http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 06.12.2015 *Year*: **2016**, *Number*: **12**, *Pages*: **23-28**
Published: 28.03.2016 *Original Article***

COMPACTIFICATION OF SOFT TOPOLOGICAL SPACES

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Abstaract − In this work, we define dense soft set and compact softsubset. We then define one point compactification on soft topological spaces.

Keywords − *Soft sets, soft topology, soft compactification.*

1 Introduction

Many problems in economics, engineering, environmental scinece and social science are highly dependent on the task of modelling uncertain data, but modelling uncertain data is usually highly complicated and difficult to characterize. There are several theories which can be used for dealing these difficulties. Some of these theories are probability theory, fuzzy set theory, rough set theory and the interval mathematics. However, these theories have their own difficulties. In 1999, the soft set theory was introduced as a new mathematical tool to solve these diffuculties by Molodtsov [17]. Following his work Maji et.al. [14] gave several basic notions and the first practical application of soft sets in decision making problems. After that, Pei Miao [18] and Chen [9] improved the work of Maji et. al.. Many researchers applied this concept on topological spaces [7, 19, 21, 3], group theory, ring theory [1, 4, 12, 11, 13], and also decison making problems [5, 6, 9, 15].

Recently, Shabir and Naz [19] introduced the soft topological spaces. They defined soft open sets, soft closed sets, soft subspace, soft closure, soft nhood, soft speration axioms and their several properties. In 2012, Zorlutuna et. al. [21] initiated the soft continuity of soft functions, soft compactness and studied some properties. Then, many reasarchers [2, 10, 8, 20, 16] improved to concept of soft topological spaces.

In this paper, we introduce a notion of one point compactification on soft topological spaces.

***Edited by* Naim Çağman *(Editor-in-Chief)*.

2 Preliminary

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$.

Definition 2.1. [17] Let X be the initial universe set and E be the set of parameters. A pair (F, A) is called a soft set over X where F is a mapping given by $F : A \to P(X)$ and $A \subseteq E$.

In the other words, the soft set is a parametrized family of subsets of the set X. Every set $F(e)$, for every $e \in A$, from this family may be considered as the set of e-elements of the soft set (F, A).

From now on, the set of all soft sets over X will be denoted by $S(X, E)$**.**

Definition 2.2. [5]Let $A \subseteq E$. A soft set F_A over universe X is mapping from the parameter set E to $P(X)$, i.e., $F_A : E \to P(X)$, where $F_A(e) \neq \varnothing$ if $e \in A \subset E$ and $F_A(e) = \varnothing$ if $e \notin A$.

Definition 2.3. [5] The soft set $F_E \in S(X, E)$ is called null soft set, denoted by $F_\emptyset$, if for all $e \in E$, $F_E(e) = \varnothing$.

Definition 2.4. [5]Let $F_E \in S(X, E)$. The soft set F_E is called universal soft set, denoted by $F_{\widetilde{E}}$, if for all $e \in E$, $F_E(e) = X$.

Definition 2.5. [5]Let $F_A, G_B \in S(X, E)$. F_A is called a soft subset of G_B if $F_A(e) \subset G_B(e)$ for every $e \in E$ and we write $F_A \widetilde{\subset} G_B$.

Definition 2.6. [5]Let $F_A, G_B \in S(X, E)$. F_A and G_B are said to be equal, denoted by $F_A = G_B$ if $F_A \widetilde{\subset} G_B$ and $G_B \widetilde{\subset} F_A$.

Definition 2.7. [5]Let $F_A, G_B \in S(X, E)$. Then the union of F_A and G_B is also a soft set H_C, defined by $H_C(e) = F_A(e) \cup G_B(e)$ for all $e \in E$, where $C = A \cup B$. Here we write $H_C = F_A \widetilde{\cup} G_B$.

Definition 2.8. [5]Let $F_A, G_B \in S(X, E)$. Then the intersection of F_A and G_B is also a soft set H_C, defined by $H_C(e) = F_A(e) \cap G_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $H_C = F_A \widetilde{\cap} G_B$.

Definition 2.9. [5]Let $F_A \in S(X, E)$. The complement of F_A, denoted by F_A^c, is a soft set defined by $F_A^c(e) = X - F_A(e)$ for every $e \in E$.

Let us call F_A^c to be soft complement function of F_A. Clearly $(F_A^c)^c = F_A$, $(F_{\widetilde{E}})^c = F_\emptyset$ and $(F_\emptyset)^c = F_{\widetilde{E}}$.

