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ON L-FUZZY INTERIOR (CLOSURE) SPACES

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Abstaract − The aim of this paper is to introduce the concept of L-fuzzy interior (closure) spaces and the L-fuzzy topological space in a complete residuated lattice. We study some relationships among those structures. Finally, we give their examples.

Keywords − *Complete residuated lattice, L-fuzzy interior operator, L-fuzzy closure operator, L-fuzzy topological space and continuous maps.*

1 Introduction

Since Chang [6] introduced fuzzy set theory to topology, many researchers have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. In Chang's I-topology on a set X, each open set was fuzzy, while the topology itself was a crisp subset of the family of all fuzzy subsets of X.

From a different direction, the fundamental idea of a topology itself being fuzzy was first defined by Höhle [14] in 1980, then was independently generalized be each of Kubiak [17] and Sôstak [25] in 1985 and independently rediscovered by Ying [26, 27] in Höhle's original setting in 1991 in Höhle's approach a topology was an L-subset of a traditional powerset.

In 1999, the axioms of many-valued L-fuzzy topological spaces and L-fuzzy continuous mappings are given a lattice-theoretical foundation by Höhle and Sôstak and a categorical foundation by Rodabaugh [23]. Sôstak [25] introduced the fuzzy

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topology as an extension of Chang's fuzzy topology, Ramadan and his colleagues [21] called it smooth topology.

Closure and interior operators on ordinary sets belongs to the very fundamental mathematical structure with direct applications, both mathematical (topology, logic, for instance) and extra mathematical (e.g. data mining, knowledge representation). In fuzzy set theory, several particular cases as well as general theory of closure operators which operate with fuzzy sets (so called fuzzy closure operators) are studied (Mashour and Ghanim [19], Bandler and Kohout [1], Bêlohàvek [2, 3], Gerla [11]).

Interior operators, however, have appeared in a few studies only (Bandler and Kohout [1], Dubois and Prade [7], Bodenhofer et al [5]), and it seem that no general theory of interior operators appeared so far. In ordinary set theory, closure and interior operators on a set in a bijective correspondence.

In this paper is, we investigate the concept of L-fuzzy interior (closure) operators using the definition of the L-fuzzy topology, which deduced an L-fuzzy (interior) closure spaces and vise versa. Continuity property and examples of those spaces are also discussed.

2 Preliminary

Definition 2.1. [4, 15] An algebra $(L, \wedge, \vee, \odot, \to, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions

(C1) $L = (L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
(C2) $(L, \odot, \top)$ is a commutative monoid;
(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

An operator $^* : L \to L$ defined by $a^* = a \to 0$ is called a *strong negation* if $a^{**} = a$.
For $\alpha \in L,\ \lambda \in L^X$, we denote $(\alpha \to \lambda),\ (\alpha \odot \lambda),\ \alpha_X,\ \top_x \in L^X$ as

$$(\alpha \to \lambda)(x) = \alpha \to \lambda(x),\ (\alpha \odot \lambda)(x) = \alpha \odot \lambda(x),\ \alpha_X(x) = \alpha,$$

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, {}^*, \perp, \top)$ be a complete residuated lattice with a strong negation *.

Lemma 2.2. [4, 15, 24] For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) $x \to y = \top$ iff $x \leq y,\ x \to \top = \top$ and $\top \to x = x$,
(2) If $y \leq z$, then $x \to y \leq x \to z,\ z \to x \leq y \to x,\ x \oplus y \leq x \oplus z$ and $x \odot y \leq x \odot z$,

(3) $x \odot y \le x \oplus y$,
(4) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $x \odot (\bigwedge_{i\in\Gamma} y_i) \le \bigwedge_{i\in\Gamma}(x \odot y_i)$,
(5) $x \oplus (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \oplus y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \oplus y = \bigvee_{i\in\Gamma}(x_i \oplus y)$,
(6) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigwedge_{i\in\Gamma} x_i) \to y \ge \bigvee_{i\in\Gamma}(x_i \to y)$,
(7) $x \to (\bigvee_{i\in\Gamma} y_i) \ge \bigvee_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$,
(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$,
(9) $(x \to y) \odot x \le y$ and $(x \to y) \odot (y \to z) \le (x \to z)$,
(10) $x \to y \le (y \to z) \to (x \to z)$, $x \to y \le (z \to x) \to (z \to y)$ and $y \to z \le x \odot y \to x \odot z$,
(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,
(12) $x \odot (y \to z) \le y \to (x \odot z)$,
(13) $(x \to y) \odot (z \to w) \le (x \odot z) \to (y \odot w)$,
(14) $(x \to y) \odot (z \to w) \le (x \oplus z) \to (y \oplus w)$,
(15) $(x \to y) \oplus (z \to w) \le (x \odot z) \to (y \oplus w)$,
(16) $x^* \to y^* = y \to x$,
(17) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$,
(18) $(x \odot y)^* = x \to y^*$ and $(x \to y)^* = x \odot y^*$,
(19) $x \odot (x^* \oplus y^*) \le y^*$.

Definition 2.3. [2, 3] Let X be a set. A function $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions
(E1) reflexive if $R(x,x) = \top$ for all $x \in X$,
(E2) transitive if $R(x,y) \odot R(y,z) \le R(x,z)$ for all $x,y,z \in X$,
(E3) if $R(x,y) = R(y,x) = \top$, then $x = y$.

Lemma 2.4. [2, 3] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda,\mu) = \bigwedge_{x\in X} (\lambda(x) \to \mu(x)).$$

Then, for each $\lambda,\mu,\rho,\nu \in L^X$ and $\alpha \in L$ the following properties hold.
(1) S is an L-partial order on L^X,
(2) $\lambda \le \mu$ iff $S(\lambda,\mu) \ge \top$,
(3) If $\lambda \le \mu$, then $S(\rho,\lambda) \le S(\rho,\mu)$ and $S(\lambda,\rho) \ge S(\mu,\rho)$ for each $\rho \in L^X$,
(4) $S(\lambda,\mu) \odot S(\nu,\rho) \le S(\lambda \odot \nu, \mu \odot \rho)$,
(5) $S(\lambda,\mu) \odot S(\nu,\rho) \le S(\lambda \oplus \nu, \mu \oplus \rho)$,
(6) $S(\lambda, \alpha \to \mu) = S(\alpha \odot \lambda, \mu) = \alpha \to S(\lambda,\mu)$ and $\alpha \odot S(\lambda,\mu) \le S(\lambda, \alpha \odot \mu)$,
(7) $\mu \odot S(\mu,\lambda) \le \lambda$, $S(\mu,\lambda) \to \lambda \ge \mu$ and $S(\lambda,\mu) = S(\mu^*,\lambda^*)$.

