http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

*Received*: 07.07.2015 *Year*: **2015**, *Number*: **6**, *Pages*: **54-65**
*Published*: 11.08.2015 *Original Article*[**]

# FUZZY OSTROWSKI TYPE INEQUALITIES FOR $(\alpha, m)-$CONVEX FUNCTIONS

**Erhan Set**[1,*] <erhanset@yahoo.com>
**Serkan Karataş**[1] <serkankaratas@odu.edu.tr>
**İlker Mumcu**[1] <mumcuilker@msn.com>

[1]*Department of Mathematics, University of Ordu, 52200, Turkey*

**Abstract** − Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior $I^{\circ}$ of $I$, and let $a, b \in I^{\circ}$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then the following inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M(b-a) \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{1}$$

for all $x \in [a, b]$.This inequality is well known in the literature as the Ostrowski inequality. In this paper, we established new Ostrowski type inequalities for $(\alpha, m)-$convex functions via fuzzy Riemann integrals.

***Keywords*** − *$(\alpha, m)-$convex function, Ostrowski inequality, Fuzzy Riemann integral.*

# 1 Introduction

In 1938, A. M. Ostrowski (see [1]) proved the following inequality, estimating the absolute value of deviation of a differentiable function by its integral mean as:

$$\left| \frac{1}{b-a} \int_a^b f(y)\,dy - f(x) \right| \leq \left( \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \, ||f'||_{\infty}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ be a diferentiable mapping on $(a, b)$ whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on $(a, b)$ that is $||f'||_{\infty} = \sup\limits_{t \in (a,b)} |f'| < \infty$.

[**]*Edited by* Oktay Muhtaroğlu *(Area Editor) and* Naim Çağman *(Editor-in-Chief).*
[*]*Corresponding Author.*

Since that time when Ostrowski proved this inequality, many mathematicians have been working on it and have been applying it in numerical analysis and probability, etc. For some applications of Ostrowski's inequality see [2]-[5] and for recent results and generalizations concerning Ostrowski's inequality see [2]-[9].

Let $I$ be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\alpha \in [0, 1]$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

(see [10], Page1). Geometrically, this means that if $A$, $B$, and $C$ are three distinct points on the graph of with $B$ between $A$ and $C$, then $B$ is on or below chord $AB$.

In [11] , Miheşan defined $(\alpha, m)$ -convexity as in the following:

The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $(\alpha, m)$ −convex, where $(\alpha, m) \in [0, 1]^2$, if one has

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Since fuzziness is a natural reality different than randomness and determinism, Anastassiou extends Ostrowski type inequalities into the fuzzy setting in 2003 [12].

The concepts of fuzzy Riemann integrals were introduced by Wu [13]. Fuzzy Riemann integral is a closed interval whose end points are the classical Riemann integrals.

# 2 Notations and Preliminaries

In this section we point out some basic definitions and notations which would help us in this work, we begin with:

**Definition 2.1.** [13] If $u : \mathbb{R} \to [0, 1]$ satisfies the following properties, then $u$ is called fuzzy number.

*i.* $u$ is normal (i.e, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$)

*ii.* $u$ is a convex fuzzy set, ie., $u(x\lambda + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, for any $x, y \in \mathbb{R}$, $\lambda \in [0, 1]$. ($u$ is called a convex fuzzy subset.)

*iii.* $u$ is upper semi continuous on $\mathbb{R}$, i.e, $\forall x_0 \in \mathbb{R}$ and $\forall \epsilon > 0$, $\exists$ neighborhood $V(x_0) : u(x) \leq u(x_0) + \epsilon$, $\forall x \in V(x_0)$.

*iv.* The set $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact where $\overline{A}$ denotes the closure of $A$.

