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# MULTIPLE SETS: A UNIFIED APPROACH TOWARDS MODELLING VAGUENESS AND MULTIPLICITY

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**Abstaract** − Multiple set is a new mathematical model to represent vagueness together with multiplicity. Multiple sets are generalization of fuzzy sets, multisets, fuzzy multisets and multi fuzzy sets. In this paper, a modified version for the definition of multiple sets is given and it is shown that the revised definition also satisfies all fundamental properties satisfied by the earlier definition. The notion of $\alpha_i - cut$ and strong $\alpha_i - cut$ are defined and their properties are studied. Finally, multiple complement function is defined and some results related to it are obtained.

***Keywords*** − *Multiple sets, Fuzzy sets, Multi sets, Multiple complement function.*

# 1 Introduction

Sets are the fundamental ideas of Mathematics. The development of set theory was mainly due to a German Mathematician Cantor (1845-1918) and it has become the language of science. A set is a well defined collection of distinct objects. The objects that make up a set can be anything: numbers, people, letters of the alphabet, other sets and so on. In this theory, a sharp, crisp and unambiguous distinction exists between a member and a nonmember for any well defined set of entities and there is a very precise and clear boundary to indicate whether an entity belongs to the set or not.

Fuzzy sets have been introduced by Zadeh [21] in 1965 as an extension of the classical notion of a set. It was specifically designed to represent uncertainty and vagueness, mathematically and to provide formalized tools for dealing with the imprecision intrinsic to many problems. In real world, there exists much fuzzy knowledge like vague, imprecise, uncertain, ambiguous, inexact etc. Since its inception, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines.

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Applications of this theory can be found in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operation research, pattern recognition and robotics.

After this, a lot of new mathematical constructions and theories treating imprecision, inexactness, ambiguity and uncertainty have been developed. Some of these constructions and theories are extensions of fuzzy set theory, while others try to mathematically model imprecision and uncertainty in a different way. The diversity of such constructions and corresponding theories includes: L-fuzzy sets by Goguen [8] in 1967, Multisets by Cerf et al. [4] in 1971, Rough sets by Pawlak [15] in 1982, Intuitionistic fuzzy sets by Atanassov [1] in 1983, Fuzzy multisets by Yager [20] in 1986, Genuine sets by Demirci [6] in 1999, Soft sets by Molodtsov [14] in 1999, Multi fuzzy sets by Sebastian and Ramakrisnan [18] in 2011 etc.

The notion of multiset (or bag) is a generalization of the notion of set in which members are allowed to appear more than once. A set takes no account of multiple occurrence of any one of its members, so when one think of the set of roots of a polynomial f(x) or the spectrum of a linear operator, we need multisets. Multiset theory was introduced by Cerf et al.[4] in 1971. Peterson [16] and Yager [20] made further contributions to it. The naive concept of multiset was formalised by Blizard [2] in 1989. Multisets have become an important tool in databases, for instance, multisets are often used to implement relations in database systems. Multisets also play an important role in computer science. A complete account of the development of multiset theory can be seen in [9, 5, 3, 7].

Fuzzy multisets were first discussed by Yager [20] as a generalization of multisets. In fuzzy multisets an element of $X$ may occur more than once with possibly same or different membership values. Later, Miyamoto established more results on fuzzy multiset theory and discussed applications of fuzzy multisets in his papers [10, 12].

The concept of multi fuzzy set was introduced by S. Sebastian and Ramakrishnan [18] in 2011. Theory of multi fuzzy sets is a generalization of theories of fuzzy sets, L-fuzzy sets and intuitionistic fuzzy sets. Theory of multi fuzzy sets deals with the multi level fuzziness and multi dimensional fuzziness. Multi fuzzy set theory is useful to characterize the problems in the fields of image processing, taste recognition, pattern recognition, decision making and approximation of vague data. Further study was carried on by the same author in his paper [17].

Fuzzy sets are useful in dealing with uncertainty of only one kind, that is only one membership function is possible. Multi fuzzy sets were introduced to handle more membership functions representing various types of uncertainties. Multisets handles repetition of elements or quantitative nature of objects. Fuzzy multisets handles quantitative and qualitative aspects together. All these ideas were developed independently and proved to be quite useful in their respective contexts. Motivated by all these concepts, one may think about a unified structure which represents all these aspects simultanously. As an attempt towards this, multiple sets [19] are introduced to model imperfect knowledge from which all the above discussed cases can be derived as particular cases. In multiple sets, multiple occurrences of elements are permitted in which each occurrence has a finite number of same or different membership values. That is, in multiple set theory, a multiple set of order $(n, k)$ gives $nk$ membership grades to each element $x$ in the universal set $X$.

In this paper, a revised definition of multiple sets is given and it is examined that new definition satisfies all the properties satisfied by the old definition. $\alpha_i - cut$ and strong $\alpha_i - cut$ are defined and their properties are studied. Then special multiple sets and strong special multiple sets are defined and representations of multiple sets in terms of special multiple sets and strong special multiple sets are mentioned as three Decomposition Theorems. Finally, multiple complement function is defined and Characterization Theorems of multiple complements are discussed.

# 2 Preliminary

## 2.1 Fuzzy Sets

The word "fuzzy" means "vagueness". Fuzzy set is very convenient method for representing some form of uncertainty or vagueness. Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval [0, 1].

**Definition 2.1.** [21] Let $X$ be a given universal set, which is always a crisp set. A fuzzy set $A$ on $X$ is characterized by a function $A : X \to [0, 1]$ called fuzzy membership function, which assigns to each object a grade of membership ranging between zero and one. A *fuzzy set* $A$ is defined as

$$A = \{(x, A(x)); x \in X\}$$

where $A(x)$ is the fuzzy membership value of $x$ in $X$.

Each fuzzy set is completely and uniquely defined by one particular membership function. Words like young, tall, good or high are fuzzy.

## 2.2 Multisets

A multiset is an unordered collection of objects in which elements may occur more than once. In other words, a multiset is a collection in which objects may appear more than once and each individual occurrence of an object is called its element. All duplicates of an object in a multiset are indistinguishable. The objects of a multiset are the distinguishable or distinct elements of the multiset.

**Definition 2.2.** [9] Let $X$ be a non empty set, called universe. A *multiset* $M$ drawn from $X$ is represented by a count function $C_M : X \to \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers. For each $x \in X$, $C_M(x)$ indicates the number of occurrences of the element $x$ in $M$. Then a multiset $M$ can be expressed as $\{C_M(x)/x; x \in X\}$.

The number of distinct elements in a multiset $M$ (which need not be finite) and their multiplicities jointly determine its cardinality, denoted by $C(M)$. In other words, the cardinality of a multiset is the sum of multiplicities of all its elements. A multiset $M$ is called finite if the number of distinct elements in $M$ and their multiplicities are both finite, it is infinite otherwise. Thus, a multiset $M$ is infinite if either the number of elements in $M$ is infinite or the multiplicity of one or more of its elements is infinite. A multiset corresponds to an ordinary set if the multiplicity of every element is one.

**Operations on Multisets [9]**

Let $M_1$ and $M_2$ be two multisets drawn from $X$.

1. **Submultiset**: $M_1$ is a sub multiset of $M_2$, denoted by $M_1 \subseteq M_2$, if $C_{M_1}(x) \leq C_{M_2}(x)$ for every $x \in X$.

2. **Equal**: $M_1$ and $M_2$ are equal, denoted by $M_1 = M_2$, if $M_1 \subseteq M_2$ and $M_2 \subseteq M_1$.

3. **Union**: The union of $M_1$ and $M_2$ is a multiset, denoted by $M = M_1 \cup M_2$, with the count function $C_M(x) = \max\{C_{M_1}(x), C_{M_2}(x)\}$, for every $x \in X$.

4. **Intersection**: The intersection of $M_1$ and $M_2$ is a multiset, denoted by $M = M_1 \cap M_2$, with the count function $C_M(x) = \min\{C_{M_1}(x), C_{M_2}(x)\}$, for every $x \in X$.

