forall\, a \in A$.

Therefore by Definition 4.1, $\mathcal{R}_W(a,a) \supset \widetilde{0}_{U\times U}$, $\forall\, a \in A$. Hence, $\mathcal{R}_W$ is semi-reflexive. □

Theorem 4.11. Let $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Then for all $a, b \in A$,

$$\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U} \text{ and } \mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U} \Leftrightarrow \mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}.$$

Proof. At first let, for all $a, b \in A$, $\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U}$ and $\mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U}$. Now $\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$.

Suppose $\mathcal{R}(a,b) \supset \widetilde{0}_{U\times U}$. This implies $\mathcal{R}(b,a) = \widetilde{0}_{U\times U}$.

Again, $\mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U} \Rightarrow \mathcal{R}(b,a) \supset \widetilde{0}_{U\times U}$ or $\mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$. But $\mathcal{R}(b,a) = \widetilde{0}_{U\times U}$. So, we must have $\mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$. Since $\mathcal{R}_I$ is semi-symmetric, hence $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$.

Conversely, let $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$ for all $a, b \in A$. Then by Definition 4.1, $\mathcal{R}_W(a,b) \supset \widetilde{0}_{U\times U}$. Since $\mathcal{R}_I$ is semi-symmetric. Hence, $\mathcal{R}_I(b,a) \supset \widetilde{0}_{U\times U}$. This implies $\mathcal{R}_W(b,a) \supset \widetilde{0}_{U\times U}$. □

Note 4.12. A fuzzy soft set weak preference relation $\mathcal{R}_W$ on $(\mathcal{F}, A)$ is semi-symmetric if and only if $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$, $\forall\, a, b \in A$.

Note 4.13. The fuzzy soft set weak preference relation $\mathcal{R}_W$ on $(\mathcal{F}, A)$ may not be connected on $(\mathcal{F}, A)$, which is reflected in the following example.

Example 4.14. Take the fuzzy soft set $(\mathcal{F}, A)$ on U as in Example 4.4. Define a fuzzy soft set relation $\mathcal{P}$ on $(\mathcal{F}, A)$ as follows:

$$\mathcal{P}(b,b) = \mathcal{P}(e,e) = \mathcal{P}(m,m) = \widetilde{0}_{U\times U};$$

$$\mathcal{P}(b,e) \supset \widetilde{0}_{U\times U}, \mathcal{P}(e,b) = \widetilde{0}_{U\times U};$$

$$\mathcal{P}(e,m) \supset \widetilde{0}_{U\times U}, \mathcal{P}(m,e) = \widetilde{0}_{U\times U};$$

$$\mathcal{P}(b,m) = \left\{ \tfrac{(s_1,s_1)}{0.6}, \tfrac{(s_1,s_2)}{0.6}, \tfrac{(s_1,s_3)}{0.6}, \tfrac{(s_2,s_1)}{0.7}, \tfrac{(s_2,s_2)}{0.7}, \tfrac{(s_2,s_3)}{0}, \tfrac{(s_3,s_1)}{0}, \tfrac{(s_3,s_2)}{0}, \tfrac{(s_3,s_3)}{0.65} \right\}$$

$$\mathcal{P}(m,b) = \widetilde{0}_{U\times U}.$$

Then by Definition 4.1, $\mathcal{P}$ is a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. Now we can define a fuzzy soft set weak preference relation $\mathcal{P}_W$ on $(\mathcal{F}, A)$ with the help of Definition 4.1. Hence we have $\mathcal{P}_W(b,b)$, $\mathcal{P}_W(e,e)$, $\mathcal{P}_W(m,m)$, $\mathcal{P}_W(b,e)$, $\mathcal{P}_W(e,m) \supset \widetilde{0}_{U\times U}$.

Since $\mathcal{P}(b,e) \supset \widetilde{0}_{U\times U}$, $\mathcal{P}(e,b) = \widetilde{0}_{U\times U}$, then $\mathcal{P}_W(e,b) \not\supset \widetilde{0}_{U\times U}$. Similarly $\mathcal{P}_W(m,e) \not\supset \widetilde{0}_{U\times U}$.

Again, Since $\mathcal{P}(b,m) \not\supset \widetilde{0}_{U\times U}$ and $\mathcal{P}(m,b) = \widetilde{0}_{U\times U}$, then by Definition 4.1, $\mathcal{P}_W(b,m) \not\supset \widetilde{0}_{U\times U}$ and $\mathcal{P}_W(m,b) \not\supset \widetilde{0}_{U\times U}$. So, the fuzzy soft set weak preference relation $\mathcal{P}_W$ is not connected on $(\mathcal{F}, A)$.

Theorem 4.15. Let $(\mathcal{F}, A)$ be a fuzzy soft set over U and $\mathcal{R}$ be a fuzzy soft set strict preference relation on $(\mathcal{F}, A)$. If $\mathcal{R}$ is negatively transitive then $\mathcal{R}_I$ is transitive relation on $(\mathcal{F}, A)$.