Denote the set of all fuzzy numbers with $\mathbb{R}_{\mathcal{F}}$. For $\alpha \in (0, 1]$ and $u \in \mathbb{R}_{\mathcal{F}}$, $[u]^{\alpha} = \{x \in \mathbb{R} : u(x) \geq \alpha\}$. Then, from $(1) - (4)$ it follows that the $\alpha-$level set $[u]^{\alpha}$ is a closed interval for all $\alpha \in [0, 1]$. Moreover, $[u]^{\alpha} = [u_{-}^{(\alpha)}, u_{+}^{(\alpha)}]$ for all $\alpha \in [0, 1]$, where $u_{-}^{(\alpha)} \leq u_{+}^{(\alpha)}$ and $u_{-}^{(\alpha)}, u_{+}^{(\alpha)} \in \mathbb{R}$, i.e, $u_{-}^{(\alpha)}$ and $u_{+}^{(\alpha)}$ are the endpoints of $[u]^{\alpha}$.

**Definition 2.2.** [14] Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$. Then, the addition and scalar multiplication are defined by the equations, respectively.

*i.* $[u \oplus v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}$

*ii.* $[k \odot u]^{\alpha} = k[u]^{\alpha}$

for all $\alpha \in [0, 1]$ where $[u]^{\alpha} + [v]^{\alpha}$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $k[u]^{\alpha}$ means the usual product between a scalar and a subset of $\mathbb{R}$.

**Proposition 2.3.** [15, 16] Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$. Then, the following properties are valid.

*i.* $1 \odot u = u$

*ii.* $u \oplus v = v \oplus u$

*iii.* $k \odot u = u \odot k$

*iv.* $[u]^{\alpha_1} \subseteq [u]^{\alpha_2}$ whenever $0 \leq \alpha_2 \leq \alpha_1 \leq 1$

*v.* For any $\alpha_n$ converging increasingly to $\alpha \in (0, 1]$, $\bigcap_{n=1}^{\infty} [u]^{\alpha_n} = [u]^{\alpha}$.

**Definition 2.4.** [14] Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{+} \cup \{0\}$ be a function defined by the equation

$$D(u, v) := \sup_{\alpha \in [0,1]} \max \left\{ |u_{-}^{(\alpha)} - v_{-}^{(\alpha)}|, |u_{+}^{(\alpha)} - v_{+}^{(\alpha)}| \right\}$$

for all $u, v \in \mathbb{R}_{\mathcal{F}}$. Then, $D$ is a metric on $\mathbb{R}_{\mathcal{F}}$.

Now, using the results of [13, 16], for all $u, v, v, w, e \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$ we have that

*i.* $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space

*ii.* $D(u \oplus w, v \oplus w) = D(u, v)$

*iii.* $D(k \odot u, k \odot v) = |k| d(u, v)$

*iv.* $D(u \oplus v, w \oplus e) = D(u, w) + D(v, e)$

*v.* $D(u \oplus v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0})$

*vi.* $D(u \oplus v, w) \leq D(u, w) + D(v, \tilde{0})$

where $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ is defined $\tilde{0}(x) = 0$ for all $x \in \mathbb{R}$.

**Definition 2.5.** [14] Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y \oplus z$, then we call $z$ the H-difference of $x$ and $y$, denoted by $z = x \ominus y$.

**Definition 2.6.** [14] Let $T := [x_0, x_0 + \beta] \subseteq \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists a $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to the metric $D$)

$$\lim_{h\to 0^+} \frac{f(x+h) \ominus f(x)}{h}, \lim_{h\to 0^+} \frac{f(x) \ominus f(x-h)}{h}$$

exist and are equal to $f'(x)$. We call $f'$ the derivative or H-derivative of f at $x$. If $f$ is H-differentiable at any $x \in T$, we call f differentiable or H-differentiable and it has H-derivative over $T$ the function $f'$.

We use a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$) introduced in [14], Definition 2.1. That is,

**Definition 2.7.** [18] Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. We say that $f$ is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]; \xi\}$ of $[a, b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_{P}^{*}(v-u) \odot f(\xi, I)\right) < \varepsilon$$

where $\sum^{*}$ denotes the fuzzy summation. We choose to write

$$I := (FR)\int_a^b f(x)dx$$

We also call an $f$ as above $(FR)$-integrable.