## 2.3 Fuzzy Multiset

In fuzzy multisets an element of $X$ may occur more than once with possibly the same or different membership values.

**Definition 2.3.** [13] For $x \in X$, the membership sequence of $x$ is defined as a non increasing sequence of membership values of $x$ and it is denoted by $(\mu_A^1(x), \mu_A^2(x), ..., \mu_A^k(x))$, such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^k(x)$, where $\mu_A$ is a membership function and $\mu_A^j$, $j = 1, 2, ..., k$ are values(same or different) of membership function $\mu_A$ . A *fuzzy multiset* is a collection of all $x$ together with its membership sequence.

## 2.4 Multi Fuzzy Sets

Multi fuzzy sets are defined in terms of ordered sequences of membership functions.

**Definition 2.4.** [18] Let $X$ be a non empty set and let $\{L_i; i \in \mathbb{N}\}$ be a family of complete lattices where $\mathbb{N}$ is the set of positive integers. A *multi fuzzy set* $A$ in $X$ is a set of ordered sequences

$$A = \{(x, \mu_1(x), \mu_2(x), ...); x \in X\}$$

where $\mu_i \in L_i^X$ for $i \in \mathbb{N}$. The function $\mu_A = (\mu_1, \mu_2, ...)$ is called a multi membership function of multi fuzzy set $A$.

If the sequences of the membership function have only $k$ terms, $k$ is called dimension of $A$. Let $L_i = [0, 1]$ for $i = 1, 2, ..., k$, then the set of all multi fuzzy sets in $X$ of dimension $k$ is denoted by $M^kFS(X)$

# 3 Multiple Sets

Multiple sets are defined in [19]. Definition of multiple sets is modified and it is investigated that new definition satisfies all the properties satisfied by the old definition.

**Definition 3.1.** Let $X$ be a non-empty crisp set called the universal set. A *multiple set* $A$ of order $(n,k)$ over $X$ is an object of the form $\{(x, A(x)); x \in X\}$, where for each $x \in X$ its membership value is an $n \times k$ matrix

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^k(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^k(x) \\ & \cdots & & \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^k(x) \end{bmatrix}$$

where $A_1, A_2, ..., A_n$ are fuzzy membership functions and for each $i = 1, 2, ..., n$, $A_i^1(x)$, $A_i^2(x), ..., A_i^k(x)$ are membership values of the membership function $A_i$ for the element $x \in X$, written in decreasing order.

The universal multiple set $X$ of order $(n,k)$ is a multiple set of order $(n,k)$ over $X$ forwhich the membership matrix for each $x \in X$ is an $n \times k$ matrix with all entries one. The empty multiple set $\Phi$ of order $(n,k)$ is a multiple set of order $(n,k)$ over $X$ forwhich the membership matrix for each $x \in X$ is an $n \times k$ matrix with all entries zero.

The set of all multiple sets of order $(n,k)$ over $X$ is denoted by $MS_{(n,k)}(X)$. It is noticed that a multiple set $A$ of order $(n,k)$ over $X$ can be viewed as a function $A : X \to \mathbb{M}$, where $\mathbb{M} = \mathbb{M}_{n\times k}([0,1])$ is the set of all matrices of order $n \times k$ with entries from $[0,1]$, which maps each $x \in X$ to its $n \times k$ membership matrix $A(x)$. It is proved that a multiple set can be viewed as a generalization of fuzzy sets, multi fuzzy sets, fuzzy multisets and multisets. The standard set operations namely, subset, intersection, union and complement are defined on multiple sets. It is proved that multiple sets satisfies the following fundemental properties of the set operations.

1. **Involution**: $\overline{\overline{A}} = A$
2. **Commutativity**: $A \cup B = B \cup A$ and $A \cap B = B \cap A$
3. **Associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
4. **Distributivity**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. **Idempotence**: $A \cup A = A$ and $A \cap A = A$
6. **Absorption**: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
7. **Absorption by $X$ and $\Phi$**: $A \cup X = X$ and $A \cap \Phi = \Phi$
8. **Identity:** $A \cup \Phi = A$ and $A \cap X = A$
9. **De Morgan's laws**: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$
10. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
11. $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Finally, it is noticed that law of contradiction and law of excluded middle are violated for multiple sets.

**Example 3.2.** Suppose $X = \{x_1, x_2, x_3\}$ is the universal set of students under consideration and there is a panel consisting of three experts evaluating the students under the criteria of intelligence, extra curricular activities, communication skill and personality. The membership functions $A_1$, $A_2$, $A_3$ and $A_4$ represents criteria intelligence, extra curricular activities, communication skill and personality respectively. For each $i = 1, 2, 3, 4$, membership values $A_i^1(x), A_i^2(x), A_i^3(x)$ of the membership function $A_i$ for the element $x \in X$ are the values given by the three experts, written in decreasing order. Then the performance of the students can be represented by a multiple set of order (4,3) as follows:

$$A = \{(x_1, A(x_1)), (x_2, A(x_2)), (x_3, A(x_3))\}$$

where $A(x_i)$ for $i = 1, 2, 3$ are $4 \times 3$ matrices given as follows;

$$A(x_1) = \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.3 \\ 0.9 & 0.9 & 0.8 \end{bmatrix}$$

$$A(x_2) = \begin{bmatrix} 0.8 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.4 \\ 0.9 & 0.8 & 0.7 \end{bmatrix}$$

$$A(x_3) = \begin{bmatrix} 0.8 & 0.7 & 0.5 \\ 0.7 & 0.6 & 0.4 \\ 0.7 & 0.4 & 0.4 \\ 0.8 & 0.8 & 0.7 \end{bmatrix}$$

Here, for the student $x_1$ the membership values corresponding to intelligence are 0.7, 0.6 and 0.5, corresponding to extra curricular activities are 0.6, 0.5 and 0.4 and so on.

**Notation**: Suppose $I$ denotes the closed interval $[0, 1]$ and $I^n$ denotes the cartesian product $[0, 1] \times [0, 1] \times ... \times [0, 1]$ $(n - times)$. A new notation can be introduced for the purpose of notational simplicity: $(\alpha_i)_1^n$ denotes the n-tuple $(\alpha_1, \alpha_2, ..., \alpha_n)$.

## 3.1 $\alpha_i - cut$ and strong $\alpha_i - cut$

In this section, the concept of $\alpha_i - cut$ and strong $\alpha_i - cut$ of multiple sets are introduced. They play a principle role in the relationship between multiple sets and crisp multisets.

**Definition 3.3.** Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$. An $\alpha_i - cut$ of $A$ is a crisp multiset $A_{[\alpha_i]} = \{C_{A_{[\alpha_i]}}(x)/x; x \in X\}$, where $C_{A_{[\alpha_i]}}(x)$ is the count of $x$ in $A_{[\alpha_i]}$, given by

$$C_{A_{[\alpha_i]}}(x) = \begin{cases} 0 & \text{if } A_i^1(x) < \alpha_i \text{ for some } i = 1, 2, ..., n. \\ j & \text{if } A_i^j(x) \geq \alpha_i \text{ for every } i = 1, 2, ..., n \text{ and} \\ & A_i^{j+1}(x) < \alpha_i \text{ for some } i = 1, 2, ..., n. \\ k & \text{if } A_i^k(x) \geq \alpha_i \text{ for every } i = 1, 2, ..., n. \end{cases}$$

A strong $\alpha_i - cut$ of $A$ is a crisp multiset $A_{[\alpha_i]+} = \{C_{A_{[\alpha_i]+}}(x)/x; x \in X\}$, where $C_{A_{[\alpha_i]+}}(x)$ is the count of $x$ in $A_{[\alpha_i]+}$, given by

$$C_{A_{[\alpha_i]+}}(x) = \begin{cases} 0 & \text{if } A_i^1(x) \leq \alpha_i \text{ for some } i = 1, 2, ..., n. \\ j & \text{if } A_i^j(x) > \alpha_i \text{ for every } i = 1, 2, ..., n \text{ and} \\ & A_i^{j+1}(x) \leq \alpha_i \text{ for some } i = 1, 2, ..., n. \\ k & \text{if } A_i^k(x) > \alpha_i \text{ for every } i = 1, 2, ..., n. \end{cases}$$

