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SOME VARIATIONS OF JANOWSKI TYPE FUNCTIONS ASSOCIATED WITH M-SYMMETRIC POINTS

Khalida Inayat Noor[1] <khalidanoor@hotmail.com>
Nasir Khan[1,*] <nasirmaths@yahoo.com>

[1]Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.

Abstaract − In the present paper, we introduce a new subclass $S_s^m(b, \gamma, A, B)$, of starlike functions with respect to m-symmetric points. Some basic properties, Integral representations, first Hankel determinant and convolution properties for the functions belonging to this class are investigated.

Keywords − *Janowski type functions, convolution, symmetric points.*

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Denote by E, the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k, \tag{2}$$

which are analytic in the open unit disk E and satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$. For two functions $f(z)$ and $g(z)$ analytic in E, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent in E then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. The idea of subordinations goes back to Lindelöf [8]. Subordination was more formally introduced and studied by Littelwood [9] and later by Rogosinski [15] and [16]. The concept of subordination was considered by Miller [12] and further investigated by Noor et al [13] and many others see [8, 17].

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* *Corresponding Author.*

Sakaguchi [18] introduced a class of functions starlike with respect to symmetric points, it consists of functions $f(z) \in S$, satisfying the inequality

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \qquad (z \in E). \tag{3}$$

Chand and Singh [2] introduced a class $\mathcal{S}_s^m$ of functions starlike with respect to m-symmetric points, which consists of functions $f(z) \in S$ satisfying the inequality

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f_m(z)}\right\} > 0, \qquad (z \in E), \tag{4}$$

where

$$f_m(z) = \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{-\upsilon} f(\epsilon^{\upsilon} z), \quad (\epsilon^m = 1 : m \in \mathbb{N}). \tag{5}$$

From (5), we can write

$$\begin{aligned} f_m(z) &= \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{-\upsilon} f(\epsilon^{\upsilon} z) = \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{-\upsilon}\left[\epsilon^{\upsilon} z + \sum_{n=2}^{\infty} a_n (\epsilon^{\upsilon} z)^n\right] \qquad (6)\\ &= z + \sum_{n=2}^{\infty} a_n \psi_n z^n, \qquad (7)\end{aligned}$$

where,

$$\psi_n = \frac{1}{m}\sum_{\upsilon=0}^{m-1} \epsilon^{(n-1)\upsilon}, \quad (m \in \mathbb{N};\ n \geq 2;\ \epsilon^m = 1). \tag{8}$$

Note that the following identities follow directly from the above definition (5),

$$f_m(\epsilon^{\upsilon} z) = \epsilon^{\upsilon} f_m(z), \tag{9}$$

$$f'_m(\epsilon^{\upsilon} z) = f_m(z) = \frac{1}{m}\sum_{\upsilon=0}^{m-1} f(\epsilon^{\upsilon} z), \quad (z \in E). \tag{10}$$

Using the concept of subordination we introduce a subclass $\mathcal{S}_s^m(b, \gamma, A, B)$ as follows.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^m(b, \gamma, A, B)$, if it satisfies the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \quad (z \in E), \tag{11}$$

where $b \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$ and $0 < \gamma \leq 1$.

Where $f_m(z)$ is defined in equation (5). To avoid repetitions, it is admitted once that $b \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$ and $0 < \gamma \leq 1$.

Special Cases;

(i) For $b = 1$, $\gamma = 1$, we obtain the class studied by Al-Shaqsi and Darus [1].

(ii) For $b = 1$, $\gamma = 1, A = \beta, B = -\alpha\beta$, we obtain the class studied by Gao and Zhou [3].

(iii) For $m = 1, b = 1, \gamma = 1$, we obtain the class studied by Janowski [4].

(iv) For $m = 1$, $b = e^{-i\lambda}\lambda$, $\gamma = 1, A = 1 - 2\delta$, $B = -1$, we obtain the class studied by Keogh and Markes [5].

(v) For $m = 1, b = (1-\rho)\, e^{-i\beta}, \gamma = 1, A = 1, B = -1$, we obtain the class studied by Libera [7].

(vi) For $b = 1$, $A = 1, B = -1$, we obtain the class studied by Ming-Sheng Liu and Yu-Can Zhu [10].

(vii) For $m = 2, b = 1$, $A = 1, B = -1$, we obtain the class studied by V. Ravichandran [14].