For some recent results connected with Fuzzy-Riemann integrals, see ([17]).
The main purpose of the this paper is to establish fuzzy Ostrowski type inequalties for fuzzy Riemann integral and $(\alpha, m)$-convex functions.

# 3 Main Results

In order to establish our main results we need the following lemma.

**Lemma 3.1.** Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on $I^{\circ}$ where $ma, mb \in I$ with $ma < mb$. If $f' \in C_F[ma, mb] \cap L_F[ma, mb]$, then we have the equality for differentiable function as follow:

$$\begin{aligned} &\frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(x-ma)^2}{b-a} \odot (FR)\int_0^1 t \odot f'(tx + m(1-t)a)\,dt \\ = \;& m \odot f(x) \oplus \frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 t \odot f'(tx + m(1-t)b)\,dt \end{aligned}$$

for $x \in (ma, mb)$.

*Proof.* By integration by parts and using properties of $\alpha$-cut of fuzzy numbers, we have following identities

$$\begin{aligned}
&\left[\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)b)dt\right]^{\alpha} \qquad (2)\\
=\ &\frac{(mb-x)^2}{b-a}\left[\left(\int_0^1 tf'(tx+m(1-t)b)dt\right)_{\alpha}^{-},\left(\int_0^1 tf'(tx+m(1-t)b)dt\right)_{\alpha}^{+}\right]\\
=\ &\frac{mb-x}{a-b}\left[\left(f(x)-\frac{1}{x-mb}\int_{mb}^{x}f(u)du\right)_{\alpha}^{-},\left(f(x)-\frac{1}{x-mb}\int_{mb}^{x}f(u)du\right)_{\alpha}^{+}\right]\\
=\ &\frac{mb-x}{a-b}\left[f(x)\oplus\frac{1}{mb-x}\odot(FR)\int_{mb}^{x}f(u)du\right]^{\alpha}
\end{aligned}$$

and

$$\begin{aligned}
&\left[\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)a)dt\right]^{\alpha} \qquad (3)\\
=\ &\frac{(x-ma)^2}{b-a}\left[\left(\int_0^1 tf'(tx+m(1-t)a)dt\right)_{\alpha}^{-},\left(\int_0^1 tf'(tx+m(1-t)a)dt\right)_{\alpha}^{+}\right]\\
=\ &\frac{x-ma}{b-a}\left[\left(f(x)-\frac{1}{ma-x}\int_{ma}^{x}f(u)du\right)_{\alpha}^{-},\left(f(x)-\frac{1}{ma-x}\int_{ma}^{x}f(u)du\right)_{\alpha}^{+}\right]\\
=\ &\frac{x-ma}{b-a}\left[f(x)\oplus\frac{1}{ma-x}\odot(FR)\int_{ma}^{x}f(u)du\right]^{\alpha}.
\end{aligned}$$

By adding (2) and (3) we have

$$\begin{aligned}
&\left[\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)b)dt\right]^{\alpha}\\
&+\left[\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 t\odot f'(tx+m(1-t)b)dt\right]^{\alpha}\\
&=\left[m\odot f(x)\oplus\frac{1}{a-b}\odot(FR)\int_{ma}^{mb}f(u)du\right]^{\alpha}
\end{aligned}$$

which the proof is completed. □

**Theorem 3.2.** Let $f : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be differentiable mapping on $I$ such that $f' \in C_F[ma, mb] \cap L_F[ma, mb]$, where $ma, mb \in I$ with $ma < mb$. If $D(f'(x), 0)$ is $(\alpha, m)$ −convex on $[ma, mb]$ for $(\alpha, m) \in [0, 1] \times [0, 1]$ and $D\left(f'(x), \tilde{0}\right) \le M$, then the following inequality holds:

$$\begin{aligned}
&D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right)\\
\le\ &M\frac{\alpha m+2}{2(\alpha+2)}\left(\frac{(x-ma)^2+(mb-x)^2}{b-a}\right)
\end{aligned}$$

for each $x \in [ma, mb]$.