Proof. Suppose there exist $a, b, c \in A$, such that $\mathcal{R}_I(a,b) \supset \widetilde{0}_{U\times U}$ and $\mathcal{R}_I(b,c) \supset \widetilde{0}_{U\times U}$. Now by Definition 4.1, this implies

$$\mathcal{R}(a,b) = \widetilde{0}_{U\times U} = \mathcal{R}(b,a) \text{ and } \mathcal{R}(b,c) = \widetilde{0}_{U\times U} = \mathcal{R}(c,b)$$

$\Rightarrow \mathcal{R}(a,b) = \widetilde{0}_{U\times U} = \mathcal{R}(b,c)$ and $\mathcal{R}(b,a) = \widetilde{0}_{U\times U} = \mathcal{R}(c,b)$

$\Rightarrow \mathcal{R}(a,c) = \widetilde{0}_{U\times U}$ and $\mathcal{R}(c,a) = \widetilde{0}_{U\times U}$, since $\mathcal{R}$ is negatively transitive

$\Rightarrow \mathcal{R}_I(a,c) \supset \widetilde{0}_{U\times U}$, by Definition of $\mathcal{R}_I$.

So, $\mathcal{R}_I$ is transitive relation on $(\mathcal{F}, A)$. □

Note 4.16. If a fuzzy soft set strict preference relation $\mathcal{R}$ on $(\mathcal{F}, A)$ is negatively transitive, then $\mathcal{R}$ and $\mathcal{R}_W$ may not be transitive on $(\mathcal{F}, A)$.

Example 4.17. Take the fuzzy soft set $(\mathcal{F}, A)$ on U and the fuzzy soft set strict preference relation $\mathcal{P}$ on $(\mathcal{F}, A)$ as in Example 4.14.

By Definition 4.9, we conclude that $\mathcal{P}$ is negatively transitive. But $\mathcal{P}(b,e) \supset \widetilde{0}_{U\times U}$, $\mathcal{P}(e,m) \supset \widetilde{0}_{U\times U}$ and $\mathcal{P}(b,m) \not\supset \widetilde{0}_{U\times U}$ implies $\mathcal{P}$ is not transitive on $(\mathcal{F}, A)$.

References

[1] F. Aleskerov, B. Monjardet, *Utility Maximization, Choice and Preference*, Springer-verlag, Heidelbergk, 2002.

[2] H. Aktas, N. Cagman, *Soft sets and soft groups*, Inform. Sci. 177, 2726-2735, 2007.

[3] P. Bhattacharya, N. P. Mukherjee, *Fuzzy relations and fuzzy groups*, Inform. Sci. 36, 267-282, 1985.

[4] K. V. Babitha, J. J. Sunil, *Soft set relations and functions*, Comput. Math. Appl. 60, 1840-1849, 2010.

[5] N. Cagman, S. Enginoglu, F. Citak, *Fuzzy soft set theory and its applications*, Iranian Journal of Fuzzy Systems, 8(3), 137-147, 2011.

[6] P. C. Fishburn, *Preference structures and their numerical presentations*, Theoretical Computer Science, 217, 359-389, 1999.

[7] J. Ghosh, B. Dinda, T. K. Samanta, *Fuzzy soft rings and fuzzy soft ideals*, Int. J. Pure Appl. Sci. Technol. 2(2), 66-74, 2011.

[8] J. Ghosh, T. K. Samanta, S. K. Roy, *A note on operations of intuitionistic fuzzy soft sets*, Journal of Hyperstructures, 2(2), 163-184, 2013.

[9] M. Roubens, Ph. Vincke, *Preference Modelling*, Springer-Verlag, Berlin, LNEMS 250, 1985.

[10] B. Dinda, T. K. Samanta, *Relations on intuitionistic fuzzy soft sets*, Gen. Math. Notes 1(2), 74-83, 2010.

[11] P. K. Maji, R. Biswas, A. R. Roy, *Fuzzy soft sets*, J. Fuzzy Math. 9(3), 589-602, 2001.

[12] D. Molodtsov, *Soft set theory-first results*, Comput. Math Appl. 37, 19-31, 1999.

[13] L. A. Zadeh, *Fuzzy Sets*, Inform. control 8, 338-353, 1965.

[14] N. Schofield, *Mathematical Methods in Economics and Social Choice*, Springer-Verlag, Berlin Heidelberg Newyork, 2003.

[15] Z. Xu, *Intuitionistic preference relations and their application is group decision making*, Inform. Sci. 177, 2363-2379, 2007.

[16] Z. Xu, R. R. Yager, *Intuitionistic and interval-valued intuitionistic fuzzy preference relations and theirs measures of similarity for the evaluation of agreement within a group*, Fuzzy Optim Decis Making, 8, 123-139, 2009.