$(viii)$ For $m = 2, b = 1$, $\gamma = 1, A = 1, B = -1$, we obtain the class studied by Sakaguchi [18].

2 Preliminary

To prove our main results we need the following Lemmas.

Lemma 2.1. [11] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic function with positive real part in E and υ is a complex number, then

$$\left|c_2 - \upsilon c_1^2\right| \leq 2 \max\{1, |2\upsilon - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z}{1-z}, \quad p(z) = \frac{1+z^2}{1-z^2}.$$

Lemma 2.2. [11] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic function with positive real part in E, then

$$\left|c_2 - \upsilon c_1^2\right| \leq \begin{cases} -4\upsilon + 2, & (\upsilon \leq 0), \\ 2, & (0 \leq \upsilon \leq 1), \\ 4\upsilon - 2, & (\upsilon \geq 1). \end{cases}$$

When $\upsilon < 0$ or $\upsilon > 0$, equality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \upsilon < 1$, then the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $\upsilon = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1+\xi}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\xi}{2}\right)\left(\frac{1-z}{1+z}\right), \quad (0 \leq \xi \leq 1),$$

or one of its rotations. For $\upsilon = 1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that equality holds in the case of $\upsilon = 0$. Although the above upper bound is sharp, it can be improved as follows when $0 < \upsilon < 1$:

$$\begin{aligned} \left|c_2 - \upsilon c_1^2\right| + \upsilon \left|c_1\right|^2 &\leq 2, \quad (0 < \upsilon \leq \frac{1}{2}), \\ \left|c_2 - \upsilon c_1^2\right| + (1-\upsilon) \left|c_1\right|^2 &\leq 2, \quad (\frac{1}{2} \leq \upsilon < 1). \end{aligned}$$

In this paper, we investigate integral representation, Feketo-Szegö inequality and convolution properties for the class $\mathcal{S}_s^m(b, \gamma, A, B)$. The motivation of this paper is to improve and generalize previously known results.

3 Main Results

3.1 Integral Representation

First we give meaningful conclusion about the class $\mathcal{S}_s^m(b, \gamma, A, B)$.

Theorem 3.1. Let $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$, then $f_m(z) \in \mathcal{S}^*(b, \gamma, A, B) \subset \mathcal{S}$.

Proof. For $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$, we can obtain

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \quad (z \in E). \tag{12}$$

Substituting z by $\epsilon^{\mu} z$ respectively $(\upsilon = 0, 1, 2, 3, \ldots, m-1)$, we have

$$1 + \frac{1}{b}\left(\frac{\epsilon^{\upsilon} z f'(\epsilon^{\upsilon} z)}{f_m(\epsilon^{\upsilon} z)} - 1\right) \prec \left(\frac{1+A(\epsilon^{\upsilon} z)}{1+B(\epsilon^{\upsilon} z)}\right)^{\gamma} \prec \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \quad (z \in E). \tag{13}$$

By definition of $f_m(z)$ and $\epsilon = \exp\left(\frac{2\pi}{m}\right)$, we know $\epsilon^{-\upsilon} f_m(\epsilon^{\upsilon} z) = f_m(z)$. Then equation (13) becomes

$$1 + \frac{1}{b}\left(\frac{zf'(\epsilon^{\upsilon} z)}{f_m(\epsilon^{\upsilon} z)} - 1\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \quad (z \in E). \tag{14}$$

Let $\upsilon = 0, 1, 2, 3, \ldots, m-1$ in (14), respectively and sum them we can get

$$1 + \frac{1}{b}\left(\frac{zf_m'(z)}{f_m(z)} - 1\right) = 1 + \frac{1}{b}\left(\frac{1}{m}\sum_{\mu=0}^{m-1}\frac{zf'(\epsilon^{\upsilon} z)}{f_m(z)} - 1\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\gamma}, \quad (z \in E).$$

That is, $f_m(z) \in \mathcal{S}_s^m(b, \gamma, A, B) \subset \mathcal{S}$. □

Putting $b = 1, \gamma = 1$ in Theorem 3.1, we can obtain the following result obtained by O. Kwon and Y. Sim [6].

Corollary 3.2. Let $f(z) \in \mathcal{S}_s^m(A, B)$, then $f_m(z) \in \mathcal{S}^*(A, B) \subset \mathcal{S}$.