**Example 3.4.** Let $A$ be a multiple set of order $(4, 3)$ over the universal set $X = \{x, y, z\}$, given by the membership matrices

$$A(x) = \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.6 & 0.5 & 0.4 \\ 0.7 & 0.5 & 0.4 \\ 0.9 & 0.8 & 0.7 \end{bmatrix}$$

$$A(y) = \begin{bmatrix} 0.8 & 0.7 & 0.5 \\ 0.7 & 0.6 & 0.4 \\ 0.7 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.7 \end{bmatrix}$$

$$A(z) = \begin{bmatrix} 0.7 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.8 & 0.7 & 0.1 \end{bmatrix}$$

For $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.5, 0.5, 0.3, 0.4) \in I^4$, the $\alpha_i - cut$ and the strong $\alpha_i - cut$ are

$$A_{[\alpha_i]} = \{2/x, 2/y, 2/z\}$$
$$A_{[\alpha_i]+} = \{1/x, 2/y, 1/z\}$$

## 3.2 Properties of $\alpha_i - cut$ and strong $\alpha_i - cut$

The various properties of $\alpha_i - cut$ and strong $\alpha_i - cut$ of multiple set are expressed in terms of theorems.

**Theorem 3.5.** Let $A, B \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n, (\beta_i)_1^n \in I^n$. Then

1. $A_{[\alpha_i]+} \subseteq A_{[\alpha_i]}$

2. If $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$, then $A_{[\beta_i]} \subseteq A_{[\alpha_i]}$ and $A_{[\beta_i]+} \subseteq A_{[\alpha_i]+}$

3. $(A \cap B)_{[\alpha_i]} = A_{[\alpha_i]} \cap B_{[\alpha_i]}$
4. $(A \cup B)_{[\alpha_i]} = A_{[\alpha_i]} \cup B_{[\alpha_i]}$
5. $(A \cap B)_{[\alpha_i]+} = A_{[\alpha_i]+} \cap B_{[\alpha_i]+}$
6. $(A \cup B)_{[\alpha_i]+} = A_{[\alpha_i]+} \cup B_{[\alpha_i]+}$

*Proof*: Let $A, B \in MS_{(n,k)}(X)$.

1. Let $(\alpha_i)_1^n \in I^n$. To prove $A_{[\alpha_i]+} \subseteq A_{[\alpha_i]}$, it is enough to prove $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   **Case 1:** $C_{A_{[\alpha_i]+}}(x) = 0$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ trivially.

   **Case 2:** $C_{A_{[\alpha_i]+}}(x) = j$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]}}(x) \geq j$ and hence $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

   **Case 3:** $C_{A_{[\alpha_i]+}}(x) = k$, then $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]}}(x) = k$ and hence $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\alpha_i]}}(x)$

   These three cases prove that $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $A_{[\alpha_i]+} \subseteq A_{[\alpha_i]}$.

2. Let $(\alpha_i)_1^n, (\beta_i)_1^n \in I^n$ such that $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$. To prove $A_{[\beta_i]} \subseteq A_{[\alpha_i]}$, it is enough to prove $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   **Case 1:** $C_{A_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) < \alpha_i$. Since $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$, we get $A_i^1(x) < \beta_i$ for some $i$. Thus $C_{A_{[\beta_i]}}(x) = 0$ and hence $C_{A_{[\beta_i]}}(x) = C_{A_{[\alpha_i]}}(x)$.

   **Case 2:** $C_{A_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^{j+1}(x) < \alpha_i$. Since $\alpha_i \leq \beta_i$ for every $i = 1, 2, ..., n$, we get $A_i^{j+1} < \beta_i$ for some $i$. Thus $C_{A_{[\beta_i]}}(x) \leq j$ and hence $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

   **Case 3:** $C_{A_{[\alpha_i]}}(x) = k$. Then $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ trivially.

   These three cases prove that $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $A_{[\beta_i]} \subseteq A_{[\alpha_i]}$. The proof of $A_{[\beta_i]+} \subseteq A_{[\alpha_i]+}$ is analogous.

3. Let $(\alpha_i)_1^n \in I^n$. To prove $(A \cap B)_{[\alpha_i]} = A_{[\alpha_i]} \cap B_{[\alpha_i]}$, it is enough to prove $C_{(A \cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]} \cap B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   **Case 1:** $C_{(A \cap B)_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $(A \cap B)_i^1 < \alpha_i$. This means that $\min\{A_i^1(x), B_i^1(x)\} < \alpha_i$ for some $i$. Then either $A_i^1(x) < \alpha_i$ or $B_i^1(x) < \alpha_i$ for some $i$. This implies that either

$C_{A_{[\alpha_i]}}(x) = 0$ or $C_{B_{[\alpha_i]}}(x) = 0$. Thus $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} = 0$ and hence $C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x) = 0$. Therefore $C_{(A\cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x)$.

**Case 2:** $C_{(A\cap B)_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $(A \cap B)_i^{j+1}(x) < \alpha_i$. This means that $\min\{A_i^{j+1}(x), B_i^{j+1}(x)\} < \alpha_i$ for some $i$. Then either $A_i^{j+1}(x) < \alpha_i$ or $B_i^{j+1}(x) < \alpha_i$ for some $i$. This implies that $C_{A_{[\alpha_i]}}(x) \le j$ or $C_{B_{[\alpha_i]}}(x) \le j$. Then $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} \le j$ and hence $C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x) \le j$. Therefore $C_{(A\cap B)_{[\alpha_i]}}(x) \ge C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x)$.

Also, $(A \cap B)_i^j(x) \ge \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\min\{A_i^j(x), B_i^j(x)\} \ge \alpha_i$ for every $i = 1, 2, ..., n$. Then $A_i^j(x) \ge \alpha_i$ and $B_i^j(x) \ge \alpha_i$ for every $i = 1, 2, ..., n$. This implies that both $C_{A_{[\alpha_i]}}(x) \ge j$ and $C_{B_{[\alpha_i]}}(x) \ge j$. Then $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} \ge j$ and hence $C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x) \ge j$. Therefore $C_{(A\cap B)_{[\alpha_i]}}(x) \le C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x)$. From two inqualities, we have $C_{(A\cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x)$.

**Case 3:** $C_{(A\cap B)_{[\alpha_i]}}(x) = k$. Then $(A \cap B)_i^k(x) \ge \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\min\{A_i^k(x), B_i^k(x)\} \ge \alpha_i$ for every $i = 1, 2, ..., n$. This implies that both $A_i^k(x) \ge \alpha_i$ and $B_i^k(x) \ge \alpha_i$ for every $i = 1, 2, ..., n$. Thus both $C_{A_{[\alpha_i]}}(x) = k$ and $C_{B_{[\alpha_i]}}(x) = k$. Then $\min\{C_{A_{[\alpha_i]}}(x), C_{B_{[\alpha_i]}}(x)\} = k$ and hence $C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x) = k$. Therefore $C_{(A\cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x)$.

These three cases prove that $C_{(A\cap B)_{[\alpha_i]}}(x) = C_{A_{[\alpha_i]}\cap B_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $(A \cap B)_{[\alpha_i]} = A_{[\alpha_i]}\cap B_{[\alpha_i]}$. The proof of (4), (5) and (6) are analogous.