Proof. We need to prove (5) by Lemma 2.2(8),(14), we have

$$\begin{aligned}
S(\lambda\oplus\nu,\mu\oplus\rho) &= \bigwedge_{x\in X} \big((\lambda\oplus\nu)(x) \to (\mu\oplus\rho)(x)\big)\\
&\ge \bigwedge_{x\in X} \big((\lambda\to\mu)(x) \odot (\nu\to\rho)(x)\big)\\
&\ge \big(\bigwedge_{x\in X} (\lambda\to\mu)(x)\big) \odot \big(\bigwedge_{x\in X} (\nu\to\rho)(x)\big)\\
&= S(\lambda,\mu) \odot S(\nu,\rho).
\end{aligned}$$

Lemma 2.5. [2, 3] Let $\phi : X \to Y$ be an ordinary mapping. Define $\phi^{\to} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ by

$$\phi^{\to}(\lambda)(y) = \bigvee_{\phi(x)=y} \lambda(x) \;\; \forall\, \lambda \in L^X,\; y \in Y,$$

$$\phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)) = \mu \circ \phi(x) \;\; \forall\, \mu \in L^Y.$$

Then for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$,

$$S(\lambda, \mu) \leq S(\phi^{\to}(\lambda), \phi^{\to}(\mu)), \qquad S(\rho, \nu) \leq S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu)),$$

and the equalities hold if ϕ is bijective.

Definition 2.6. [15] A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions.

(LO1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$,
(LO2) $\mathcal{T}(\lambda \odot \mu) \geq \mathcal{T}(\lambda) \odot \mathcal{T}(\mu), \;\; \forall\, \lambda, \mu \in L^X$,
(LO3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i), \;\; \forall\, \{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

An L-fuzzy topology is enriched if (R) $\mathcal{T}(\alpha \odot \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X, \alpha \in L$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two L-fuzzy topological spaces. A mapping $\phi : X \to Y$ is said to be LF-fuzzy continuous iff for each $\lambda \in L^Y$, we have

$$\mathcal{T}_Y(\lambda) \leq \mathcal{T}_X(\phi^{\leftarrow}(\lambda)).$$

Definition 2.7. [15] A map $\mathcal{F} : L^X \to L$ is called an L-fuzzy co-topology on X if it satisfies the following conditions.

(LF1) $\mathcal{F}(\bot_X) = \mathcal{F}(\top_X) = \top$,
(LF2) $\mathcal{F}(\lambda \oplus \mu) \geq \mathcal{F}(\lambda) \odot \mathcal{F}(\mu), \;\; \forall\, \lambda, \mu \in L^X$,
(LF3) $\mathcal{F}(\bigwedge_i \lambda_i) \leq \bigvee_i \mathcal{F}(\lambda_i), \;\; \forall\, \{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

The pair $(X, \mathcal{F})$ is called an L-fuzzy co-topological space. An L-fuzzy co-topology is called enriched if (S) $\mathcal{F}(\alpha \to \lambda) \geq \mathcal{F}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two L-fuzzy co-topological spaces. A mapping $\phi : X \to Y$ is said to be LF-fuzzy continuous iff for each $\lambda \in L^Y$, we have

$$\mathcal{F}_Y(\lambda) \leq \mathcal{F}_X(\phi^{\leftarrow}(\lambda)).$$

Definition 2.8. [22] A map $\mathcal{I} : L^X \times L_\bot \to L^X,\; L_\bot = L - \{\bot\}$ is called an L-fuzzy interior operator on X if $\mathcal{I}$ satisfies the following conditions

(I1) $\mathcal{I}(\top_X, r) = \top_X$,
(I2) $\mathcal{I}(\lambda, r) \leq \lambda$, or equivalently, $S(\mathcal{I}(\lambda, r), \lambda) \geq \top$ for all $\lambda \in L^X$,

(I3) $S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$ for all $\lambda,\mu \in L^X$,
(I4) If $r \le s$, then $\mathcal{I}(\lambda,s) \le \mathcal{I}(\lambda,r)$,
(I5) $\mathcal{I}(\lambda \odot \mu, r \odot s) \ge \mathcal{I}(\lambda,r) \odot \mathcal{I}(\mu,s)$.

The pair $(X,\mathcal{I})$ is called an L-fuzzy interior space. An L-fuzzy interior space $(X,\mathcal{I})$ is topological if
(T) $\mathcal{I}(\mathcal{I}(\lambda,r),r) = \mathcal{I}(\lambda,r) \ \ \forall\, \lambda \in L^X, r \in L_\perp$.

Let $(X,\mathcal{I}_X)$ and $(X,\mathcal{I}_Y)$ be two L-fuzzy interior spaces. A map $\phi : X \to Y$ is called $\mathcal{I}$-map if

$$\phi^{\leftarrow}(\mathcal{I}_Y(\mu,r)) \le \mathcal{I}_X(\phi^{\leftarrow}(\mu),r) \ \ \forall\, \mu \in L^Y, r \in L_\perp.$$

Lemma 2.9. Let $\mathcal{I} : L^X \times L_\perp \to L^X$, $L_\perp = L - \{\perp\}$ be a map. It satisfies $S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$ for all $\lambda,\mu \in L^X$ iff $\mathcal{I}(\alpha \odot \lambda, r) \ge \alpha \odot \mathcal{I}(\lambda,r)$ and $\mathcal{I}(\lambda,r) \le \mathcal{I}(\mu,r)$ if $\lambda \le \mu$.

Proof. If $\lambda \le \mu$, $\top = S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$, then $\mathcal{I}(\lambda,r) \le \mathcal{I}(\mu,r)$. Moreover, $S(\mathcal{I}(\lambda,r),\mathcal{I}(\alpha \odot \lambda,r)) \ge S(\lambda,\alpha \odot \lambda) \ge \alpha$. That is,

$$\alpha \odot \mathcal{I}(\lambda,r) \le \mathcal{I}(\alpha \odot \lambda, r).$$

On the other hand, put $\alpha = S(\lambda,\mu)$, then

$$S(\lambda,\mu) \odot \mathcal{I}(\lambda,r) \le \mathcal{I}(S(\lambda,\mu) \odot \lambda, r) \le \mathcal{I}(\mu,r).$$

Hence, $S(\lambda,\mu) \le S(\mathcal{I}(\lambda,r),\mathcal{I}(\mu,r))$.