Putting $b = 1$, $\gamma = 1$, $A = 1$, $B = -1$ and $m = 2$, in Theorem 3.1, we can obtain the Corollary 3.3, below which is comparable to the corollary of O. Kwon and Y. Sim [6, Cor.2.2].

Corollary 3.3. Let $f(z) \in \mathcal{S}_s^m$, defined as (3). Then the odd function,

$$\frac{1}{2}\left(f(z) - f(-z)\right),$$

is a starlike function.

Now we give the integral representations of the functions belonging to the class $\mathcal{S}_s^m(b, \gamma, A, B)$.

Theorem 3.4. Let $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$. Then

$$f_m(z) = z \cdot \exp\int_0^{\epsilon^{\upsilon} z}\left\{\frac{b}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{t}\left(\left(\frac{1+Aw(t)}{1+Bw(t)}\right)^{\gamma} - 1\right)\right\}\mathrm{d}t. \tag{15}$$

where $f_m(z)$ is define by (5), $w(z)$ is analytic in E with $w(0) = 0$, and $|w(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$. It is easy to know that condition (11), can be written as

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) = \left(\frac{1 + Aw(z)}{1 + Bw(z)}\right)^{\gamma}, \tag{16}$$

where $w(z)$ is analytic in E and $w(0) = 0$, $|w(z)| < 1$. Substituting z by $\epsilon^{\upsilon} z$ respectively $(\upsilon = 0, 1, 2, \ldots, m - 1)$, we have

$$\frac{zf'(\epsilon^{\upsilon} z)}{\epsilon^{-\upsilon} f_m(\epsilon^{\upsilon} z)} = b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon} z)}{1 + Bw(\epsilon^{\upsilon} z)}\right)^{\gamma} - 1\right) + 1. \tag{17}$$

Letting $\upsilon = 0, 1, 2, \ldots, m - 1$ in (17), respectively and summing them we can obtain

$$\frac{zf'_m(z)}{f_m(z)} = \frac{1}{m}\sum_{\upsilon=0}^{m-1}\left\{b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon} z)}{1 + Bw(\epsilon^{\upsilon} z)}\right)^{\gamma} - 1\right) + 1\right\}. \tag{18}$$

From the equality (18), we can obtain

$$\frac{f'_m(z)}{f_m(z)} - \frac{1}{z} = \frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{z}\left\{b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon} z)}{1 + Bw(\epsilon^{\upsilon} z)}\right)^{\gamma} - 1\right) + 1\right\} - \frac{1}{z}. \tag{19}$$

Integrating equality (19), we have

$$\begin{aligned} \log\frac{f_m(z)}{z} &= \int_0^z \left\{\frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{\zeta}\left\{b\left(\left(\frac{1 + Aw(\epsilon^{\upsilon}\zeta)}{1 + Bw(\epsilon^{\upsilon}\zeta)}\right)^{\gamma} - 1\right) + 1\right\} - \frac{1}{\zeta}\right\} d\zeta \\ &= \int_0^{\epsilon^{\upsilon} z} \left\{\frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{t}\left\{b\left(\left(\frac{1 + Aw(t)}{1 + Bw(t)}\right)^{\gamma} - 1\right) + 1\right\} - \frac{1}{t}\right\} dt. \end{aligned} \tag{20}$$

That is

$$f_m(z) = z \cdot \exp\int_0^{\epsilon^{\upsilon} z}\left\{\frac{1}{m}\sum_{\upsilon=0}^{m-1}\frac{b}{t}\left(\left(\frac{1 + Aw(t)}{1 + Bw(t)}\right)^{\gamma} - 1\right)\right\} dt.$$

Hence the proof of the Theorem 3.4 is complete. □

Putting $b = 1$ and $\gamma = 1$ in Theorem 3.4, we can obtain the following result obtained by O. Kwon and Y. Sim [6].

Corollary 3.5. Let $f(z) \in \mathcal{S}_m(A, B)$. Then

$$f_m(z) = z \cdot \exp\left\{\frac{A - B}{m}\sum_{\upsilon=0}^{m-1}\int_0^{\epsilon^{\upsilon} z}\frac{w(t)}{t(1 + Bw(t))} dt\right\},$$

where $f_m(z)$ is define by (5) and $w(z)$ is analytic in E with $w(0) = 0$, $|w(z)| < 1$.