*Proof.* From Lemma 3.1

$$
\begin{aligned}
& D\left(m \odot f(x), \frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx\right) \\
= \; & D\left(m \odot f(x) \oplus \frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 t \odot f'(tx+m(1-t)b)\,dt,\right. \\
& \frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(mb-x)^2}{b-a} \\
& \left.\odot (FR)\int_0^1 t \odot f'(tx+m(1-t)b)\,dt\right) \\
= \; & D\left(\frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(x-ma)^2}{b-a}\right. \\
& \odot (FR)\int_0^1 t \odot f'(tx+m(1-t)a)\,dt, \\
& \frac{1}{b-a} \odot (FR)\int_{ma}^{mb} f(x)\,dx \oplus \frac{(mb-x)^2}{b-a} \\
& \left.\odot (FR)\int_0^1 t \odot f'(tx+m(1-t)b)\,dt\right) \\
= \; & D\left(\frac{(x-ma)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\right. \\
& \left.\frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right) \\
\leq \; & D\left(\frac{(x-ma)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)a)\,dt, \widetilde{0}\right) \\
& + D\left(\frac{(mb-x)^2}{b-a} \odot (FR)\int_0^1 tf'(tx+m(1-t)b)\,dt, \widetilde{0}\right) \\
= \; & \frac{(x-ma)^2}{b-a} D\left((FR)\int_0^1 tf'(tx+m(1-t)a)\,dt, \widetilde{0}\right) \\
& + \frac{(mb-x)^2}{b-a} D\left((FR)\int_0^1 tf'(tx+m(1-t)b)\,dt, \widetilde{0}\right) \\
\leq \; & \frac{(x-ma)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)a), \tilde{0}\right)dt \\
& + \frac{(mb-x)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)b), \tilde{0}\right)dt \qquad (4)
\end{aligned}
$$

Since $D\left(f'(x),\tilde{0}\right)$ is $(\alpha,m)$ −convex and $D\left(f'(x),\tilde{0}\right)\leq M$, then we have

$$\begin{aligned} D\left(f'(tx+m(1-t)a),\tilde{0}\right) &\leq t^{\alpha}D\left(f'(x),\widetilde{0}\right)+m(1-t^{\alpha})D\left(f'(a),\tilde{0}\right) \\ &\leq M\left[t^{\alpha}+m(1-t^{\alpha})\right] \end{aligned} \tag{5}$$

$$\begin{aligned} D\left(f'(tx+m(1-t)b),\tilde{0}\right) &\leq t^{\alpha}D\left(f'(x),\widetilde{0}\right)+m(1-t^{\alpha})D\left(f'(b),\tilde{0}\right) \\ &\leq M\left[t^{\alpha}+m(1-t^{\alpha})\right] \end{aligned} \tag{6}$$

By using (5)and (6) in (4), we get

$$\begin{aligned} & D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right) \\ \leq\ & M\frac{\alpha m+2}{2(\alpha+2)}\left(\frac{(x-ma)^2+(mb-x)^2}{b-a}\right) \end{aligned}$$

□

**Theorem 3.3.** Let $f:I\subset\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ be differentiable mapping on $I$ such that $f'\in C_F[ma,mb]\cap L_F[ma,mb]$, where $ma,mb\in I$ with $ma<mb$. If $\left[D\left(f'(x),0\right)\right]^q$ is $(\alpha,m)$ −convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]\times[0,1]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $D\left(f'(x),\widetilde{0}\right)\leq M$, then the following inequality holds:

$$\begin{aligned} & D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right) \\ \leq\ & \left(\frac{1}{p+1}\right)^{1/p}M\left(\frac{1+\alpha m}{\alpha+1}\right)\left(\frac{(x-ma)^2+(mb-x)^2}{b-a}\right) \end{aligned}$$

for each $x\in[ma,mb]$.