Definition 2.10. A map $\mathcal{C} : L^X \times L_\perp \to L^X$ is called an L-fuzzy closure operator on X if $\mathcal{C}$ satisfies the following conditions

(C1) $\mathcal{C}(\perp_X, r) = \perp_X$,
(C2) $\mathcal{C}(\lambda,r) \ge \lambda$, or equivalently, $S(\lambda,\mathcal{C}(\lambda,r)) = \top_X$ for all $\lambda \in L^X$,
(C3) $S(\lambda,\mu) \le S(\mathcal{C}(\lambda,r),\mathcal{C}(\mu,r))$ for all $\lambda,\mu \in L^X$,
(C4) If $r \le s$, then $\mathcal{C}(\lambda,r) \le \mathcal{C}(\lambda,s)$,
(C5) $\mathcal{C}(\lambda \oplus \mu, r \odot s) \le \mathcal{C}(\lambda,r) \oplus \mathcal{C}(\mu,s)$.

The pair $(X,\mathcal{C})$ is called an L-fuzzy closure space. An L-fuzzy closure space $(X,\mathcal{C})$ is topological if
(T) $\mathcal{C}(\mathcal{C}(\lambda,r),r) = \mathcal{C}(\lambda,r) \ \forall\, \lambda \in L^X, r \in L_\perp$.

Let $(X,\mathcal{C}_X)$ and $(X,\mathcal{C}_Y)$ be two L-fuzzy closure spaces. A map $\phi : X \to Y$ is called a $\mathcal{C}$-map if $\phi^{\leftarrow}(\mathcal{C}_Y(\lambda,r)) \ge \mathcal{C}_X(\phi^{\leftarrow}(\lambda),r), \ \ \forall\, \lambda \in L^Y, r \in L_\perp$.

Lemma 2.11. Let $\mathcal{C} : L^X \times L_\perp \to L^X$, $L_\perp = L - \{\perp\}$ be a map. It satisfies $S(\lambda,\mu) \le S(\mathcal{C}(\lambda,r),\mathcal{C}(\mu,r))$ for all $\lambda,\mu \in L^X$ iff $\mathcal{C}(\alpha \odot \lambda, r) \ge \alpha \odot \mathcal{C}(\lambda,r)$ and $\mathcal{C}(\lambda,r) \le \mathcal{C}(\mu,r)$ if $\lambda \le \mu$.

3 L-fuzzy Interior Space Induced by L-fuzzy Topological Space

Theorem 3.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Define the mapping $\mathcal{I}_{\mathcal{T}} : L^X \times L_{\perp} \to L^X$ as follows

$$\mathcal{I}_{\mathcal{T}}(\lambda, r) = \bigvee_{\mu} \{\mu \odot S(\mu, \lambda) \mid \mathcal{T}(\mu) \geq r\}.$$

Then we have the following properties.

(1) $(X, \mathcal{I}_{\mathcal{T}})$ is an L-fuzzy interior space,

(2) If $(X, \mathcal{T})$ is enriched, then $(X, \mathcal{I}_{\mathcal{T}})$ is a strong L-fuzzy interior space,

(3) $\mathcal{I}_{\mathcal{T}}(\lambda, r) \leq \bigvee\{\mu \mid \mu \leq \lambda, \mathcal{T}(\mu) \geq r\}$,

(4) If $(X, \mathcal{T})$ is enriched, then the equality in (3) holds.

Proof. (1) (I1) For each $\mathcal{T}(\mu) \geq r$, $S(\top_X, \top_X) = \top$. Thus, $\mathcal{I}_{\mathcal{T}}(\top_X, r) \geq \top_X \odot \top = \top_X$. Therefore, $\mathcal{I}_{\mathcal{T}}(\top_X, r) = \top_X$.

(I2) By Lemma 2.4(7), we have $\mathcal{I}_{\mathcal{T}}(\lambda, r) = \bigvee_{\mu}\{\mu \odot S(\mu, \lambda) \mid \mathcal{T}(\mu) \geq r\} \leq \lambda$ for all $\lambda \in L^X$.

(I3) Using Lemma 2.2(8),(10), we can get

$$\begin{aligned}
S(\mathcal{I}_{\mathcal{T}}(\lambda, r), \mathcal{I}_{\mathcal{T}}(\mu, r)) &= \bigwedge_{x \in X} \big(\mathcal{I}_{\mathcal{T}}(\lambda, r)(x) \to \mathcal{I}_{\mathcal{T}}(\mu, r)(x)\big) \\
&= \bigwedge_{x \in X} \Big(\bigvee_{\mathcal{T}(\rho) \geq r} \rho(x) \odot S(\nu, \lambda) \to \bigvee_{\mathcal{T}(\rho) \geq r} \rho(x) \odot S(\rho, \mu)\Big) \\
&\geq \bigwedge_{x \in X} \bigwedge_{\mathcal{T}(\rho) \geq r} \big(\rho(x) \odot S(\rho, \lambda) \to \rho(x) \odot S(\rho, \mu)\big) \\
&\geq \bigwedge_{x \in X} \bigwedge_{\mathcal{T}(\rho) \geq r} (S(\rho, \lambda) \to S(\rho, \mu)) \geq S(\lambda, \mu).
\end{aligned}$$

(I4) If $r \leq s$, then

$$\mathcal{I}_{\mathcal{T}}(\lambda, s) = \bigvee_{\mathcal{T}(\mu) \geq s} \mu \odot S(\mu, \lambda) \leq \bigvee_{\mathcal{T}(\mu) \geq r} \mu \odot S(\mu, \lambda) = \mathcal{I}_{\mathcal{T}}(\lambda, r).$$

(I5) By Lemma 2.4(4), we have

$$\begin{aligned}
\mathcal{I}_{\mathcal{T}}(\lambda, r) \odot \mathcal{I}_{\mathcal{T}}(\mu, s) &= \bigvee_{\mathcal{T}(\rho_1) \geq r} \rho_1 \odot S(\rho_1, \lambda) \odot \bigvee_{\mathcal{T}(\rho_2) \geq s} \rho_2 \odot S(\rho_2, \mu) \\
&= \bigvee_{\mathcal{T}(\rho_1) \geq r} \bigvee_{\mathcal{T}(\rho_2) \geq s} (\rho_1 \odot \rho_2) \odot S(\rho_1, \lambda) \odot S(\rho_2, \mu) \\
&\leq \bigvee_{\mathcal{T}(\rho_1) \odot \mathcal{T}(\rho_2) \geq r \odot s} (\rho_1 \odot \rho_2) \odot S(\rho_1 \odot \rho_2, \lambda \odot \mu) \\
&= \mathcal{I}_{\mathcal{T}}(\lambda \odot \mu, r \odot s).
\end{aligned}$$