Theorem 3.6. Let $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$. Then

$$f(z) = \int_0^z \left\{\begin{array}{c} \exp\int_0^{\epsilon^{\upsilon} z}\left\{\frac{b}{m}\sum_{\upsilon=0}^{m-1}\frac{1}{t}\left(\left(\frac{1+Aw(t)}{1+Bw(t)}\right)^{\gamma} - 1\right)\right\} dt \\ \times\left(1 + b\left(\left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)}\right)^{\gamma} - 1\right)\right) \end{array}\right\} d\zeta, \tag{21}$$

where $f_m(z)$ is define by (5) and $w(z)$ is analytic in E with $w(0) = 0$, $|w(z)| < 1$.

Proof. Let $f(z) \in \mathcal{S}_s^m(b, \gamma, A, B)$. Then from equalities (15) and (16) we have

$$f'(z) = \frac{f_m(z)}{z} \cdot \left(1 + b\left(\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma - 1\right)\right)$$

$$\begin{aligned} f'(z) &= \exp \int_0^{\epsilon^\upsilon z} \left\{ \frac{1}{m} \sum_{\upsilon=0}^{m-1} \frac{1}{t} \left\{ b\left(\left(\frac{1+Aw(t)}{1+Bw(t)}\right)^\gamma - 1\right) + 1 \right\} - \frac{1}{t} \right\} \mathrm{d}t \\ & \times \left(1 + b\left(\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma - 1\right)\right). \end{aligned}$$

Integrating the above equality, we can obtain (21). Hence the proof of the Theorem 3.6 is complete. □

Putting $b = 1$ and $\gamma = 1$ in Theorem 3.6, we can obtain the following result obtained by O. Kwon and Y. Sim [6].

Corollary 3.7. Let $f(z) \in S_s^m(A, B)$. Then

$$f(z) = \int_0^z \exp\left\{ \frac{A-B}{m} \sum_{\upsilon=0}^{m-1} \int_0^{\epsilon^\upsilon \zeta} \frac{w(t)}{t(1+Bw(t))} \mathrm{d}t \right\} \cdot \left(\frac{1+Aw(\zeta)}{1+Bw(\zeta)} \right) \mathrm{d}\zeta,$$

where $f_m(z)$ is define by (5) and $w(z)$ is analytic in E with $w(0) = 0$, $|w(z)| < 1$.

3.2 Coefficient Problems

A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. Usually, there is a parameter over which the extremal value of the functional is needed. Here we deals with one important functional of this type the Fekete-Szegö functional. In this sections, we proved the first Hankel determinant and Feketo-Szegö inequality for the functions belonging to the class $S_s^m(b, \gamma, A, B)$.

Theorem 3.8. Let $f(z) \in S_s^m(b, \gamma, A, B)$. Then

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\le \frac{|b|\,\gamma\,(A-B)}{(3-\psi_3)} \\ & \max\left\{1, \left| \begin{array}{c} \frac{2}{(2-\psi_2)^2}\{(2-\psi_2^2)\left[B + \frac{1}{2}(1-\gamma)(A-B)\right] - b\gamma(A-B) \\ ((2-\psi_2)\psi_2 - (3-\psi_3)\mu) - (2-\psi_2)^2\} \end{array} \right| \right\}, \end{aligned}$$

where ψ_n, is given in (8). The result is sharp.

Proof. Suppose that $f(z) \in S_s^m(b, \gamma, A, B)$. It is easy to know that condition (11), can be written as

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma, \tag{22}$$

where $w(z)$ is analytic in E and $w(0) = 0$, $|w(z)| < 1$. We can write equation (22), as

$$\frac{zf'(z)}{f_m(z)} = 1 + b\left(\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\gamma - 1\right). \tag{23}$$

By expanding the L.H.S of (23), we can obtain

$$1 + (2 - \psi_2)\, a_2 z + \left((3 - \psi_3)\, a_3 + \left(\psi_2^2 - 2\psi_2\right) a_2^2\right) z^2 + \cdots, \tag{24}$$

similarly expanding the R.H.S of (23), we can obtain

$$1+b\times(\gamma\,(A-B)\,c_1 z$$
$$+(\gamma\,(A-B)\,c_2 - \gamma\,(A-B)\,[B + \frac{1}{2}\,(1-\gamma)\,(A-B)]c_1^2)z^2 + \cdots). \tag{25}$$