**Theorem 3.6.** Let $A_t \in MS_{(n,k)}(X)$ for all $t \in T$, where $T$ is an index set and let $(\alpha_i)_1^n \in I^n$. Then

1. $\bigcup_{t\in T} (A_t)_{[\alpha_i]} \subseteq \left(\bigcup_{t\in T} A_t\right)_{[\alpha_i]}$
2. $\bigcap_{t\in T} (A_t)_{[\alpha_i]} \supseteq \left(\bigcap_{t\in T} A_t\right)_{[\alpha_i]}$
3. $\bigcup_{t\in T} (A_t)_{[\alpha_i]+} \supseteq \left(\bigcup_{t\in T} A_t\right)_{[\alpha_i]+}$
4. $\bigcap_{t\in T} (A_t)_{[\alpha_i]+} \subseteq \left(\bigcap_{t\in T} A_t\right)_{[\alpha_i]+}$

*Proof*: Let $A_t \in MS_{(n,k)}(X)$ for all $t \in T$ and let $(\alpha_i)_1^n \in I^n$.

1. To prove $\bigcup_{t\in T} (A_t)_{[\alpha_i]} \subseteq \left(\bigcup_{t\in T} A_t\right)_{[\alpha_i]}$, it is enough to prove $C_{\bigcup_{t\in T}(A_t)_{[\alpha_i]}}(x) \le C_{(\bigcup_{t\in T} A_t)_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

**Case 1:** $C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $\left(\bigcup\limits_{t\in T} A_t\right)_i^1 (x)$
$< \alpha_i$. This means that $\sup\limits_{t\in T}\{(A_t)_i^1(x)\} < \alpha_i$ for some $i$. This implies that $(A_t)_i^1(x) < \alpha_i$ for some $i$ and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) = 0$ for every $t \in T$. This implies that $\sup\limits_{t\in T}\left\{C_{(A_t)_{[\alpha_i]}}(x)\right\} = 0$ and thus $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) = 0$. Therefore $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) = C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

**Case 2:** $C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $\left(\bigcup\limits_{t\in T} A_t\right)_i^{j+1} (x) < \alpha_i$. This means that $\sup\limits_{t\in T}\{(A_t)_i^{j+1}(x)\} < \alpha_i$ for some $i$. This implies that $(A_t)_i^{j+1}(x) < \alpha_i$ for some $i$ and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) \leq j$ for every $t \in T$. This implies that $\sup\limits_{t\in T}\left\{C_{(A_t)_{[\alpha_i]}}(x)\right\} \leq j$ and thus $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq j$. Therefore $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

**Case 3:** $C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = k$. Then $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ trivially.

These three cases prove that $C_{\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \leq C_{(\bigcup\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore, $\bigcup\limits_{t\in T}(A_t)_{[\alpha_i]} \subseteq \left(\bigcup\limits_{t\in T} A_t\right)_{[\alpha_i]}$.

2. To prove $\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]} \supseteq \left(\bigcap\limits_{t\in T} A_t\right)_{[\alpha_i]}$, it is enough to prove $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$, for every $x \in X$. Let $x \in X$. There are three cases;

   **Case 1:** $C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = 0$. Then $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ trivially.

   **Case 2:** $C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = j$. Then $\left(\bigcap\limits_{t\in T} A_t\right)_i^j (x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\inf\limits_{t\in T}\{(A_t)_i^j(x)\} \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $(A_t)_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$ and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) \geq j$ for every $t \in T$. This implies that $\inf\limits_{t\in T}\left\{C_{(A_t)_{[\alpha_i]}}\right\} \geq j$ and thus $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq j$. Therefore $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

   **Case 3:** $C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x) = k$. Then $\left(\bigcap\limits_{t\in T} A_t\right)_i^k (x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This means that $\inf\limits_{t\in T}\{(A_t)_i^k(x)\} \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $(A_t)_i^k(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$ and for every $t \in T$. Thus $C_{(A_t)_{[\alpha_i]}}(x) = k$ for every $t \in T$. This implies that $\inf\limits_{t\in T}\left\{C_{(A_t)_{[\alpha_i]}}\right\} = k$ and thus $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) =$

$k$. Therefore $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) = C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$.

These three cases prove that $C_{\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]}}(x) \geq C_{(\bigcap\limits_{t\in T} A_t)_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore, $\bigcap\limits_{t\in T}(A_t)_{[\alpha_i]} \supseteq \left(\bigcap\limits_{t\in T} A_t\right)_{[\alpha_i]}$.

The proof of (3) and (4) are analogous.

**Theorem 3.7.** Let $A, B \in MS_{(n,k)}(X)$ and let $(\alpha_i)_1^n \in I^n$. Then

1. $A \subseteq B$ iff $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$
2. $A \subseteq B$ iff $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$
3. $A = B$ iff $A_{[\alpha_i]} = B_{[\alpha_i]}$
4. $A = B$ iff $A_{[\alpha_i]+} = B_{[\alpha_i]+}$

*Proof*: Let $A, B \in MS_{(n,k)}(X)$ and let $(\alpha)_n \in I^n$.

1. Suppose $A \subseteq B$. Then $A_i^j(x) \leq B_i^j(x)$ for every $i = 1, 2, ..., n,\ j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$, it is enough to prove $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   **Case 1:** $C_{B_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^1(x) < \alpha_i$. This implies that $A_i^1(x) \leq B_i^1(x) < \alpha_i$ for some $i$. Therefore $C_{A_{[\alpha_i]}}(x) = 0$. Thus $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   **Case 2:** $C_{B_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^{j+1}(x) < \alpha_i$. This implies that $A_i^{j+1}(x) \leq B_i^{j+1}(x) < \alpha_i$ for some $i$. Therefore $C_{A_{[\alpha_i]}}(x) \leq j$. Thus $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$.

   **Case 3:** Suppose $C_{B_{[\alpha_i]}}(x) = k$. Then $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ trivially.

   These three cases prove that $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Therefore $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$.

   Conversely, suppose that $A_{[\alpha_i]} \subseteq B_{[\alpha_i]}$. This means that $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x)$ for $i = 1, 2, ..., n$. Then $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]}}(x) \leq C_{B_{[\alpha_i]}}(x)$. That is $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A \subseteq B$.

2. Suppose $A \subseteq B$. Then $A_i^j(x) \leq B_i^j(x)$ for every $i = 1, 2, ..., n,\ j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$ , it is enough to prove $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   **Case 1:** $C_{B_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ...n\}$ such that $B_i^1(x) \leq \alpha_i$. This implies that $A_i^1(x) \leq B_i^1(x) \leq \alpha_i$ for some $i$. Therefore

$C_{A_{[\alpha_i]+}}(x) = 0$. Thus $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

**Case 2:** $C_{B_{[\alpha_i]+}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^{j+1}(x) \leq \alpha_i$. This implies that $A_i^{j+1}(x) \leq B_i^{j+1}(x) \leq \alpha_i$ for some $i$. Therefore $C_{A_{[\alpha_i]+}}(x) \leq j$. Thus $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$.

**Case 3:** $C_{B_{[\alpha_i]+}}(x) = k$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ trivially.

These three cases prove that $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Hence $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$.

Conversely, suppose that $A_{[\alpha_i]+} \subseteq B_{[\alpha_i]+}$. This means that $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]+}}(x) \leq C_{B_{[\alpha_i]+}}(x)$. That is $B_i^j(x) > A_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A \subseteq B$.

3. Suppose $A = B$. Then $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]} = B_{[\alpha_i]}$, it is enough to prove $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. there are three cases;

   **Case 1:** $C_{B_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^1(x) < \alpha_i$. This implies that $A_i^1(x) = B_i^1(x) < \alpha_i$ for some $i$. Therefore $C_{A_{[\alpha_i]}}(x) = 0$. Thus $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   **Case 2:** $C_{B_{[\alpha_i]}}(x) = j$. Then $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]}}(x) \geq j$. Also, there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^{j+1}(x) < \alpha_i$. This implies that $A_i^{j+1}(x) = B_i^{j+1}(x) < \alpha_i$ for some $i$. Therefore $C_{A_{[\alpha_i]}}(x) \leq j$. Thus $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   **Case 3:** $C_{B_{[\alpha_i]}}(x) = k$. Then $B_i^k(x) \geq \alpha_i$ for every i=1,2,...,n. This implies that $A_i^k(x) \geq \alpha_i$. Thus $C_{A_{[\alpha_i]}}(x) = k$ and therefore $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$.