(2) Since $\mathcal{T}$ is enriched, $\mathcal{T}(\mathcal{I}_{\mathcal{T}}(\lambda, r)) \geq r$. Thus,

$$\begin{aligned}\mathcal{I}_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}(\lambda, r), r) &= \bigvee_{\mathcal{T}(\mu)\geq r} \mu \odot S(\mu, \mathcal{I}_{\mathcal{T}}(\lambda, r)) \\ &\geq \mathcal{I}_{\mathcal{T}}(\lambda, r) \odot S(\mathcal{I}_{\mathcal{T}}(\lambda, r), \mathcal{I}_{\mathcal{T}}(\lambda, r)) = \mathcal{I}_{\mathcal{T}}(\lambda, r).\end{aligned}$$

(3) For each $\mathcal{T}(\mu) \geq r$ with $\mu \leq \lambda$, we have $\mu = \top \odot \mu \leq S(\mu, \lambda) \odot \mu$, it follows that

$$\bigvee_{\mathcal{T}(\mu)\geq r} \{\mu \mid \mu \leq \lambda\} \leq \bigvee_{\mathcal{T}(\mu)\geq r} S(\mu, \lambda) \odot \mu = \mathcal{I}_{\mathcal{T}}(\lambda, r).$$

(4) For any $\mathcal{T}(\mu) \geq r$, $\mathcal{T}(S(\mu, \lambda) \odot \mu) \geq \mathcal{T}(\mu) \geq r$, because $\mathcal{T}$ is enriched. Thus, $\mathcal{I}_{\mathcal{T}}(\lambda, \mu) = \bigvee_{\mathcal{T}(\mu)\geq r} S(\mu, \lambda) \odot \mu \leq \bigvee_{\mathcal{T}(\mu)\geq r} \{\mu \mid \mu \leq \lambda\}$.

Theorem 3.2. Let $(X, \mathcal{I})$ be an L-fuzzy interior space. Define the mapping $\mathcal{T}_{\mathcal{I}} : L^X \to L$ by

$$\mathcal{T}_{\mathcal{I}}(\lambda) = \bigvee \{r \in L \mid S(\lambda, \mathcal{I}(\lambda, r)) = \top\}.$$

Then, $\mathcal{T}_{\mathcal{I}}$ is an enriched L-fuzzy topology on X.

Proof. (LO1) $\mathcal{T}_{\mathcal{I}}(\top_X) = \bigvee \{r \in L \mid S(\top_X, \mathcal{I}(\top_X, r)) = \top\}$, and
$\mathcal{T}_{\mathcal{I}}(\bot_X) = \bigvee \{r \in L \mid S(\bot_X, \mathcal{I}(\bot_X, r)) = \top\}$.

(LO2) By Lemma 2.4(4) and Definition 2.8(I5), we have

$$\begin{aligned}S(\lambda_1, \mathcal{I}(\lambda_1, r)) \odot S(\lambda_2, \mathcal{I}(\lambda_2, s)) &\leq S(\lambda_1 \odot \lambda_2, \mathcal{I}(\lambda_1, r) \odot \mathcal{I}(\lambda_2, s)) \\ &\leq S(\lambda_1 \odot \lambda_2, \mathcal{I}(\lambda_1 \odot \lambda_2, r \odot s)).\end{aligned}$$

If $S(\lambda_1, \mathcal{I}(\lambda_1, r)) = \top$ and $S(\lambda_2, \mathcal{I}(\lambda_2, s)) = \top$, then
$S(\lambda_1 \odot \lambda_2, \mathcal{I}(\lambda_1 \odot \lambda_2, r \odot s)) = \top$. Thus, $\mathcal{T}_{\mathcal{I}}(\lambda_1 \odot \lambda_2) \geq \mathcal{T}_{\mathcal{I}}(\lambda_1) \odot \mathcal{T}_{\mathcal{I}}(\lambda_2)$.

(LO3) For a family of $\{\lambda_i \mid i \in I\} \subseteq L^X$, we have

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(\bigvee_{i\in I} \lambda_i) &= \bigvee \{r \in L \mid S(\bigvee_{i\in I} \lambda_i, \mathcal{I}(\bigvee_{i\in I} \lambda_i, r)) = \top\} \\ &\geq \bigwedge_{i\in I} \bigvee \{r \in L \mid S(\lambda_i, \mathcal{I}(\bigvee_{i\in I} \lambda_i, r)) = \top\} \\ &\geq \bigwedge_{i\in I} \bigvee \{r \in L \mid S(\lambda_i, \mathcal{I}(\lambda_i, r)) = \top\} = \bigwedge_{i\in I} \mathcal{T}_{\mathcal{I}}(\lambda_i).\end{aligned}$$

Finally, for $\alpha \in L_{\bot}$ and $\lambda \in L^X$, we have

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(\alpha \odot \lambda) &= \bigvee \{r \in L \mid S(\alpha \odot \lambda, \mathcal{I}(\alpha \odot \lambda, r)) = \top\} \\ &\geq \bigvee \{r \in L \mid S(\alpha \odot \lambda, \alpha \odot \mathcal{I}(\lambda, r)) = \top\} \\ &\geq \bigvee \{r \in L \mid S(\lambda, \mathcal{I}(\lambda, r)) = \top\} = \mathcal{T}_{\mathcal{I}}(\lambda).\end{aligned}$$

Hence, $\mathcal{T}_{\mathcal{I}}$ is an enriched L-fuzzy topology on X.

Theorem 3.3. (1) If $(X,\mathcal{I})$ is an L-fuzzy interior space, then $\mathcal{I}_{\mathcal{T}_{\mathcal{I}}} \leq \mathcal{I}$.
(2) If $(X,\mathcal{T})$ is an L-fuzzy topological space, then $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}} \geq \mathcal{T}$.

Proof. (1) By Lemma 2.4(7), we have

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{\mathcal{I}}}(\lambda,r) &= \bigvee_{\mu}\{\mu \odot S(\mu,\lambda) \mid \mathcal{T}_{\mathcal{I}}(\mu) \geq r\}\\ &= \bigvee_{\mu}\{\mu \odot S(\mu,\lambda) \odot S(\lambda,\mathcal{I}(\lambda,r)) \mid \mathcal{T}_{\mathcal{I}}(\mu) \geq r\}\\ &\leq \bigvee_{\mu}\{\mu \odot S(\mu,\mathcal{I}(\lambda,r)) \mid \mathcal{T}_{\mathcal{I}}(\mu) \geq r\} \leq \mathcal{I}(\lambda,r).\end{aligned}$$

(2) Let $\mathcal{T}(\lambda) \geq r$. Then, $\mathcal{I}_{\mathcal{T}}(\lambda,r) = \lambda$. Thus, $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\lambda) \geq r$. Hence, $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}} \geq \mathcal{T}$.