Equating the coefficients of z and z^2 in (24) and (25) we have

$$a_2 = \frac{b\gamma\,(A-B)}{(2-\psi_2)} c_1, \tag{26}$$

and

$$\begin{aligned} a_3 &= \frac{b\gamma\,(A-B)}{(3-\psi_3)} c_2 - \frac{b\gamma\,(A-B)}{(3-\psi_3)\,(2-\psi_2)} \\ &\quad \times \left((2-\psi_2)\,[B + \frac{1}{2}\,(1-\gamma)\,(A-B)] - b\psi_2\gamma\,(A-B)\right) c_1^2. \end{aligned} \tag{27}$$

For any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{b\gamma\,(A-B)}{(3-\psi_3)} c_2 - \frac{b\gamma\,(A-B)}{(2-\psi_2)^2\,(3-\psi_3)}$$
$$\times \left\{(2-\psi_2)^2\,[B + \frac{1}{2}\,(1-\gamma)\,(A-B)] - b\gamma\,(A-B)\,((2-\psi_2)\,\psi_2 - (3-\psi_3)\,\mu)\right\} c_1^2$$

or

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{b\gamma\,(A-B)}{3-\psi_3} \\ &\quad \times \left[c_2 - \frac{1}{(2-\psi_2)^2} \left\{ \begin{array}{c} (2-\psi_2)^2\,[B + \frac{1}{2}\,(1-\gamma)\,(A-B)] \\ -b\gamma\,(A-B)\,((2-\psi_2)\,\psi_2 - (3-\psi_3)\,\mu) \end{array} \right\} c_1^2 \right], \end{aligned}$$

we can write

$$a_3 - \mu a_2^2 = \frac{b\gamma\,(A-B)}{(3-\psi_3)} \left\{c_2 - \nu c_1^2\right\},$$

where

$$\nu = \frac{1}{(2-\psi_2)^2}$$
$$\times \left\{(2-\psi_2)^2 \times [B + \frac{1}{2}\,(1-\gamma)\,(A-B)] - b\gamma\,(A-B)\,((2-\psi_2)\,\psi_2 - (3-\psi_3)\,\mu)\right\},$$

our result now follows directly by an applications of Lemma 2.1. Equality can be attained by the function

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f_m(z)} - 1\right) = \left(\frac{1+Az}{1+Bz}\right)^{\gamma},$$

or

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f_m(z)}-1\right)=\left(\frac{1+Az^2}{1+Bz^2}\right)^{\gamma}.$$

Hence the proof of the Theorem 3.8 is complete. □

Putting $b=1,\gamma=1,A=1$ and $B=-1$ in Theorem 3.8, we can obtain the following Corollary.

Corollary 3.9. Let $f(z)\in S_s^m$. Then

$$\left|a_3-\mu a_2^2\right|\leq\frac{2}{(3-\psi_3)}$$

$$\max\left\{1,\left|\frac{2}{(2-\psi_2)^2}\left\{\left(2-\psi_2^2\right)+2\left((2-\psi_2)\psi_2-(3-\psi_3)\mu\right)\right\}+1\right|\right\},$$

where ψ_n, is given by (8). The result is sharp.

Setting $m=1$, $b=1$, $A=1$ and $B=-1$ in Theorem 3.8, we can obtain Corollary 2.2, below which is comparable to the result obtain by Cho and Owa [19, Th. 2.1,$\alpha=0$].

Corollary 3.10. Let $f(z)\in\widetilde{\mathcal{S}}_{\gamma}^{*}$. Then

$$\left|a_3-\mu a_2^2\right|\leq\gamma\max\left\{1\ ,\left|(3-4\mu)\gamma\right|\right\},$$

Putting $b=1,\gamma=1,\mu=1,A=1$ and $B=-1$ in Theorem 3.8, we can obtain the following Corollary.

Corollary 3.11. Let $f(z)\in S_s^m$. Then

$$\left|a_3-a_2^2\right|\leq\frac{2}{(3-\psi_3)}$$

$$\max\left\{1,\left|\frac{2}{(2-\psi_2)^2}\left\{\left(2-\psi_2^2\right)+2\left((2-\psi_2)\psi_2-(3-\psi_3)\right)\right\}+1\right|\right\},$$

where ψ_n, is given by (8). The result is sharp.