Definition 2.10. Let $F_A \in S(X, E)$ and $x \in X$. Then $F_A \widetilde{\cup} x$ is soft set in $S(X, E)$, defined by $(F_A \widetilde{\cup} x)(e) = F_A(e) \cup \{x\}$ for all $e \in E$.

Example 2.11. Let $E = \{e_1, e_2, e_3\}$, $X = \{x_1, x_2, x_3\}$ and $F_A = \{(e_1, \{x_1\}), (e_3, \{x_2, x_3\})\}$. Then $F_A \widetilde{\cup} x_2 = \{(e_1, \{x_1,\ x_2\}),(e_2, \{x_2\})\ (e_3, \{x_2, x_3\})\}$.

Definition 2.12. (see [19]) A soft topological space is a triple (X, τ, E) where X is a nonempty set and τ is a family of soft sets over X satisfying the following properties:
(1) $F_{\widetilde{E}}, F_{\varnothing} \in \tau$
(2) If F_A, $G_B \in \tau$, then $F_A \widetilde{\cap} G_B \in \tau$
(3) If $F_{A_i} \in \tau$, $\forall i \in J$, then $\widetilde{\bigcup\limits_{i \in I}} F_{A_i} \in \tau$.
Then τ is called a topology of soft sets on X. Every member of τ is called soft open. G_B is called soft closed in (X, τ, E) if $(G_B)^c \in \tau$.

Example 2.13. Let $E = \{e_1, e_2, ..., e_k\}$ set of parameter, $X = [0, 1)$,

$$F_{An} = \{(e_i, [0, 1 - \frac{1}{n})) : e_i \in E, n \in \mathbb{N} \backslash \{0, 1\}\}$$

and $\tau = \{F_{A_n}\}_{n \in \mathbb{N} \backslash \{0,1\}} \cup F_{\varnothing} \cup F_{[0,1)}$ Then (X, τ, E) soft topological space on X.

Definition 2.14. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. Then $\tau_{F_A} = \{F_A \widetilde{\cap} G_B : G_B \in \tau\}$.

Example 2.15. Let $E = \{e_1, e_2, e_3\}$, $X = \{x_1, x_2, x_3\}$, $F_A = \{(e_1, \{x_1\}), (e_3, \{x_2, x_3\})\}$ and $\tau = \{F_{\varnothing},\ F_E,\ G_B\}$, where $G_B = \{(e_1, \{x_1, x_2\}), (e_2, X)\}$. Then $\tau_{F_A} = \{F_{\varnothing},\ F_A,\ F_A \widetilde{\cap} G_B\}$, where $F_A \widetilde{\cap} G_B = \{(e_1, \{x_1\})\}$.

Definition 2.16. [19]Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. The soft closure of F_A denoted by $\overline{F_A}$ is the intersection of all soft closed supersets of F_A.

Clearly, $\overline{F_A}$ is the smallest soft closed set over X which contains F_A.

Definition 2.17. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. F_A is called dense soft set in X if $\overline{F_A} = F_E$.

Definition 2.18. Let (X, τ, E) be a soft topological space and $\mathcal{U} = \{F_{A_i} : i \in I\}$. A family $\mathcal{U}$ of soft sets is a cover of a soft set F_A if $F_A \widetilde{\subset} \widetilde{\cup} \{F_{A_i} : i \in I\}$.

Definition 2.19. A soft topological space (X, τ, E) is compact if each soft open cover of $F_{\widetilde{E}}$ has a finite subcover.

Example 2.20. Let us consider the soft topological space (X, τ, E) in example 2.13. Then (X, τ, E) is not compact topological space because $\{F_{A_n}\}_{n \in \mathbb{N} \backslash \{0,1\}}$ is soft open cover of $F_{\widetilde{E}}$ but there is no finite subcover.

Definition 2.21. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. F_A is called compact soft subset if $(F_A, \tau_{F_A}.E)$ is compact.

Proposition 2.22. Let (X, τ, E) be a soft topological space and $F_A \in S(X, E)$. F_A is a compact if and only if each soft open cover of F_A has a finite subcover.

Proof. Let F_A be a compact and $\mathcal{U}^* = \{G_{B_i} : G_{B_i} \in \tau_{F_A}, i \in I\}$ be a soft open cover of F_A. Since $G_{B_i} \in \tau_{F_A}$, then there exist F_{A_i} soft open sets such that $G_{B_i} = F_{A_i} \widetilde{\cap} F_A$. Since F_A is compact, F_A has a finite subcover of U^*. □

Theorem 2.23. Soft closed set of the compact soft topological space is compact.