Theorem 3.4. Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be two L-fuzzy topological spaces. If $\phi:(X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ is an LF-continuous map, then $\phi:(X,\mathcal{I}_{\mathcal{T}_X}) \to (Y,\mathcal{I}_{\mathcal{T}_Y})$ is an I-map.

Proof. By Lemma 2.5 and Definition 2.6, we have

$$\begin{aligned}\phi^{\leftarrow}(\mathcal{I}_{\mathcal{T}_Y}(\lambda,r)) &= \phi^{\leftarrow}\big(\bigvee_{\mu}\{\mu \odot S(\mu,\lambda) \mid \mathcal{T}_Y(\mu) \geq r\}\big)\\ &= \bigvee_{\phi^{\leftarrow}(\mu)}\{\phi^{\leftarrow}(\mu) \odot S(\mu,\lambda) \mid \mathcal{T}_Y(\mu) \geq r\}\\ &\leq \bigvee_{\phi^{\leftarrow}(\mu)}\{\phi^{\leftarrow}(\mu) \odot S(\phi^{\leftarrow}(\mu),\phi^{\leftarrow}(\lambda)) \mid \mathcal{T}_X(\phi^{\leftarrow}(\mu)) \geq r\}\\ &\leq \bigvee_{\rho}\{\rho \odot S(\rho,\phi^{\leftarrow}(\lambda)) \mid \mathcal{T}_X(\rho) \geq r\} = \mathcal{I}_{\mathcal{T}_X}(\phi^{\leftarrow}(\lambda),r).\end{aligned}$$

Theorem 3.5. Let $(X,\mathcal{I}_X)$ and $(Y,\mathcal{I}_Y)$ be two L-fuzzy interior spaces. If $\phi:(X,\mathcal{I}_X) \to (Y,\mathcal{I}_Y)$ is an I-map, then $\phi:(X,\mathcal{T}_{\mathcal{I}_X}) \to (Y,\mathcal{T}_{\mathcal{I}_Y})$ is LF-continuous.

Proof. From Theorem 3.4 and Lemma 2.5, we have

$$S(\phi^{\leftarrow}(\lambda),\mathcal{I}_X(\phi^{\leftarrow}(\lambda),r)) \geq S(\phi^{\leftarrow}(\lambda),\phi^{\leftarrow}(\mathcal{I}_Y(\lambda,r))) \geq S(\lambda,\mathcal{I}_Y(\lambda,r)).$$

So, $\mathcal{T}_{\mathcal{I}_X}(\phi^{\leftarrow}(\lambda)) \geq \mathcal{T}_{\mathcal{I}_Y}(\lambda)$.

4 L-fuzzy Closure Space Induced by L-fuzzy Co-topological Space

Theorem 4.1. Let $(X,\mathcal{F})$ be an L-fuzzy co-topological space. Define the mapping $\mathcal{C}_{\mathcal{F}}: L^X \times L_{\perp} \to L^X$ by

$$\mathcal{C}_{\mathcal{F}}(\lambda,r)(x) = \bigwedge_{\mathcal{F}(\mu)\geq r}\big(S(\lambda,\mu) \to \mu(x)\big).$$

Then we have the following properties.

(1) $(X, \mathcal{C}_{\mathcal{F}})$ is an L-fuzzy closure space,
(2) If $(X, \mathcal{F})$ is enriched, then $(X, \mathcal{C}_{\mathcal{F}})$ is a topological L-fuzzy closure space,
(3) $\mathcal{C}^*_{\mathcal{F}}(\lambda^*, r) = \mathcal{I}_{\mathcal{T}}(\lambda, r)$,
(4) $\mathcal{C}_{\mathcal{F}}(\lambda, r) \leq \bigwedge_{\mathcal{F}(\mu) \geq r}\{\mu \mid \lambda \leq \mu\}$,
(5) If $(X, \mathcal{F})$ is enriched, $\mathcal{C}_{\mathcal{F}}(\lambda, r) = \bigwedge_{\mathcal{F}(\mu) \geq r}\{\mu \mid \lambda \leq \mu\}$.

Proof. (1) (C1) By Lemma 2.4(7), we have

$$\mathcal{C}_{\mathcal{F}}(\perp_X, r)(x) = \bigwedge_{\mathcal{F}(\mu) \geq r} \big(S(\perp_X, \mu) \to \mu(x)\big) \geq \perp_X(x) = \perp.$$

(C2) By Lemma 2.2(11), we have

$$\begin{aligned} S(\lambda, \mathcal{C}_{\mathcal{F}}(\lambda, r)) &= \bigwedge_{x \in X} \big(\lambda(x) \to \mathcal{C}_{\mathcal{F}}(\lambda, r)(x)\big) \\ &= \bigwedge_{x \in X} (\lambda(x) \to \bigwedge_{\mathcal{F}(\mu) \geq r} \big(S(\lambda, \mu) \to \mu(x))\big) \\ &= \bigwedge_{x \in X} \bigwedge_{\mathcal{F}(\mu) \geq r} \big(\lambda(x) \to ((\bigwedge_{x \in X} \lambda(x) \to \mu(x)) \to \mu(x))\big) \\ &\geq \bigwedge_{x \in X} \bigwedge_{\mathcal{F}(\mu) \geq r} \big(\lambda(x) \to ((\lambda(x) \to \mu(x)) \to \mu(x))\big) \\ &= \bigwedge_{x \in X} \bigwedge_{\mathcal{F}(\mu) \geq r} \big((\lambda(x) \to \mu(x)) \to (\lambda(x) \to \mu(x))\big) = \top. \end{aligned}$$

(C3) By Lemma 2.2(10), we have

$$\begin{aligned} S(\mathcal{C}_{\mathcal{F}}(\lambda, r), \mathcal{C}_{\mathcal{F}}(\rho, r)) &= \bigwedge_{x \in X} \big(\mathcal{C}_{\mathcal{F}}(\lambda, r)(x) \to \mathcal{C}_{\mathcal{F}}(\rho, r)(x)\big) \\ &= \bigwedge_{x \in X} \big((\bigwedge_{\mathcal{F}(\mu) \geq r} S(\lambda, \mu) \to \mu(x)) \to (\bigwedge_{\mathcal{F}(\mu) \geq r} S(\rho, \mu) \to \mu(x))\big) \\ &\geq \bigwedge_{x \in X} \bigwedge_{\mathcal{F}(\mu) \geq r} \big((S(\lambda, \mu) \to \mu(x)) \to (S(\rho, \mu) \to \mu(x))\big) \\ &\geq \bigwedge_{\mathcal{F}(\mu) \geq r} \big(S(\rho, \mu) \to S(\lambda, \mu)\big) \geq S(\lambda, \rho). \end{aligned}$$

(C4) It follows from the definition of $\mathcal{C}_{\mathcal{F}}$.