   These three cases prove that $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Hence $A_{[\alpha_i]} = B_{[\alpha_i]}$.

   Conversely, suppose that $A_{[\alpha_i]} = B_{[\alpha_i]}$. This means that $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x)$ for every $i = 1, 2, ..., n$. Then $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$. That is $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Also, take $\alpha_i = B_i^j(x)$ for every $i = 1, 2, ..., n$. Then $B_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$. This implies $A_i^j(x) \geq \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]}}(x) = C_{B_{[\alpha_i]}}(x)$. That is $A_i^j(x) \geq B_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. Thus $A = B$.

4. Suppose $A = B$. Then $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. To prove $A_{[\alpha_i]+} = B_{[\alpha_i]+}$, it is enough to prove $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

   **Case 1:** $C_{B_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $B_i^1(x) \leq \alpha_i$. This implies that $A_i^1(x) = B_i^1(x) \leq \alpha_i$ for some $i$. Therefore $C_{A_{[\alpha_i]+}}(x) = 0$. Thus $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

   **Case 2:** $C_{B_{[\alpha_i]+}}(x) = j$. Then $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]+}}(x) \geq j$. Also, there exist some $i \in \{1, 2, .., n\}$ such that $B_i^{j+1}(x) \leq \alpha_i$. This implies that $A_i^{j+1}(x) = B_i^{j+1}(x) \leq \alpha_i$ for some $i$. Therefore $C_{A_{[\alpha_i]+}}(x) \leq j$. Thus $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

   **Case 3:** $C_{B_{[\alpha_i]+}}(x) = k$. Then $B_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Thus $C_{A_{[\alpha_i]+}}(x) = k$ and therefore $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$.

   These three cases prove that $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Hence $A_{[\alpha_i]+} = B_{[\alpha_i]+}$.

   Conversely, suppose that $A_{[\alpha_i]+} = B_{[\alpha_i]+}$. This means that $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$ for every $x \in X$. Let $x \in X$ and $j \in \{1, 2, ..., k\}$. Take $\alpha_i = A_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. This implies that $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$. That is $B_i^j(x) > A_i^j(x) - \epsilon$ for every $i = 1, 2, ..., n$. Thus $B_i^j(x) \geq A_i^j(x)$ for every $i = 1, 2, ..., n$. Also, take $\alpha_i = B_i^j(x) - \epsilon$ for any $\epsilon > 0$ and for every $i = 1, 2, ..., n$. Then $B_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$. That implies $A_i^j(x) > \alpha_i$ for every $i = 1, 2, ..., n$, since $C_{A_{[\alpha_i]+}}(x) = C_{B_{[\alpha_i]+}}(x)$. That is $A_i^j(x) > B_i^j(x) - \epsilon$ for every $i = 1, 2, ..., n$. Thus $A_i^j(x) \geq B_i^j(x)$ for every $i = 1, 2, ..., n$. Therefore $A_i^j(x) = B_i^j(x)$ for every $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and $x \in X$. Thus $A = B$.

**Theorem 3.8.** For any $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 0$ for every $i = 1, 2, ..., n$, the following property holds:

$$A_{[\alpha_i]} = \bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i, i=1,2,...,n}} A_{[\beta_i]} = \bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i, i=1,2,...,n}} A_{[\beta_i]+}$$

*Proof*: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 0$ for every $i = 1, 2, ..., n$. To prove $A_{[\alpha_i]} = \bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}$, it is enough to prove $C_{A_{[\alpha_i]}}(x) = C_{\bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$. We have $A_{[\alpha_i]} \subseteq A_{[\beta_i]}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$, for every $i = 1, 2, ..., n$. This means that $C_{A_{[\alpha_i]}}(x) \leq C_{A_{[\beta_i]}}(x)$ for every $x \in X$. This implies that $C_{A_{[\alpha_i]}}(x) \leq \inf_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]}}(x) \right\}$ and therefore $C_{A_{[\alpha_i]}}(x) \leq$

$C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$.

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Then $\inf\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]}}(x) \right\} \leq C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Hence $C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. From two inequalities, we have $C_{A_{[\alpha_i]}}(x) = C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

**Case 1:** $C_{A_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^1(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]}}(x) = C_{A_{[\alpha_i]}}(x)$.

**Case 2:** $C_{A_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^{j+1}(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^{j+1}(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]}}(x) \leq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

**Case 3:** $C_{A_{[\alpha_i]}}(x) = k$. Take $\beta_i = \alpha_i/2$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]}}(x) \leq k$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

Next, to prove $A_{[\alpha_i]} = \bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}$, it is enough to prove

$$C_{A_{[\alpha_i]}}(x) = C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$$

for every $x \in X$. We have $A_{[\alpha_i]} \subseteq A_{[\beta_i]+}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$. This means that $C_{A_{[\alpha_i]}}(x) \leq C_{A_{[\beta_i]+}}(x)$ for every $x \in X$. This implies that $C_{A_{[\alpha_i]}}(x) \leq \inf\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]+}}(x) \right\}$ and therefore $C_{A_{[\alpha_i]}}(x) \leq C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$.

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Then $\inf\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} \left\{ C_{A_{[\beta_i]+}}(x) \right\} \leq C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Hence $C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. From two inequalities, we have

$C_{A_{[\alpha_i]}}(x) = C_{\bigcap\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

**Case 1:** $C_{A_{[\alpha_i]}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^1(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]+}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]+}}(x) = C_{A_{[\alpha_i]}}(x)$.

**Case 2:** $C_{A_{[\alpha_i]}}(x) = j$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^{j+1}(x) < \alpha_i$. For those $i's$ there exist $\beta_i \in I$ such that $A_i^{j+1}(x) < \beta_i < \alpha_i$ and for other $i's$ take $\beta_i = \alpha_i/2$. Then $C_{A_{[\beta_i]+}}(x) \leq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

**Case 3:** $C_{A_{[\alpha_i]}}(x) = k$. Take $\beta_i = \alpha_i/2$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]+}}(x) \leq k$. That is, for $(\beta_i)_1^n$ we have $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i < \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]}}(x)$.

**Theorem 3.9.** For any $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 1$ for every $i = 1, 2, ..., n$, the following property holds:

$$A_{[\alpha_i]+} = \bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i, i=1,2,...,n}} A_{[\beta_i]} = \bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i, i=1,2,...,n}} A_{[\beta_i]+}$$

*Proof*: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$ such that $\alpha_i \neq 1$ for every $i = 1, 2, ..., n$. To prove $A_{[\alpha_i]+} = \bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}$, it is enough to prove $C_{A_{[\alpha_i]+}}(x) = C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$. We have $A_{[\beta_i]} \subseteq A_{[\alpha_i]+}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. This means that $C_{A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]+}}(x)$ for every $x \in X$. This implies that $\sup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{ C_{A_{[\beta_i]}}(x) \right\} \leq C_{A_{[\alpha_i]+}}(x)$ and therefore

$$C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}}(x) \leq C_{A_{[\alpha_i]+}}(x)$$

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$ for every $x \in X$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x) \leq \sup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{ C_{A_{[\beta_i]}}(x) \right\}$ for every $x \in X$. Hence $C_{A_{[\alpha_i]+}}(x) \leq C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$. From two inequalities, we have $C_{A_{[\alpha_i]+}}(x) = C_{\bigcup\limits_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

**Case 1:** $C_{A_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) \leq \alpha_i$. Since $\alpha_i \neq 1$ take $\beta_i \in I$ such that $\alpha_i < \beta_i < 1$. Then $C_{A_{[\beta_i]}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]}}(x)$

**Case 2:** $C_{A_{[\alpha_i]+}}(x) = j$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, .., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, .., n$ . Then $C_{A_{[\beta_i]}}(x) \geq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$

**Case 3:** $C_{A_{[\alpha_i]+}}(x) = k$. Then $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]}}(x) = k$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]}}(x)$

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]}}(x)$.