Proof. Let (X,τ,E) be a soft topological space, F_A is a soft closed set in X and $\mathcal{U}=\{F_{A_i} : i\in I\}$ soft open cover of F_A. Then $F_A\widetilde{\subset}\underset{i\in I}{\widetilde{\cup}}F_{A_i}$. Since F_A^c soft open set, $\mathcal{U}^*=\mathcal{U}\widetilde{\cup}(F_A^c)$ is a soft open cover of $F_{\widetilde{E}}$. Again since, (X,τ,E) is a compact soft topological space, then $\mathcal{U}^*$ has a finite subfamily such that $F_{\widetilde{E}}=\overset{n}{\underset{i=1}{\widetilde{\cup}}}F_{A_i}$, hence $F_A\widetilde{\subset}\overset{n}{\underset{i=1}{\widetilde{\cup}}}(F_{A_i}\widetilde{\cap}F_A)=\overset{n}{\underset{i=1}{\widetilde{\cup}}}F_{A_i}$. Thus F_A is compact. $\square$

Proposition 2.24. Let (X,τ,E) be a noncompact soft topological space, $X^*=X\cup\{x\}$ and $\mathcal{W}=\{F_A\widetilde{\cup}x : F_A^c \text{ compact}, F_A\in\tau\}$. Then $\tau^*=\tau\cup\mathcal{W}$ is soft topology on X^*.

Proof. T1) Since $F_\varnothing$ and $F_{\widetilde{E}}$ elements of τ, then $F_\varnothing$,$F^*_{\widetilde{E}}\in\tau^*$.

T2) Let $F_{A_1},F_{A_2}\in\tau^*$. Then

Case I. If $F_{A_1},F_{A_2}\in\tau$, then the proof is clear.

Case II. If $F_{A_1}\in\tau$ and $F_{A_2}\in\mathcal{W}$, then there exists $G_B\in\tau$ such that $F_{A_2}=G_B\widetilde{\cup}x$ and G_B^c is compact. Since $F_{A_1}\widetilde{\cap}F_{A_2}=F_{A_1}\widetilde{\cap}(G_B\widetilde{\cup}x)=F_{A_1}\widetilde{\cap}G_B$, then $F_{A_1}\widetilde{\cap}F_{A_2}\in\tau$. Thus, we have $F_{A_1}\widetilde{\cap}F_{A_2}\in\tau^*$.

Case III. If $F_{A_1},F_{A_2}\in\mathcal{W}$, then there exist $F_{A_1}=G_{B_1}\widetilde{\cup}x$ and $F_{A_2}=G_{B_2}\widetilde{\cup}x$ such that G_{B_1},$G_{B_2}\in\tau$ and $G^c_{B_1}$,$G^c_{B_2}$ are compact. Since $F_{A_1}\widetilde{\cap}F_{A_2}=(G_{B_1}\widetilde{\cup}x)\widetilde{\cap}(G_{B_2}\widetilde{\cup}x)=(G_{B_1}\widetilde{\cap}G_{B_2})\widetilde{\cup}x$, $(G_{B_1}\widetilde{\cap}G_{B_2})^c$ is compact, then $F_{A_1}\widetilde{\cap}F_{A_2}\in\tau^*$.

T3) Let I be an arbitrary index set and $F_{A_i}\in\tau^*$ for all $i\in I$. Then

Case I. If $F_{A_i}\widetilde{\in}\tau$ for all $i\in I$, then $\underset{i\in I}{\widetilde{\cup}}F_{A_i}\widetilde{\in}\tau$.