(C5) By Lemma 2.4(5) and Lemma 2.2(15), we have

$$
\begin{aligned}
\mathcal{C}_{\mathcal{F}}(\lambda,r)(x)\oplus\mathcal{C}_{\mathcal{F}}(\rho,s)(x) &= \big(\bigwedge_{\mathcal{F}(\mu_1)\geq r} S(\lambda,\mu_1)\to\mu_1(x)\big)\oplus\big(\bigwedge_{\mathcal{F}(\mu_2)\geq s} S(\rho,\mu_2)\to\mu_2(x)\big)\\
&= \bigwedge_{\mathcal{F}(\mu_1)\geq r}\ \bigwedge_{\mathcal{F}(\mu_2)\geq s}\big((S(\lambda,\mu_1)\to\mu_1(x))\oplus(S(\rho,\mu_2)\to\mu_2(x))\big)\\
&\geq \bigwedge_{\mathcal{F}(\mu_1\oplus\mu_2)\geq r\odot s}\big((S(\lambda,\mu_1)\odot S(\rho,\mu_2))\to(\mu_1\oplus\mu_2)(x)\big)\\
&\geq \bigwedge_{\mathcal{F}(\mu_1\oplus\mu_2)\geq r\odot s}\big(S(\lambda\oplus\rho,\mu_1\oplus\mu_2)\to(\mu_1\oplus\mu_2)(x)\big)\\
&= \mathcal{C}_{\mathcal{F}}(\lambda\oplus\rho,r\odot s)(x).
\end{aligned}
$$

(2) Since $\mathcal{F}$ is enriched, then $\mathcal{F}(\mathcal{C}_{\mathcal{F}}(\lambda,r)\geq r$. Thus,

$$
\begin{aligned}
\mathcal{C}_{\mathcal{F}}(\mathcal{C}_{\mathcal{F}}(\lambda,r),r)(x) &= \bigwedge_{\mathcal{F}(\mu)\geq r}\big(S(\mathcal{C}_{\mathcal{F}}(\lambda,r),\mu)\to\mu(x)\big)\\
&\leq \bigwedge_{\mathcal{F}(\mathcal{C}_{\mathcal{F}}(\lambda,r))\geq r}\big(S(\mathcal{C}_{\mathcal{F}}(\lambda,r),\mathcal{C}_{\mathcal{F}}(\mu,r))\to\mathcal{C}_{\mathcal{F}}(\lambda,r)(x)\big)\\
&= \mathcal{C}_{\mathcal{F}}(\lambda,r)(x).
\end{aligned}
$$

(3)

$$
\begin{aligned}
\mathcal{C}^{*}_{\mathcal{F}}(\lambda^{*},r) &= \big\{\bigwedge_{\mathcal{F}(\mu^{*})\geq r}\big(S(\lambda^{*},\mu^{*})\to\mu^{*}\big)\big\}^{*}\\
&= \bigvee_{\mathcal{F}(\mu^{*})\geq r}\big(S(\lambda^{*},\mu^{*})\odot\mu\big) = \bigvee_{\mathcal{T}(\mu)\geq r}\mu\odot S(\mu,\lambda)=\mathcal{I}_{\mathcal{T}}(\lambda,r).
\end{aligned}
$$

(4) If $\mu\leq\lambda$, then $S(\lambda,\mu)=\top$ and $S(\lambda,\mu)\to\mu\leq\mu$. Thus,

$$
\bigwedge_{\mathcal{F}(\mu)\geq r}\big(S(\lambda,\mu)\to\mu\big)\leq\bigwedge_{\mathcal{F}(\mu)\geq r}\{\mu\mid\lambda\leq\mu\}.
$$

(5) For any $\mathcal{F}(\mu)\geq r$, $\mathcal{F}\big(S(\lambda,\mu)\to\mu\big)\geq\mathcal{F}(\mu)$, i.e., $\mathcal{F}\big(S(\lambda,\mu)\to\mu\big)\geq r$, because $\mathcal{F}$ is enriched. Thus,

$$
\bigwedge_{\mathcal{F}(\mu)\geq r}\big(S(\lambda,\mu)\to\mu\big)\geq\bigwedge_{\mathcal{F}(\mu)\geq r}\{\mu\mid\lambda\leq\mu\}.
$$

Theorem 4.2. If $\mathcal{C}:L^X\times L_{\perp}$ is an L-fuzzy closure operator. Define the mapping $\mathcal{F}_{\mathcal{C}}:L^X\to L$ by

$$
\mathcal{F}_{\mathcal{C}}(\lambda)=\bigvee\{r\in L\mid S(\mathcal{C}(\lambda,r),\lambda)=\top\}.
$$

Then, $\mathcal{F}_{\mathcal{C}}$ is an enriched L-fuzzy co-topology on X.

Proof. (LF1) $\mathcal{F}_{\mathcal{C}}(\top_X) = \bigvee\{r \in L \mid S(\mathcal{C}(\top_X, r), \top_X) = \top\}$ by (C2), and
$\mathcal{F}_{\mathcal{C}}(\bot_X) = \bigvee\{r \in L \mid S(\mathcal{C}(\bot_X, r), \bot_X) = \top\}$ by (C1).

(LF2) By Lemma 2.4(5) and (C4), we have

$$\begin{aligned} S(\mathcal{C}(\lambda_1, r), \lambda_1) \odot S(\mathcal{C}(\lambda_2, r), \lambda_2) &\leq S(\mathcal{C}(\lambda_1, r) \oplus \mathcal{C}(\lambda_2, r), \lambda_1 \oplus \lambda_2) \\ &\leq S(\mathcal{C}(\lambda_1 \oplus \lambda_2, r), \lambda_1 \oplus \lambda_2). \end{aligned}$$

If $S(\mathcal{C}(\lambda_1, r), \lambda_1) = \top$ and $S(\mathcal{C}(\lambda_2, r), \lambda_2) = \top$, then $S(\mathcal{C}(\lambda_1 \oplus \lambda_2, r), \lambda_1 \oplus \lambda_2) = \top$. Thus, $\mathcal{F}_{\mathcal{C}}(\lambda_1 \oplus \lambda_2) \geq \mathcal{F}_{\mathcal{C}}(\lambda_1) \odot \mathcal{F}_{\mathcal{C}}(\lambda_2)$.