Next, to prove $A_{[\alpha_i]+} = \bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}$, it is enough to prove

$$C_{A_{[\alpha_i]+}}(x) = C_{\bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}}(x)$$

for every $x \in X$. We have $A_{[\beta_i]+} \subseteq A_{[\alpha_i]+}$ for any $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. This means that $C_{A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]+}}(x)$ for every $x \in X$. This implies that $\sup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{ C_{A_{[\beta_i]+}}(x) \right\} \leq C_{A_{[\alpha_i]+}}(x)$ and therefore $C_{\bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}}(x) \leq C_{A_{[\alpha_i]+}}(x)$.

To prove the reverse inequality, it is enough to prove there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$ for every $x \in X$. Then $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x) \leq \sup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} \left\{ C_{A_{[\beta_i]+}}(x) \right\}$ for every $x \in X$. Hence $C_{A_{[\alpha_i]+}}(x) \leq C_{\bigcap_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i < \alpha_i}} A_{[\beta_i]+}}(x)$ for every $x \in X$. From two inequalities, we have $C_{A_{[\alpha_i]+}}(x) = C_{\bigcup_{\substack{(\beta_i)_1^n \in I^n \\ \beta_i > \alpha_i}} A_{[\beta_i]+}}(x)$ for every $x \in X$.

It remains to prove the existence of $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$ for every $x \in X$. Let $x \in X$. There are three cases;

**Case 1:** $C_{A_{[\alpha_i]+}}(x) = 0$. Then there exist some $i \in \{1, 2, ..., n\}$ such that $A_i^1(x) \leq \alpha_i$. Choose $\beta_i \in I$ such that $\alpha_i < \beta_i < 1$ and then $A_i^1(x) \leq \alpha_i < \beta_i$ for some $i \in \{1, 2, ..., n\}$. Then $C_{A_{[\beta_i]+}}(x) = 0$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]+}}(x)$.

**Case 2:** $C_{A_{[\alpha_i]+}}(x) = j$. Then $A_i^j(x) > \alpha_i$ for every $i = 1, 2, .., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, .., n$. Then $C_{A_{[\beta_i]+}}(x) \geq j$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$.

**Case 3:** $C_{A_{[\alpha_i]+}}(x) = k$. Then $A_i^k(x) > \alpha_i$ for every $i = 1, 2, ..., n$. Then there exist $\beta_i \in I$ such that $A_i^j(x) > \beta_i > \alpha_i$ for every $i = 1, 2, ..., n$. Then $C_{A_{[\beta_i]+}}(x) = k$. That is, for $(\beta_i)_1^n$ we have $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ and $C_{A_{[\alpha_i]+}}(x) = C_{A_{[\beta_i]+}}(x)$.

These three cases prove that, for every $x \in X$, there exist $(\beta_i)_1^n \in I^n$ such that $\beta_i > \alpha_i$ for every $i = 1, 2, ..., n$ satisfying $C_{A_{[\alpha_i]+}}(x) \leq C_{A_{[\beta_i]+}}(x)$.

**Definition 3.10.** The *level set* of a multiple set $A$ is a crisp set, denoted by $\Lambda(A)$, is defined as

$$\begin{aligned} \Lambda(A) \;=\; & \left\{(\alpha_i)_1^n \in I^n; \alpha_i = A_i^j(x), 1 \leq i \leq n, \right. \\ & \left. \text{for some } x \in X \text{ and for some } j \in \{1, 2, ..., k\}\right\} \end{aligned}$$

## 3.3 Representations of Multiple Sets

The principal role of $\alpha_i - cuts$ and strong $\alpha_i - cuts$ in multiple set theory is their capability to represent corresponding multiple sets. In this section, it is shown that each multiple set can uniquely be represented by either the family of all its $\alpha_i - cuts$ or the family of all its strong $\alpha_i - cuts$.

**Definition 3.11.** Let $A \in MS_{(n,k)}(X)$, $(\alpha_i)_1^n \in I^n$ and $A_{[\alpha_i]} = \left\{C_{A_{[\alpha_i]}}(x)/x; x \in X\right\}$ be the $\alpha_i - cut$ of $A$. Then the special multiple set of $A$ with respect to $(\alpha_i)_1^n$ is a multiple set $S[A; (\alpha_i)_1^n]$ over $X$, with the membership matrix $S[A; (\alpha_i)_1^n](x)$ in which the first $C_{A_{[\alpha_i]}}(x)$ columns are $((\alpha_i)_1^n)^T$ and remaining are zero columns. The strong special multiple set of $A$ with respect to $(\alpha_i)_1^n$ is a multiple set $S^+[A; (\alpha_i)_1^n]$ over $X$, with the membership matrix $S^+[A; (\alpha_i)_1^n](x)$ for which the first $C_{A_{[\alpha_i]+}}(x)$ columns are $((\alpha_i)_1^n)^T$ and remaining are zero columns. $((\alpha_i)_1^n)^T$ denotes the column vector $(\alpha_i)_1^n$.

The representation of an arbitrary multiple set in terms of special multiple sets or in terms of strong special multiple sets is formulated as three basic decomposition theorems of multiple sets:

**Theorem 3.12. (First Decomposition Theorem for Multiple sets)**
For every $A \in MS_{(n,k)}(X)$, we have

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S[A; (\alpha_i)_1^n]$$

where $S[A; (\alpha_i)_1^n]$ is the special multiple set of $A$ with respect to $(\alpha_i)_1^n$ and $\bigcup$ denotes the standard union.

*Proof*: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$. To prove $A = \bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]$, it is enough to prove $A_i^j(x) = \left(\bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]\right)_i^j (x)$ for every $x \in X$. Let $x \in X,\ i \in \{1,2,...,n\}$ and $j \in \{1,2,...,k\}$. Take $a = A_i^j(x)$. Then

$$\begin{aligned}\left(\bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]\right)_i^j (x) &= \sup_{(\alpha_i)_1^n \in I^n} \left\{(S[A;(\alpha_i)_1^n])_i^j(x)\right\} \\ &= \max\left\{\sup_{\alpha_i \in [0,a]} \left\{(S[A;(\alpha_i)_1^n])_i^j(x)\right\},\right. \\ &\qquad \left.\sup_{\alpha_i \in (a,1]} \left\{(S[A;(\alpha_i)_1^n])_i^j(x)\right\}\right\}\end{aligned}$$

Since, for every $\alpha_i \in (a,1]$, $(S[A;(\alpha_i)_1^n])_i^j(x) = 0$ and for every $\alpha_i \in [0,a]$, $(S[A;(\alpha_i)_1^n])_i^j(x) = \alpha_i$ Therefore,

$$\left(\bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]\right)_i^j (x) = a = A_i^j(x)$$

for every $x \in X$. Therefore,

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S[A;(\alpha_i)_1^n]$$

**Theorem 3.13. (Second Decomposition Theorem for Multiple sets)**
For every $A \in MS_{(n,k)}(X)$, we have

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]$$

where $S^+[A;(\alpha_i)_1^n]$ is the strong special multiple set of $A$ with respect to $(\alpha_i)_1^n$ and $\bigcup$ denotes the standard union.