Theorem 3.12. Let $f(z)\in S_s^m\left(b,\gamma,A,B\right)$. If $b>0$, then

$$\left|a_3-\mu a_2^2\right|\leq$$
$$\begin{cases}\frac{2b\gamma(A-B)}{(3-\psi_3)(2-\psi_2)^2}[2b\gamma\left(A-B\right)\left((2-\psi_2)\psi_2-(3-\psi_3)\mu\right) \\ \qquad -2\left(2-\psi_2\right)^2\left(B+\frac{1}{2}\left(1-\gamma\right)\left(A-B\right)\right)+\left(2-\psi_2\right)^2], & (\mu\leq\sigma_1), \\ \qquad \frac{4b\gamma(A-B)}{(3-\psi_3)}, & (\sigma_1\leq\mu\leq\sigma_2), \\ \frac{2b\gamma(A-B)}{(3-\psi_3)(2-\psi_2)^2}[2\left(2-\psi_2\right)^2\left[\left(B+\frac{1}{2}\left(1-\gamma\right)\left(A-B\right)\right)\right. \\ \qquad -2b\gamma\left(A-B\right)\left((2-\psi_2)\psi_2-(3-\psi_3)\mu\right)-\left(2-\psi_2\right)^2], & (\mu\geq\sigma_2),\end{cases}$$

where

$$\sigma_1 = \frac{b\gamma\,(A-B)\,(2-\psi_2)\,\psi_2 - (2-\psi_2)^2\,[B+\frac{1}{2}\,(1-\gamma)\,(A-B)]}{b\gamma\,(A-B)\,(3-\psi_3)}, \tag{28}$$

and

$$\sigma_2 = \frac{(2-\psi_2)^2 + b\gamma\,(A-B)\,(2-\psi_2)\,\psi_2 - (2-\psi_2)^2\,[B+\frac{1}{2}\,(1-\gamma)\,(A-B)]}{b\gamma\,(A-B)\,(3-\psi_3)}, \tag{29}$$

where ψ_n, is given by (8). The result is sharp.

Proof. Since

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\gamma b\,(A-B)}{(3-\psi_3)}$$

$$\times \left[c_2 - \frac{1}{(2-\psi_2)^2}\left\{\begin{array}{c} (2-\psi_2)^2\left(B+\frac{1}{2}\,(1-\gamma)\,(A-B)\right) \\ -b\gamma\,(A-B)\,((2-\psi_2)\,\psi_2 - (3-\psi_3)\,\mu) \end{array}\right\} c_1^2\right],$$

therefore using Lemma (2.2), we can get the required result. To show that the bounds are sharp, we defined the functions $F(z)$, as follows:

$$1+\frac{1}{b}\left(\frac{zF'(z)}{F_m(z)}-1\right) = \left(\frac{1+Az}{1+Bz}\right)^{\gamma},\ \text{if } \mu<\sigma_1, \mu>\sigma_2,$$

$$1+\frac{1}{b}\left(\frac{zF'(z)}{F_m(z)}-1\right) = \left(\frac{1+Az^2}{1+Bz^2}\right)^{\gamma},\ \text{if } \sigma_1<\mu<\sigma_1,$$

$$1+\frac{1}{b}\left(\frac{zF'(z)}{F_m(z)}-1\right) = \left(\frac{1+A\phi(z)}{1+B\phi(z)}\right)^{\gamma},\ \text{if } \mu=\sigma_1,$$

$$1+\frac{1}{b}\left(\frac{zF'(z)}{F_m(z)}-1\right) = \left(\frac{1-A\phi(z)}{1-B\phi(z)}\right)^{\gamma},\ \text{if } \mu=\sigma_2,$$

where $\phi(z) = \frac{z(z+\eta)}{1+\eta z}$ with $0 \leq \eta \leq 1$. □

Setting $m=1$, $b=1$, $A=1$ and $B=-1$ in Theorem 3.12, we can obtain Corollary 3.13, below which is comparable to the result obtain by Cho and Owa [19, Th.2.1, $\alpha=0$].

Corollary 3.13. Let $f(z) \in \widetilde{\mathcal{S}}_{\gamma}^{*}$. Then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} (3-4\mu)\,\gamma^2 & \left(\mu \leq \frac{3\gamma-1}{4\gamma}\right), \\ \gamma, & \left(\frac{3\gamma-1}{4\gamma} \leq \mu \leq \frac{3\gamma+1}{4\gamma}\right), \\ (4\mu-3)\,\gamma^2 & \left(\mu \geq \frac{3\gamma+1}{4\gamma}\right). \end{cases}$$

The result is sharp.