*Proof.* From Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned} & D\left(m\odot f(x),\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\right) \\ =\ & D\left(m\odot f(x)\oplus\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\right. \\ & \frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(mb-x)^2}{b-a} \\ & \left.\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right) \\ =\ & D\left(\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\right. \end{aligned}$$

$$\frac{1}{b-a}\odot(FR)\int_{ma}^{mb}f(x)\,dx\oplus\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\Bigg)$$

$$= D\left(\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\right.$$

$$\left.\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right)$$

$$\begin{aligned}
&\le D\left(\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\widetilde{0}\right)\\
&\quad+D\left(\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\widetilde{0}\right)\\
&= \frac{(x-ma)^2}{b-a}D\left((FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\widetilde{0}\right)\\
&\quad+\frac{(mb-x)^2}{b-a}D\left((FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\widetilde{0}\right)\\
&\le \frac{(x-ma)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)a),\tilde{0}\right)dt\\
&\quad+\frac{(mb-x)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)b),\tilde{0}\right)dt\\
&\le \frac{(x-ma)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1\left[D\left(f'(tx+m(1-t)a),\tilde{0}\right)\right]^q dt\right)^{1/q}\\
&\quad+\frac{(mb-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1\left[D\left(f'(tx+m(1-t)b),\tilde{0}\right)\right]^q dt\right)^{1/q}
\end{aligned}$$

Since $\left[D\left(f'(x),\tilde{0}\right)\right]^q$ is $(\alpha,m)-$convex and $D\left(f'(x),\tilde{0}\right)\le M$, then we have

$$\begin{aligned}
\left[D\left(f'(tx+m(1-t)a),\tilde{0}\right)\right]^q &\le t^\alpha D\left(f'(x),\tilde{0}\right)^q+m(1-t^\alpha)D\left(f'(a),\tilde{0}\right)^q\\
&\le M^q\left[t^\alpha+m(1-t^\alpha)\right]
\end{aligned}\tag{7}$$

$$\begin{aligned}
\left[D\left(f'(tx+m(1-t)b),\tilde{0}\right)\right]^q &\le t^\alpha D\left(f'(x),\tilde{0}\right)^q+m(1-t^\alpha)D\left(f'(b),\tilde{0}\right)^q\\
&\le M^q\left[t^\alpha+m(1-t^\alpha)\right]
\end{aligned}\tag{8}$$

By using (5)and (6) in (4), we get

$$\begin{aligned}
& D\left(m\odot f\left(x\right),\frac{1}{b-a}\odot\left(FR\right)\int_{ma}^{mb}f\left(x\right)dx\right)\\
\leq\ & \frac{\left(x-ma\right)^{2}}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(M^{q}\int_{0}^{1}\left[t^{\alpha}+m\left(1-t^{\alpha}\right)\right]dt\right)^{1/q}\\
& +\frac{\left(mb-x\right)^{2}}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(M^{q}\int_{0}^{1}\left[t^{\alpha}+m\left(1-t^{\alpha}\right)\right]dt\right)^{1/q}\\
=\ & M\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1+\alpha m}{\alpha+1}\right)^{1/q}\left(\frac{\left(x-ma\right)^{2}+\left(mb-x\right)^{2}}{b-a}\right)
\end{aligned}$$