*Proof*: Let $A \in MS_{(n,k)}(X)$ and $(\alpha_i)_1^n \in I^n$. We have to prove $A = \bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]$, it is enough to prove $A_i^j(x) = \left(\bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]\right)_i^j (x)$ for every $x \in X$. For, let $x \in X,\ i \in \{1,2,...,n\}$ and $j \in \{1,2,...,k\}$. Take $a = A_i^j(x)$. Then

$$\begin{aligned}\left(\bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]\right)_i^j (x) &= \sup_{(\alpha_i)_1^n \in I^n} \left\{\left(S^+[A;(\alpha_i)_1^n]\right)_i^j(x)\right\} \\ &= \max\left\{\sup_{\alpha_i \in [0,a)} \left\{\left(S^+[A;(\alpha_i)_1^n]\right)_i^j(x)\right\},\right. \\ &\qquad \left.\sup_{\alpha_i \in [a,1]} \left\{\left(S^+[A;(\alpha_i)_1^n]\right)_i^j(x)\right\}\right\}\end{aligned}$$

Since, for every $\alpha_i \in [a,1]$, $(S^+[A;(\alpha_i)_1^n])_i^j(x) = 0$ and for every $\alpha_i \in [0,a)$, $(S^+[A;(\alpha_i)_1^n])_i^j(x) = \alpha_i$. Therefore,

$$\left(\bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]\right)_i^j (x) = a = A_i^j(x)$$

for every $x \in X$. Therefore,

$$A = \bigcup_{(\alpha_i)_1^n \in I^n} S^+[A;(\alpha_i)_1^n]$$

**Theorem 3.14. (Third Decomposition Theorem for Multiple sets)**
For every $A \in MS_{(n,k)}(X)$, we have

$$A = \bigcup_{(\alpha)_n \in \Lambda(A)} S[A;(\alpha_i)_1^n]$$

where $\Lambda(A)$ is the level set of $A$, $S[A;(\alpha_i)_1^n]$ is the special multiple set of $A$ and $\bigcup$ denotes the standard union.

The proof is analogous to the proof of theorem 3.12.

# 4 Multiple Complements

**Definition 4.1.** [11] A *fuzzy complement function* is a function $c : [0,1] \to [0,1]$ satisfying the axioms;
(c1) Boundary conditions: $c(0) = 1$ and $c(1) = 0$.
(c2) Monotonicity: For all $a, b \in [0,1]$, if $a \leq b$ then $c(a) \geq c(b)$.
A fuzzy complement function is said to be *continuous* if it satisfies the axiom;
(c3) $c$ is a continuous function.
and is said to be *involutive*, if it satisfies the axiom;
(c4) $c(c(a)) = a$ for every $a \in [0,1]$.

**Definition 4.2.** [11] The *equilibrium point* of a fuzzy complement $c$ is defined as any value $a \in [0,1]$ for which $c(a) = a$.

**Theorem 4.3.** [11] Every fuzzy complement has at most one equilibrium.

**Theorem 4.4.** [11] Assume that a given fuzzy complement $c$ has an equilibrium $e_c$, which is unique by Theorem 4.3. Then

$$a \leq c(a) \text{ if and only if } a \leq e_c$$
$$a \geq c(a) \text{ if and only if } a \geq e_c$$

**Theorem 4.5.** [11] If $c$ is continuous fuzzy complement, then $c$ has a unique equilibrium.

**Definition 4.6.** [11] Let $c$ be any fuzzy complemet function and $a \in [0,1]$ be any membership grade, then any real number $d_a \in [0,1]$ such that

$$c(d_a) - d_a = a - c(a)$$

is called a dual point of a with respect to $c$.

**Theorem 4.7.** [11] If a fuzzy complement $c$ has an equilibrium $e_c$, then

$$d_{e_c} = e_c$$

**Theorem 4.8.** [11] For each $a \in [0,1]$, $d_a = c(a)$ if and only if $c$ is involutive.

**Theorem 4.9.** [11]**(First Characterization Theorem of Fuzzy Complements)** Let $c$ be a function from $[0,1]$ to $[0,1]$. Then, $c$ is a fuzzy complement(involutive) if and only if there exists a continuous function $g$ from $[0,1]$ to $\mathbb{R}$ such that $g(0) = 0$, $g$ is strictly increasing and

$$c(a) = g^{-1}(g(1) - g(a))$$

for all $a \in [0,1]$.

**Theorem 4.10.** [11]**(Second Characterization Theorem of Fuzzy Complements)** Let $c$ be a function from $[0,1]$ to $[0,1]$. Then, $c$ is a fuzzy complement if and only if there exists a continuous function $f$ from $[0,1]$ to $\mathbb{R}$ such that $f(1) = 0$, $f$ is strictly decreasing and

$$c(a) = f^{-1}(f(0) - f(a))$$

for all $a \in [0,1]$.

**Definition 4.11.** Let $\mathbb{M} = \mathbb{M}_{n\times k}([0,1])$ be the set of all matrices of order $n \times k$ with entries from $[0,1]$. A *multiple complement function* is a function $c : \mathbb{M} \to \mathbb{M}$, where $c$ is characterized by fuzzy complement functions $c_{ij}$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}$ such that $b_{ij} = c_{ij}(a_{ij})$ for every $i = 1,2,...,n$ and $j = 1,2,...,k$. In this case we represent multiple complement function $c$ as a matrix $(c_{ij})$.

**Remark 4.12.** Using multiple complement function we can define complement of a multiple set as follows: Given a multiple set $A$ in $MS_{(n,k)}(X)$, we obtain the complement of $A$, denoted by $c(A)$, by applying function $c$ to matrix $A(x)$ for all $x \in X$.

**Definition 4.13.** A multiple complement function $c = (c_{ij})$ is said to be continuous if $c_{ij}$ is continuous and is said to be involutive if $c_{ij}$ is involutive for every $i = 1,2,...,n$ and $j = 1,2,...,k$.

**Example 4.14.**

1. Threshold Type Multiple Complement: Let $t_{ij} \in [0,1)$ for every $i = 1,2,...,n$ and $j = 1,2,...,k$. The function $c = (c_{ij})$ given by

$$c_{ij}(a) = \begin{cases} 1 & \text{for } a \leq t_{ij} \\ 0 & \text{for } a > t_{ij} \end{cases}$$

for $a \in [0,1]$ is a multiple complement. The matrix $T = (t_{ij})$ is called the threshold of $c$. This function is neither continuous and nor involutive.

2. The function $c = (c_{ij})$, where $c_{ij}$ is defined by $c_{ij}(a) = \dfrac{1}{2}(1 + \cos \pi a)$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, is a multiple complement. This is continuous, but not involutive.

3. Sugeno Class Multiple Complement: The function $c = (c_{ij})$, where $c_{ij}$ is defined by $c_{ij}(a) = \dfrac{1-a}{1+\lambda_{ij}a}$, where $\lambda_{ij} \in (-1, \infty)$, is an involutive multiple complement. Clearly, Sugeno class multiple complement is characterized by the matrix $\Lambda = (\lambda_{ij})$ and is represented by $c_\Lambda$. For $\lambda_{ij} = 0$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, this function becomes the standard multiple complement.

4. Yager Class Multiple Complement: The function $c = (c_{ij})$, where $c_{ij}$ is defined by $c_{ij}(a) = (1 - a^{\omega_{ij}})^{1/\omega_{ij}}$ where $\omega_{ij} \in (0, \infty)$, is an involutive multiple complement. Clearly, Yager class multiple complement is characterized by the matrix $\Omega = (\omega_{ij})$ and is represented by $c_\Omega$. When $\omega_{ij} = 1$, for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, this function becomes the standard multiple complement.

**Definition 4.15.** An equilibrium matrix of a multiple complement function $c$ is defined as any matrix $A$ in $\mathbb{M}$ for which $c(A) = A$.

**Note:** If $c = (c_{ij})$, there exists an equilibrium matrix $E_c$ for c if and only if there exist $e_{ij} \in [0, 1]$ such that $c_{ij}(e_{ij}) = e_{ij}$ (that is $e_{ij}$ is the equilibrium point of $c_{ij}$) for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then $E_c = (e_{ij})$.

**Example 4.16.** The equilibrium matrix of the Sugeno class multiple complement $c_\Lambda$ characterized by the matrix $\Lambda = (\lambda_{ij})$ is given by $E_\Lambda = (e_{ij})$, where

$$e_{ij} = \begin{cases} ((1+\lambda_{ij})^{1/2} - 1)/\lambda_{ij} & \text{for } \lambda_{ij} \neq 0 \\ 1/2 & \text{for } \lambda_{ij} = 0 \end{cases}$$

**Theorem 4.17.** Every multiple complement has atmost one equilibrium matrix.

*Proof.* By Theorem 4.3, $c_{ij}$ has at most one equilibrium point for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Hence a multiple complement has at most one equilibrium matrix.