(LF3) For a family of $\{\lambda_i \mid i \in I\} \subseteq L^X$, we have

$$\begin{aligned} \mathcal{F}_{\mathcal{C}}(\bigwedge_{i \in I} \lambda_i) &= \bigvee\{r \in L \mid S(\mathcal{C}(\bigwedge_{i \in I} \lambda_i, r), \bigwedge_{i \in I} \lambda_i) = \top\} \\ &\leq \bigvee_{i \in I} \bigvee\{r \in L \mid S(\bigwedge_{i \in I} \mathcal{C}(\lambda_i, r), \lambda_i) = \top\} \\ &\leq \bigvee_{i \in I} \bigvee\{r \in L \mid S(\mathcal{C}(\lambda_i, r), \lambda_i) = \top\} = \bigvee_{i \in I} \mathcal{F}_{\mathcal{C}}(\lambda_i). \end{aligned}$$

Hence, $\mathcal{F}_{\mathcal{C}}$ is an L-fuzzy co-topology on X. By Lemma 2.4(3), (6), we have

$$\begin{aligned} \mathcal{F}_{\mathcal{C}}(\alpha \to \lambda) &= \bigvee\{r \in L \mid S(\mathcal{C}(\alpha \to \lambda, r), \alpha \to \lambda) = \top\} \\ &= \bigvee\{r \in L \mid S(\alpha \odot \mathcal{C}(\alpha \to \lambda, r), \lambda) = \top\} \\ &\geq \bigvee\{r \in L \mid S(\mathcal{C}(\alpha \odot (\alpha \to \lambda), r), \lambda) = \top\} \\ &\geq \bigvee\{r \in L \mid S(\mathcal{C}(\lambda, r), \lambda) = \top\} = \mathcal{F}_{\mathcal{C}}(\lambda). \end{aligned}$$

Theorem 4.3. Let $(X, \mathcal{C}_{\mathcal{F}})$ be an L-fuzzy closure space, then $\mathcal{C}_{\mathcal{F}_{\mathcal{C}}} \geq \mathcal{C}$.

Proof. By Lemma 2.4(7), we have

$$\begin{aligned} \mathcal{C}_{\mathcal{F}_{\mathcal{C}}}(\lambda, r) = \bigwedge_{\mathcal{F}_{\mathcal{C}}(\mu) \geq r} \big(S(\lambda, \mu) \to \mu\big) &= \bigwedge_{\mathcal{F}_{\mathcal{C}}(\mu) \geq r} \big((S(\mathcal{C}(\lambda, r), \lambda) \odot S(\lambda, \mu)) \to \mu\big) \\ &\geq \bigwedge_{\mathcal{F}_{\mathcal{C}}(\mu) \geq r} \big(S(\mathcal{C}(\lambda, r), \mu) \to \mu\big) \geq \mathcal{C}(\lambda, r). \end{aligned}$$

Theorem 4.4. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two L-fuzzy co-topological spaces. If $\phi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is an LF-continuous map, then $\phi : (X, \mathcal{C}_{\mathcal{F}_X}) \to (Y, \mathcal{C}_{\mathcal{F}_Y})$ is a C-map.

Proof. By Lemma 2.11, we have

$$\begin{aligned} \phi^{\leftarrow}(\mathcal{C}_{\mathcal{F}_Y}(\lambda, r)) = \phi^{\leftarrow}\big(\bigwedge_{\mathcal{F}_Y(\mu) \geq r} (S(\lambda, \mu) \to \mu)\big) &= \bigwedge_{\mathcal{F}_Y(\mu) \geq r} (S(\lambda, \mu) \to \phi^{\leftarrow}(\mu)) \\ &\geq \bigwedge_{\mathcal{F}_X(\phi^{\leftarrow}(\mu)) \geq r} \big(S(\phi^{\leftarrow}(\lambda), \phi^{\leftarrow}(\mu)) \to \phi^{\leftarrow}(\mu)\big) = \mathcal{C}_{\mathcal{F}_X}(\phi^{\leftarrow}(\lambda), r). \end{aligned}$$

Theorem 4.5. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be two L-fuzzy closure spaces. If $\phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is a C-map, then $\phi : (X, \mathcal{F}_{\mathcal{C}_X}) \to (Y, \mathcal{F}_{\mathcal{C}_Y})$ is LF-continuous.

Proof. From Theorem 4.3, we have

$$\begin{aligned}
\mathcal{F}_{\mathcal{C}_X}(\phi^{\leftarrow}(\lambda)) &= \bigvee\{r \in L \mid S(\mathcal{C}_X(\phi^{\leftarrow}(\lambda), r), \phi^{\leftarrow}(\lambda)) = \top\} \\
&\geq \bigvee\{r \in L \mid S(\phi^{\leftarrow}(\mathcal{C}_Y(\lambda, r)), \phi^{\leftarrow}(\lambda)) = \top\} \\
&= \bigvee\{r \in L \mid \bigwedge_{x \in X} \big(\mathcal{C}_Y(\lambda, r)(\phi(x)) \to \lambda(\phi(x))\big) = \top\} \\
&\geq \bigvee\{r \in L \mid \bigwedge_{y \in Y} \big(\mathcal{C}_Y(\lambda, r)(y) \to \lambda(y)\big) = \top\} \\
&= \bigvee\{r \in L \mid S(\mathcal{C}_Y(\lambda, r), \lambda) = \top\} = \mathcal{F}_{\mathcal{C}_Y}(\lambda).
\end{aligned}$$

Example 4.6. Let $(L = [0, 1], \odot, \to, *)$ be a complete residuated lattice defined as

$$x \odot y = (x + y - 1) \vee 0, \;\; x \to y = (1 - x + y) \wedge 1, \;\; x* = 1 - x.$$

Let $X = \{x, y, z\}$ be a set and let $\mu \in [0, 1]^X$ be a fuzzy set as follow

$$\mu(x) = 0.5, \quad \mu(y) = 0.3, \quad \mu(z) = 0.6.$$

We define the $[0, 1]$-fuzzy topology $\mathcal{T} : [0, 1]^X \to [0, 1]$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bot_X \; or \; \top_X, \\ 0.3, & \text{if } \lambda = \mu \odot \mu, \\ 0.6, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Also, we define the $[0, 1]$-fuzzy co-topology $\mathcal{F} : [0, 1]^X \to [0, 1]$ as follows

$$\mathcal{F}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bot_X \; or \; \top_X, \\ 0.2, & \text{if } \lambda = \mu \oplus \mu, \\ 0.6, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

(1) By Theorem 3.1, we have $\mathcal{I}_{\mathcal{T}} : [0, 1]^X \times (0, 1] \to [0, 1]^X$ as a $[0, 1]$-fuzzy interior space as follows

$$\mathcal{I}_{\mathcal{T}}(\lambda, r) = \begin{cases} (\bigwedge \lambda(x)), & \text{if } r > 0.6, \\ (\bigwedge \lambda(x)) \vee (\mu \odot S(\mu, \lambda)), & \text{if } 0.3 < r \leq 0.6, \\ (\bigwedge \lambda(x)) \vee (\mu \odot S(\mu, \lambda)), & \text{if } 0 < r \leq 0.3, \\ \vee(\mu \odot \mu \odot S(\mu \odot \mu, \lambda)). & \end{cases}$$