□

**Theorem 3.4.** Let $f : I\subset\mathbb{R}\to R_F$ be differentiable mapping on $I$ such that $f'\in C_F\left[ma,mb\right]\cap L_F\left[ma,mb\right]$, where $ma,mb\in I$ with $ma<mb$.If $\left[D\left(f'\left(x\right),\tilde{0}\right)\right]^{q}$ is $\left(\alpha,m\right)-$convex on $\left[ma,mb\right]$ for $\left(\alpha,m\right)\in\left[0,1\right]\times\left(0,1\right]$, $q\in\left[1,\infty\right)$ and $D\left(f'\left(x\right),\tilde{0}\right)\leq M$, then the following inequality holds:

$$\begin{aligned}
& D\left(m\odot f\left(x\right),\frac{1}{b-a}\odot\left(FR\right)\int_{ma}^{mb}f\left(x\right)dx\right)\\
\leq\ & M\left(\frac{2+\alpha m}{\alpha+2}\right)^{1/q}\frac{\left(x-ma\right)^{2}+\left(mb-x\right)^{2}}{2\left(b-a\right)}
\end{aligned}$$

*Proof.* From Lemma 3.1 and using the well-known power-mean inequality, we get

$$\begin{aligned}
& D\left(m\odot f\left(x\right),\frac{1}{b-a}\odot\left(FR\right)\int_{ma}^{mb}f\left(x\right)dx\right)\\
=\ & D\left(m\odot f\left(x\right)\oplus\frac{\left(mb-x\right)^{2}}{b-a}\odot\left(FR\right)\int_{0}^{1}tf'\left(tx+m\left(1-t\right)b\right)dt,\right.\\
& \frac{1}{b-a}\odot\left(FR\right)\int_{ma}^{mb}f\left(x\right)dx\oplus\frac{\left(mb-x\right)^{2}}{b-a}\\
& \left.\odot\left(FR\right)\int_{0}^{1}tf'\left(tx+m\left(1-t\right)b\right)dt\right)\\
=\ & D\left(\frac{1}{b-a}\odot\left(FR\right)\int_{ma}^{mb}f\left(x\right)dx\oplus\frac{\left(x-ma\right)^{2}}{b-a}\right.\\
& \odot\left(FR\right)\int_{0}^{1}tf'\left(tx+m\left(1-t\right)a\right)dt,\frac{1}{b-a}\odot\left(FR\right)\int_{ma}^{mb}f\left(x\right)dx
\end{aligned}$$

$$
\begin{aligned}
&\oplus\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\Bigg) \\
= \;& D\left(\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\right. \\
&\left.\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt\right) \\
\le \;& D\left(\frac{(x-ma)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\widetilde{0}\right) \\
&+D\left(\frac{(mb-x)^2}{b-a}\odot(FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\widetilde{0}\right) \\
= \;& \frac{(x-ma)^2}{b-a}D\left((FR)\int_0^1 tf'(tx+m(1-t)a)\,dt,\widetilde{0}\right) \\
&+\frac{(mb-x)^2}{b-a}D\left((FR)\int_0^1 tf'(tx+m(1-t)b)\,dt,\widetilde{0}\right)
\end{aligned}
$$

$$
\begin{aligned}
\le \;& \frac{(x-ma)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)a),\tilde{0}\right)dt \\
&+\frac{(mb-x)^2}{b-a}\int_0^1 tD\left(f'(tx+m(1-t)b),\tilde{0}\right)dt \\
\le \;& \frac{(x-ma)^2}{b-a}\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t\left[D\left(f'(tx+m(1-t)a),\tilde{0}\right)\right]^q dt\right)^{1/q} \\
&+\frac{(mb-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-1/q}\left(\int_0^1 t\left[D\left(f'(tx+m(1-t)b),\tilde{0}\right)\right]^q dt\right)^{1/q}
\end{aligned}
$$