Case II. If $F_{A_{i_0}}\widetilde{\in}\mathcal{W}$ for some $i_0\in I$, then there exists $G_{B_{i_0}}\in\tau$ such that $F_{A_{i_0}}=G_{B_{i_0}}\widetilde{\cup}x$ and $G^c_{B_{i_0}}$ is compact. Therefore, we have $\underset{i\in I}{\widetilde{\cup}}F_{A_i}=(\underset{i\neq i_0}{\widetilde{\cup}}F_{A_i})\widetilde{\cup}(G_{B_{i_0}}\widetilde{\cup}x)=((\underset{i\neq i_0}{\widetilde{\cup}}F_{A_i})\widetilde{\cup}G_{B_{i_0}})\widetilde{\cup}x$. Then $((\underset{i\neq i_0}{\widetilde{\cup}}F_{A_i})\widetilde{\cup}G_{B_{i_0}})^c=(\underset{i\neq i_0}{\widetilde{\cap}}F^c_{A_i})\widetilde{\cap}(G^c_{B_{i_0}})$. Since $\underset{i\neq i_0}{\widetilde{\cap}}F^c_{A_i}$ is soft closed and $G^c_{B_{i_0}}$ is compact, $((\underset{i\neq i_0}{\widetilde{\cup}}F_{A_i})\widetilde{\cup}G_{B_{i_0}})^c$ is compact.

Case III. If $F_{A_i}\widetilde{\in}\mathcal{W}$ for all $i\in I$, then there exist $G_{B_i}\in\tau$ such that $F_{A_i}=G_{B_i}\widetilde{\cup}x$ and $G^c_{B_i}$ is compact. Therefore, we have $\underset{i\in I}{\widetilde{\cup}}F_{A_i}=\underset{i\in I}{\widetilde{\cup}}(G_{B_i}\widetilde{\cup}x)=(\underset{i\in I}{\widetilde{\cup}}G_{B_i})\widetilde{\cup}x$. Then $((\underset{i\in I}{\widetilde{\cup}}G_{B_i})\widetilde{\cup}x)^c=(\underset{i\in I}{\widetilde{\cap}}G^c_{B_i})$. Since $G^c_{B_i}$ is compact for all $i\in I$, then $\underset{i\in I}{\widetilde{\cap}}G^c_{B_i}$ is compact. Hence $\underset{i\in I}{\widetilde{\cup}}F_{A_i}\widetilde{\in}\tau^*$. $\square$

Proposition 2.25. (X^*,τ^*,E) soft topological space is compact.

Proof. Let $\mathcal{U}=\{F_{A_i} : i\in I\}$ be a cover of $F^*_{\widetilde{E}}$. Since $x\in X^*$, then there exists $i_0\in I$ such that $x\in F_{A_{i_0}}\in\mathcal{U}$. Then there exists $G_B\in\tau$ such that $F_{A_{i_0}}=G_B\widetilde{\cup}x$ where G_B^c is compact. Since G_B^c is compact, then there exist $F_{A_1},F_{A_2},...,F_{A_n}\in\mathcal{U}$ such that $G_B^c\widetilde{\subset}F_{A_1}\widetilde{\cup}F_{A_2}\widetilde{\cup}...\widetilde{\cup}F_{A_n}$. Then $F^*_{\widetilde{E}}=(F_{\widetilde{E}}\widetilde{\setminus}G_B)\widetilde{\cup}F_{A_{i_0}}\widetilde{\subset}F_{A_1}\widetilde{\cup}F_{A_2}\widetilde{\cup}...\widetilde{\cup}F_{A_n}\widetilde{\cup}F_{A_{i_0}}$. Hence (X^*,τ^*,E) topological space is compact. $\square$

Proposition 2.26. $F_{\widetilde{E}}$ is dense soft subset in (X^*, τ^*, E) topological space.

Proof. Since $\overline{F_{\widetilde{E}}}$ is the intersection of all soft closed supersets of $F_{\widetilde{E}}$ in $S(X^*, E)$, $\overline{F_{\widetilde{E}}} = F^*_{\widetilde{E}}$. Hence we have $F_{\widetilde{E}}$ is dense soft subset in (X^*, τ^*, E). $\square$

Example 2.27. Let us consider the soft topological space (X, τ, E) in example 2.13 and $X^* = X \cup \{1\} = [0, 1]$. Since $F_\varnothing$ only compact soft set in τ, then

$$\mathcal{W} = \{F_A \widetilde{\cup} x : F_A^c compact, F_A \in \tau\} = \{F_E \widetilde{\cup} x\}$$

Again since $\tau^* = \tau \cup \mathcal{W}$, then (X^*, τ^*, E) soft compact. Hence (X^*, τ^*, E) soft compactification of (X, τ, E).

3 Conclusion

In the present work, we have continued to study soft topological spaces. We introduce soft compactifiation. We hope that the findings in this paper will help researcher enhance and promote the further study soft topology to carry out a general framework for their applications in practical life.

Acknowledgement

The authors are grateful for financial support from the Research Fund of Cumhuriyet University under grand no: F-458.

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