Theorem 3.14. Let $f(z) \in S_s^m(b,\gamma,A,B)$. If $b>0$, then

$$\left|a_3-\mu a_2^2\right|+\left(\mu-\frac{b\gamma\left(A-B\right)\left(2-\psi_2\right)\psi_2-\left(2-\psi_2\right)^2\left[B+\frac{1}{2}\left(1-\gamma\right)\left(A-B\right)\right]}{b\gamma\left(A-B\right)\left(3-\psi_3\right)}\right)\left|a_2\right|^2$$

$$\leq\frac{4b\gamma\left(A-B\right)}{\left(3-\psi_3\right)},\qquad\left(\sigma_1\leq\mu\leq\sigma_3\right),$$

$$\left|a_3-\mu a_2^2\right|$$

$$+\left(\frac{\left(2-\psi_2\right)^2+b\gamma\left(A-B\right)\left(2-\psi_2\right)\psi_2-\left(2-\psi_2\right)^2\left[B+\frac{1}{2}\left(1-\gamma\right)\left(A-B\right)\right]}{b\gamma\left(A-B\right)\left(3-\psi_3\right)}-\mu\right)\left|a_2\right|^2$$

$$\leq\frac{4b\gamma\left(A-B\right)}{\left(3-\psi_3\right)},\qquad\left(\sigma_3\leq\mu\leq\sigma_2\right),$$

where σ_1 and σ_2 are given by (28) and (29) and

$$\begin{aligned}\sigma_3 &= \frac{\left(2-\psi_2\right)^2}{2b\gamma\left(A-B\right)\left(3-\psi_3\right)}\\ &\quad\times\left(b\gamma\left(A-B\right)\left(2-\psi_2\right)\psi_2-\left(2-\psi_2\right)^2\left[B+\frac{1}{2}\left(1-\gamma\right)\left(A-B\right)\right]\right).\end{aligned}$$

The result is sharp.

Proof. The proof of Theorem 3.14 is similar to the proof of Theorem 3.12 so the details are omitted. □

Putting $b=1$, $\gamma=1$, $A=1$ and $B=-1$ in Theorem (3.14), we can obtain the following Corollary.

Corollary 3.15. Let $f(z) \in S_s^m(b,\gamma,A,B)$. If $b>0$, then

$$\left|a_3-\mu a_2^2\right|+\left(\mu-\frac{2\left(2-\psi_2\right)\psi_2+\left(2-\psi_2\right)^2}{2\left(3-\psi_3\right)}\right)\left|a_2\right|^2$$

$$\leq\frac{8}{\left(3-\psi_3\right)},\qquad\left(\sigma_1\leq\mu\leq\sigma_3\right),$$

$$\left|a_3-\mu a_2^2\right|+\left(\frac{\left(2-\psi_2\right)^2+2\left(2-\psi_2\right)\psi_2+\left(2-\psi_2\right)^2}{2\left(3-\psi_3\right)}-\mu\right)\left|a_2\right|^2$$

$$\leq\frac{8}{\left(3-\psi_3\right)},\qquad\left(\sigma_3\leq\mu\leq\sigma_2\right),$$

where σ_1 and σ_2 are given by (28) and (29) and

$$\sigma_3=\frac{\left(2+\psi_2\right)\left(2-\psi_2\right)^3}{4\left(3-\psi_3\right)}.$$

The result is sharp.

3.3 Convolution Properties

In this sections the necessary and sufficient conditions are given in terms of convolution operators for a function to be in the class $S_s^m(b,\gamma,A,B)$.

Theorem 3.16. A function $f(z)\in S_s^m(b,\gamma,A,B)$, if and only if

$$\frac{1}{z}\left\{f(z)*\left(\frac{z}{(1-z)^2}\left(\left(1+B(e^{i\theta})\right)^{\gamma}\right)-\left(\left(1+A(e^{i\theta})\right)^{\gamma}\right)h(z)\right)\right\}\neq 0, \tag{30}$$

for all $z\in E$ and $0\leq\theta<2\pi$, where $h(z)$ is given by (36).