For $\lambda = (0.1, 0, 2, 0, 3)$, we have

$$\mathcal{I}_{\mathcal{T}}(\lambda, 0.5) = (\bigwedge \lambda(x)) \vee (\mu \odot S(\mu, \lambda)) = (0.1, 0.1, 0.2).$$

Since $\mathcal{I}_{\mathcal{T}}((0.1, 0.1, 0.2), r) = (0.1, 0.1, 0.2)$ for $0 < r \le 0.6$, then we have

$$\mathcal{T}(\mathcal{I}_{\mathcal{T}}(0.1, 0.1, 0.2)) = 0.6.$$

(2) By Theorem 4.1, we have $\mathcal{C}_{\mathcal{F}} : [0,1]^X \times (0,1] \to [0,1]^X$ as a $[0,1]$-fuzzy closure space as follows

$$\mathcal{C}_{\mathcal{F}}(\lambda, r) = \begin{cases} \bigvee_{x\in X} \lambda(x), & \text{if } r > 0.6, \\ (\bigvee \lambda(x)) \wedge (S(\lambda,\mu) \to \mu), & \text{if } 0.3 < r \le 0.6, \\ (\bigvee \lambda(x)) \wedge (S(\lambda,\mu) \to \mu), & \text{if } 0 < r \le 0.3, \\ \wedge(S(\lambda, \mu \oplus \mu) \to \mu \oplus \mu), & \end{cases}$$

because $S(\lambda, 0) \to 0 = \bigwedge_{x\in X}(\lambda^*(x)) \to 0 = \bigvee_{x\in X} \lambda(x)$.

For $\lambda = (0.7, 0, 6, 0, 8)$, $\mathcal{C}_{\mathcal{F}}(\lambda, 0.5) = (\bigvee \lambda(x)) \wedge (S(\lambda,\mu) \to \mu) = (0.8, 0.8, 0.9)$. Since $(0.9, 0.8, 0.9) = \mathcal{C}_{\mathcal{F}}(\mathcal{C}_{\mathcal{F}}(\lambda, 0.5), 0.5) \ne \mathcal{C}_{\mathcal{F}}(\lambda, 0.5) = (0.8, 0.8, 0.9)$.

5 Conclusion

In this paper, we managed to deduce a new form of an L-fuzzy interior space (L-fuzzy closure space) through an L-fuzzy topological space (L-fuzzy co-topological space) and vise versa in a complete residuated lattice. We gave an example on [0,1] interval and finally we proved that the continuity property is compatible with the introduced spaces.

References

[1] Bandler W., Kohout L., *Special properties, closures and interiors of crisp and fuzzy relations*, Fuzzy sets and Systems 26(3) (1988) 317-331.

[2] Bělohlávek R., *Fuzzy closure operators I*, J. Math. Anal. Appl. 262 (2001) 473-489.

[3] Bělohlávek R., *Fuzzy closure operators II*, Soft Comput. 7(1) (2002) 53-64.

[4] Bělohlávek R., *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York 156 (2002) 369.

[5] Bodenhofer U., De Cock M., Kerre E. E., *Openings and closures of fuzzy preorderings:theoretical basics and applications to fuzzy rule-based systems*, International Journal of General Systems 32(4) (2003) 343-360.

[6] Chang C.L., *Fuzzy topological spaces*, J.Math.Anal.Appl. 24 (1968) 182-190.

[7] Dubois D., Prade H., *"Putting rough sets and fuzzy sets together". In : Slowinsiki, R. ed., Intelligent Decision Support*, Handbook of Applications and Advances of the Rough Set Theory (Kluwer, Dordecht) 996 (1990) 203-232.

[8] Fang J., *I-fuzzy Alexandrov topologies and specialization orders*, Fuzzy Sets and Systems 161 (2007) 2359-2374.

[9] Fang J., Yue Y., *L-fuzzy closure systems*, Fuzzy Sets and Systems 161 (2010) 1242-1252.

[10] Fang J., *The relationship between L-ordered convergence structures and strong L-topologies*, Fuzzy Sets and Systems 161 (2010) 2923-2944.

[11] Gerla G., *Fuzzy Logic Mathematical Tools for Approximate Reasoning*, Kluwer, Dordecht (2001).

[12] Gutierrez Garcia J., Mardones Perez I., Burton M. H., *The relationship between various filter notions on a GL-monoid*, J. Math. Anal. Appl. 230 (1999) 291-302.

[13] Hajek P., *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998).

[14] Hohle U., *Upper semi continuous fuzzy sets and applications*, J.Math.Anal.Appl. 78 (1980) 659-673.

[15] Hohle U., Rodabaugh S. E., *Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory*, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht 3 (1999) 273-388.

[16] Kotze W., *Uniform spaces*, in: Hohle U., Rodabaugh S. E.(Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Handbook Series, Kluwer Academic Publishers, Boston, Dordrecht, London, Chapter 8, 3 (1999) 553-580.

[17] Kubiak T., *On fuzzy topologies*, Ph.D. Thesis, Adam Mickiewicz Uniformity, Poznan, Poland (1985).

[18] Lai H., Zhang D., *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems 157 (2006) 1865-1885.

[19] Mashour A. S., Ghanim M. H., *Fuzzy closure spaces*, J.Math. Anal. Appl. 106 (1985) 154-170.

[20] Radzikowska A. M., Kerre E. E., *A comparative study of fuzzy rough sets*, Fuzzy Sets and Systems 126 (2002) 137-155.

[21] Ramadan A. A., *Smooth topological Spaces*, Fuzzy Sets and Systems 48(3) (1992) 371-357.

[22] Ramadan A. A., *L-fuzzy interior systems*, Comp. and Math. with Appl. 62 (2011) 4301-4307.

[23] Rodabaugh S. E., *Categorial foundations of variable-basis fuzzy topology*, In: Hohle U., Rodabaugh S. E.(Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, Handbook series, Kluwer Academic Publishers, Chapter 4 (1999).

[24] Rodabaugh S. E., Klement E. P., *Topological and Algebraic Structures In Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London (2003) 467 pp.

[25] Sostak A., *On a fuzzy topological structure*, Suppl. Rend Circ. Matem. Palermo, Ser. II [11] (1985) 125-186.

[26] Ying M.S., *A new approach to fuzzy topology*, Part I, Fuzzy Sets Syst. 39 (1991) 303-321.

[27] Ying M.S., *A new approach to fuzzy topology*, Part II, Fuzzy Sets Syst. 47 (1992) 221-232.