**Notation:** In this context, for $A, B \in \mathbb{M}$, we say $A \leq B$, if $a_{ij} \leq b_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$

**Theorem 4.18.** Let $E_c$ be the equilibrium matrix of multiple complement function $c$. Then

$$A \leq c(A) \text{ if and only if } A \leq E_c$$
$$A \geq c(A) \text{ if and only if } A \geq E_c$$

*Proof.* By Theorem 4.4, for every $c_{ij}$, we have

$$a \leq c_{ij}(a) \text{ if and only if } a \leq e_{c_{ij}}$$
$$a \geq c_{ij}(a) \text{ if and only if } a \geq e_{c_{ij}}$$

for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Hence the theorem.

**Theorem 4.19.** If $c$ is a continuous multiple complement, then $c$ has a unique equilibrium matrix.

*Proof.* Since $c_{ij}$ is continuous, by Theorem 4.5, $c_{ij}$ has a unique equilibrium point, say $e_{ij}$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.. Then the matrix $E_c = (e_{ij})$ is the unique equilibrium matrix of $c$.

**Definition 4.20.** Let $c$ be any multiple complement function and $A \in \mathbb{M}$, then any matrix $D_A \in \mathbb{M}$ such that

$$c(D_A) - D_A = A - c(A)$$

is called a dual matrix of $A$ with respect to $c$.

**Note:** If $c = (c_{ij})$, there exists a dual matrix $D_A$ for $A = (a_{ij})$ with respect to c if and only if there exist $d_{ij} \in [0, 1]$ such that $c_{ij}(d_{ij}) - d_{ij} = a_{ij} - c(a_{ij})$ (that is $d_{ij}$ is the dual point of $a_{ij}$ with respect to $c_{ij}$) for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then $D_A = (d_{ij})$.

**Theorem 4.21.** If a multiple complement $c$ has an equilibrium matrix $E_c$, then

$$D_{E_c} = E_c$$

*Proof.* We have $E_c = (e_{ij})$, where $e_{ij}$ is the equilibrium point of $c_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then by Theorem 4.7, $d_{ij} = e_{ij}$, where $d_{ij}$ is the dual point of $e_{ij}$ with respect to $c_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Thus

$$D_{E_c} = (d_{ij}) = (e_{ij}) = E_c$$

**Theorem 4.22.** For each $A \in \mathbb{M}$, $D_A = c(A)$ if and only if $c$ is involutive.

*Proof.* We have, for each complement function $c_{ij}$, by Theorem 4.8, $d_{a_{ij}} = c_{ij}(a_{ij})$ if and only if $c_{ij}$ is involutive, for every $i = 1, 2, ..., n$ and $j = 1, 2, ...k$. Hence the theorem.

**Theorem 4.23. (First Characterization Theorem of Multiple Complements)** Let $c$ be a function from $\mathbb{M}$ to $\mathbb{M}$, where $c$ is characterized by functions $c_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}$ such that $b_{ij} = c_{ij}(a_{ij})$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then $c$ is a multiple complement(involutive) if and only if there exists a function $G : \mathbb{M} \to \mathbb{M}_{n\times k}(\mathbb{R})$ where $G$ is characterized by continuous functions $g_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}_{n\times k}(\mathbb{R})$ such that $b_{ij} = g_{ij}(a_{ij})$, $g_{ij}(0) = 0$, $g_{ij}$ is strictly increasing and $c_{ij}(a) = g_{ij}^{-1}(g_{ij}(1) - g_{ij}(a))$ for every $a \in [0, 1]$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. In this case we represent function $G$ as a matrix $(g_{ij})$.

*Proof.* For every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, by Theorem 4.9, $c_{ij}$ is a fuzzy complement(involutive) if and only if there exists a continuous function $g_{ij}$ from $[0, 1]$ to $\mathbb{R}$ such that $g_{ij}(0) = 0$, $g_{ij}$ is strictly increasing and

$$c_{ij}(a) = g_{ij}^{-1}(g_{ij}(1) - g_{ij}(a))$$

for all $a \in [0, 1]$. Hence the theorem.

**Remark 4.24.** Functions $G$, defined in the Theorem 4.23, are usually called increasing generators. Each function $G$ that qualifies as an increasing generator determines a multiple complement.
For standard multiple complement, the increasing generator is $G = (g_{ij})$, where $g_{ij}$ is defined as $g_{ij}(a) = a$ for evrey $a \in [0, 1]$.
For Sugeno class of multiple complements, the increasing generators are $G = (g_{ij})$, where $g_{ij}$ is defined as $g_{ij}(a) = \dfrac{1}{\lambda_{ij}} \ln(1 + \lambda_{ij} a)$ for every $a \in [0, 1]$ and for $\lambda_{ij} > -1$.
For Yager class of multiple complements, the increasing generators are $G = (g_{ij})$, where $g_{ij}$ is defined as $g_{ij}(a) = a^{\omega_{ij}}$ for every $a \in [0, 1]$ and for $\omega_{ij} > 0$.

**Theorem 4.25. (Second Characterization Theorem of Multiple Complements)** Let $c$ be a function from $\mathbb{M}$ to $\mathbb{M}$, where $c$ is characterized by functions $c_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $\mathbb{M}$ such that $b_{ij} = c_{ij}(a_{ij})$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then $c$ is a multiple complement if and only if there exists a function $F : \mathbb{M} \to M_{n\times k}(\mathbb{R})$ where $F$ is characterized by continuous functions $f_{ij}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$ in such a way that $A = (a_{ij})$ in $\mathbb{M}$ is mapped to $B = (b_{ij})$ in $M_{n\times k}(\mathbb{R})$ such that $b_{ij} = f_{ij}(a_{ij})$, $f_{ij}(1) = 0$, $f_{ij}$ is strictly decreasing and $c_{ij}(a_{ij}) = f_{ij}^{-1}(f_{ij}(0) - f_{ij}(a))$ for every $a \in [0, 1]$, i=1,2,...,n and j=1,2,...,k. In this case we represent function $F$ as a matrix $(f_{ij})$.

*Proof.* For every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, by Theorem 4.9, $c_{ij}$ is a fuzzy complement if and only if there exists a continuous function $f_{ij}$ from $[0, 1]$ to $\mathbb{R}$ such that $f_{ij}(1) = 0$, $f_{ij}$ is strictly decreasing and

$$c_{ij}(a) = f_{ij}^{-1}(f_{ij}(0) - f_{ij}(a))$$

for all $a \in [0, 1]$. Hence the theorem.

**Remark 4.26.** Functions $F$ defined in the Theorem 4.24 are usually called decreasing generators. Each function $F$ that qualifies as an decreasing generator also determines a multiple complement.
For standard multiple complement, the decreasing generator is $F = (f_{ij})$, where $f_{ij}$ is defined as $f_{ij}(a) = -ka + k$ for every $a \in [0, 1]$, where $k > 0$.
For Yager class of multiple complements, the decreasing generators are $F = (f_{ij})$, where $f_{ij}$ is defined as $f_{ij}(a) = 1 - a^{\omega_{ij}}$ for every $a \in [0, 1]$ and for $\omega_{ij} > 0$.

# 5 Conclusion

In this paper, a modified definition of multiple sets is given and it is shown that the revised definition also satisfies all fundamental properties satisfied by the earlier definition. Then, the ideas of $\alpha_i - cut$ and strong $\alpha_i - cut$ and representation of multiple set by using $\alpha_i - cut$ and strong $\alpha_i - cut$ are proposed . Then, the concept of multiple complement is introduced as an extension of fuzzy complement and its properties are discussed.

Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set. So one can think about aggregation operations on multiple sets, which yields a single multiple set by combining two or more multiple sets. Fuzzy numbers are special type of fuzzy sets, which play an important role in many applications, including fuzzy control, decision making, approximate reasoning, optimization and statistics with imprecise probabilities. So in future work, it is possible to think multiple number as a special type of multiple set.

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