Proof. Assume that $f(z)\in S_s^m(b,\gamma,A,B)$, then we have

$$\frac{zf'(z)}{f_m(z)}\prec\left(\frac{1+Az}{1+Bz}\right)^{\gamma},\qquad (z\in E). \tag{31}$$

If and only if

$$\frac{zf'(z)}{f_m(z)}\neq\left(\frac{1+A(e^{i\theta})}{1+B(e^{i\theta})}\right)^{\gamma}, \tag{32}$$

for all $z\in E$, and $0\leq\theta<2\pi$. The condition (32), can be written as

$$\frac{1}{z}\left\{zf'(z)\left[\left(1+B(e^{i\theta})\right)^{\gamma}\right]-f_m(z)\left[\left(1+A(e^{i\theta})\right)^{\gamma}\right]\right\}\neq 0. \tag{33}$$

On the other hand it is well known that

$$zf'(z)=f(z)*\frac{z}{(1-z)^2}. \tag{34}$$

And from the definition of $f_m(z)$, we have

$$f_m(z)=z+\sum_{n=2}^{\infty}a_n\psi_nz^n=(f*h)(z), \tag{35}$$

where

$$h(z)=z+\sum_{n=2}^{\infty}\psi_nz^n, \tag{36}$$

for

$$\psi_n=\begin{cases}1, & n=lm+1\\ 0, & n\neq lm+1\end{cases}\qquad (l\in\mathbb{N}_0),$$

substituting (34) and (35) in (33) we can get (30). This completes the proof of the Theorem 3.16. □

4 Conclusion

In this paper, we have used the techniques of differential subordination and convolution to obtain inclusion theorems and subordination theorems. Many interesting particular cases of the main theorems are emphasized in the form of corollaries. The ideas and techniques of this work may motivate and inspire the others to explore this interesting field further.

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References

[1] K. Al-Shaqsi, M. Darus, Fekete-Szegö problem for univalent functions with respect to k-symmetric points, Aust. J. Math. Anal. Appl. 5(2008) 1–12.

[2] R. Chand, P. Singh, On certain schlicht mapping, Ind. J. Pure App. Math. 10(1979) 1167–1174.

[3] C. Gao, S. Zhou, A new subclass of close to convex functions, Soochow. J. Math. 31(2005) 41–49.

[4] W, Janowski, Some extremal problems for certain families of analytic functions, Bull. Acad. Polon. Soc. 21(1973), 17–25.

[5] F. R. Keogh, E. P. Markes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20(1969) 8–12.

[6] O. Kwon, Y. Sim, A certain subclass of Janowski type functions associated with k-symmetric points, Comm. Kor. Math. Soc. 28(2013) 143–154.

[7] R. J. Libera, Univalent α -spiral functions, Cand. J. Math. 19(1967) 449–456.

[8] E. Lindelöf, Mémorie sur certaines inequalitiés dans la théorie des fonctions monogènes et sur quelques propiétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel, Acta Soc. Sci. Fenn. 35(1908) 1-35.

[9] J. E. Littelwood, On inequalities in the theory of functions, Proc. London Math. Soc. 23(1925) 481—519.

[10] M. S. Liu, Y. C. Zhu, Criteria for strongly starlike and Φ-like functions, Complex Var. Ellip. Eq. 53(2008) 485–500.

[11] W. Ma, D. Minda, A unified treatment of some special classes of univalent functions, in proceedings of the conference on Complex Analysis, Z. Li, F. Ren, L. Yang and S. Zhang, Eds, Int. Press (1994) 157–169.

[12] S. S. Miller, P. T. Mocanu, Differential subordination and univalent functions, Michigan. Math. J. 28(2)(1981) 157–172.

[13] K. I. Noor, S. N. Malik, M. Arif, M. Raza, On bounded boundary and bounded radius rotation related with Janowski function, World. Appl. Sci. J. 12(6) (2011), 895–902.

[14] V. Ravichandran, Starlike and convex functions with respect to conjugate points, Acta. Math. Acad. Peda. Ny. 20(2004) 31–37.

[15] W. Rogosinski, On subordinate functions, Proc. Camb. Phil. Soc. 35(1939) 1—26.

[16] W. Rogosinski, On the coefficients of subordinate functions, Proc. London Math. Soc. 48(1943) 48—82.

[17] M. A. Rosihan, V. Ravichandran, Convolution and Differential subordination for mulitivalent functions, Bull. Malays. Math. Sci. 32(3) (2009) 351–360.

[18] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan 11(1959) 72–75.

[19] N. E Cho, S. Owa. On the feketo-Szegö problem for strongly α -logarithmic quasi convex functions, Ressearch Institute for Math. Sci. 1341(2003) 1–11.