$\left[D\left(f'(x),\tilde{0}\right)\right]^q$ is $(\alpha,m)$ −convex and $D\left(f'(x),\tilde{0}\right)\le M$, so we know

$$
\begin{aligned}
&\int_0^1 t\left[D\left(f'(tx+m(1-t)a),\tilde{0}\right)\right]^q dt \\
\le \;& \int_0^1 t[t^\alpha\left(D\left(f'(x),\tilde{0}\right)\right)^q+m(1-t^\alpha)\left(D\left(f'(a),\tilde{0}\right)\right)^q]dt \\
\le \;& \frac{M^q}{\alpha+2}\left(1+\frac{\alpha m}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 t\left[D\left(f'(tx+m(1-t)b),\tilde{0}\right)\right]^q dt \\
\le \;& \int_0^1 t[t^\alpha\left(D\left(f'(x),\tilde{0}\right)\right)^q+m(1-t^\alpha)\left(D\left(f'(b),\tilde{0}\right)\right)^q]dt \\
\le \;& \frac{M^q}{\alpha+2}\left(1+\frac{\alpha m}{2}\right)
\end{aligned}
$$

Therefore, we know

$$\begin{aligned} & D\left(m \odot f(x), \frac{1}{b-a} \odot (FR) \int_{ma}^{mb} f(x)\, dx\right) \\ \leq \quad & M\left(\frac{2+\alpha m}{\alpha+2}\right)^{1/q} \frac{(x-ma)^2+(mb-x)^2}{2(b-a)} \end{aligned}$$

□

# References

[1] A. Ostrowski, Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert,Comment. Math. Helv., 10 (1938), 226–227.

[2] M. Alomari, M. Darus, Some Ostrowski's type inequalities for convex functions with applications, RGMIA 13(1) (2010), Article 3. [ONLINE: http://ajmaa.org/RGMIA/ v13n1.php]

[3] M. Alomari, M. Darus, S. S. Dragomir, P. Cerone, Ostrowski's inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett. 23 (9) (2010), 1071-1076.

[4] P. Cerone, S. S. Dragomir J. Roumeliotis, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications, RGMIA Res. Rep. Coll. 1(1) (1998), Article 4.

[5] S. S. Dragomir, S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules , Appl. Math. Lett. 11 (1998), 105–109.

[6] M. Z. Sarıkaya, On the Ostrowski type integral inequality, Acta Math. Univ. Comenianae 79(1) (2010), 129–134.

[7] E. Set, M. E. Özdemir, M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications, Facta Univ. Ser. Math. Inform. 27(1) (2012), 67–82.

[8] M. E. Özdemir, H. Kavurmaci, E. Set, Ostrowski's type inequalities for ($\alpha$, m)-convex functions, Kyungpook Math. J. 50 (2010), 371–378.

[9] M. Z. Sarıkaya, E. Set, M. E. Özdemir, On some integral inequalities for twice differentiable mappings, Stud. Univ. Babeş-Bolyai Math. 59(1) (2014), 11–24.

[10] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, (1993).

[11] V. G. Miheşan, A generalization of the convexity, Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, (Romania), (1993).

[12] G.A. Anastassiou, Fuzzy Ostrowski type inequalities, Computational and Applied Mathematics, Vol. 22, No. 2 (2003), pp. 279-292.

[13] W. Congxin, M. Ming, On embedding problem of fuzzy number space: Part 1, Fuzzy Sets and Systems 44 (1991) 33–38.

[14] W.Congxin and G. Zengtai, On Henstock integral of fuzzy-number-valued functions (I), Fuzzy Sets and Systems 120 (2001) 523–532.

[15] O. Kaleva, Fuzzy diferential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[16] M. Ming, On embedding problem of fuzzy number space: Part 4, Fuzzy Sets and Systems 58 (1993) 185–193.

[17] M. A. Latif and S. Hussain, Two-point fuzzy Ostrowski type inequalities, International Journal of Analysis and Applications, Volume 3, Number 1 (2013), 35–46.

[18] S. Gal, Approximation theory in fuzzy setting, Chapter 13 in Handbook of Analytic Computational Methods in Applied Mathematics (edited by G. Anastassiou), Chapman & Hall, CRC Press, Boca Raton, New York, 2000, pp. 617–666.