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SOME RESULTS ON SEMI OPEN SETS IN FUZZIFYING BITOPOLOGICAL SPACES

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Abstaract − In the present paper, we introduce and study the concepts of $(i, j)-$semi open set and $(i, j)-$semi neighborhood system in fuzzifying bitopological spaces. Also, the concepts of $(i, j)-$semi derived set and $(i, j)-$semi closure, $(i, j)-$semi interior, $(i, j)-$semi exterior, $(i, j)-$semi boundary operators in fuzzifying bitopological spaces are introduced and studied. Furthermore, we introduce and study the concepts of $(i, j)-$semi convergence of nets and $(i, j)-$semi convergence of filters in fuzzifying bitopological spaces.

Keywords − *Semiopen sets, Fuzzifying topology, fuzzifying bitopological space.*

1 Introduction

In 1965 [13], Zadeh introduced the fundamental concept of fuzzy sets which to formed the backbone of fuzzy mathematics. Since Chang introduced fuzzy sets theory into topology in 1968 [1]. Wong, Lowen, Hutton, Pu and Liu, etc., discussed respectively various aspects of fuzzy topology [3, 7, 8].

In 1991-1993 [10, 11, 12], Ying introduced the concept of the fuzzifying topology with the sematic method of continuous valued logic. In 1999 Khedr et al. [6] introduced the concept of semiopen sets and semicontinuity in fuzzifying topology.

The study of bitopological spaces was first initiated by Kelley [5] in 1963. In 2003 Zhang et al. [14], studied the concept of fuzzy $\theta_{i,j}$-closed, $\theta_{i,j}$-open sets in fuzzifying bitopological spaces. Also in [2], Gowrisankar et al. studied the concepts of $(i, j)-$pre open sets in fuzzifying bitopological spaces.

The contains of this paper are arranged as follows: In section (3) we introduce the concepts of $(i, j)-$semiopen sets in fuzzifying bitopological spaces. In section (4) we introduce and study the concepts of $(i, j)-$semi neighborhood system in fuzzifying bitopological spaces. In section (5) we introduce and study the concepts of

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$(i,j)-$semi derived sets and $(i,j)-$semi closure operator in fuzzifying bitopological space. In section (6) we introduce and study the concepts of $(i,j)-$semi interior and $(i,j)-$semi exterior, and $(i,j)-$semi boundary operators in fuzzifying bitopological spaces. In section (7) we introduce and study $(i,j)-$semi convergence of nets in fuzzifying bitopological spaces. Finally in section (8) we study $(i,j)-$semi convergence of filters in fuzzifying bitopological spaces.

2 Preliminary

Firstly, we display the fuzzy logical and corresponding set-theoretical notations used in this paper.

For formula φ, the symbol $[\varphi]$ means the truth of φ, where the set of truth values is the unit interval $[0,1]$. A formula φ is valid, we write $\models \varphi$ if and only if $[\varphi] = 1$ for every interpretation.

(1) $[\alpha] := \alpha$ $(\alpha \in [0,\ 1])$; $[\alpha \wedge \beta] = \min([\alpha],[\beta])$; $[\alpha \to \beta] = \min(1, 1 - [\alpha] + [\beta])$, $[\forall x\ \alpha(x)] = \inf\limits_{x\in X}[\alpha(x)]$, where X is the universe of discourse.

(2) If $\widetilde{A} \in \Im(X)$, where $\Im(X)$ is the family of fuzzy sets of X, then $[x \in \widetilde{A}] := \widetilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x\ \alpha(x)] = \inf\limits_{x\in X}[\alpha(x)]$.

In addition, the following derived formulae are given:

(1) $[\neg\alpha] := [\alpha \to 0] = 1 - [\alpha]$.

(2) $[\alpha \vee \beta] := [\neg(\neg\alpha \wedge \neg\beta)] = \max([\alpha],[\beta])$.

(3) $[\alpha \leftrightarrow \beta] := [(\alpha \to \beta) \wedge (\beta \to \alpha)]$.

(4) $[\alpha \dot{\wedge} \beta] := [\neg(\alpha \to \neg\beta)] = \max(0, [\alpha] + [\beta] - 1)$.

(5) $[\alpha \dot{\vee} \beta] := [\neg\alpha \to \beta] = \min(1, [\alpha] + [\beta])$.

(6) $[\exists x\ \alpha(x)] := [\neg(\forall x\ \neg\alpha(x))]$.

(7) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then

(a) $[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \to x \in \widetilde{B})] = \inf\limits_{x\in X}\min(1, 1 - \widetilde{A}(x) + \widetilde{B}(x))$;

(b) $[A \equiv B] := [(\widetilde{A} \subseteq \widetilde{B}) \wedge (\widetilde{B} \subseteq \widetilde{A})]$.

Secondly, we give the following definitions which are used in the sequel.

Definition 2.1. [10] Let X be a universe of discourse, $P(X)$ is the family of subsets of X and $\tau \in \Im(P(X))$ satisfy the following conditions:
(1) $\tau(X) = 1$ and $\tau(\phi) = 1$;
(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau(\bigcup\limits_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge\limits_{\lambda\in\Lambda} \tau(A_\lambda)$.

Then τ is a fuzzifying topology and (X,τ) a fuzzifying topological space.

Definition 2.2. [10] The family of fuzzifying closed sets is denoted by $F \in \Im(P(X))$, and defined as $A \in F := X \sim A \in \tau$,where $X \sim A$ is the complement of A.

Definition 2.3. [10] Let $x \in X$. The neighborhood system of x is denoted by $N_x \in \Im(P(X))$ and defined as $N_x(A) = \sup_{x \in B \subseteq A} \tau(B)$.

Definition 2.4. [10] The closure $cl(A)$ of A is defined as $cl(A)(x) = 1 - N_x(X \sim A)$. In Theorem 5.3 [10], M.S. Ying proved that the closure $cl : P(X) \to \Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [10]) since its extension
$cl : \Im(X) \to \Im(X)$, $cl(\widetilde{A}) = \bigcup_{\alpha \in [0,1]} \alpha cl(\widetilde{A}_\alpha)$, $\widetilde{A} \in \Im(X)$ satisfies the following Kuratowski closure axioms:
(1) $\models cl(\phi) \equiv \phi$;
(2) for any $\widetilde{A} \in \Im(X)$, $\models \widetilde{A} \subseteq cl(\widetilde{A})$;
(3) for any $\widetilde{A}$, $\widetilde{B} \in \Im(X)$, $\models cl(\widetilde{A} \cup \widetilde{B}) = cl(\widetilde{A}) \cup cl(\widetilde{B})$;
(4) for any $\widetilde{A} \in \Im(X)$, $\models cl(cl(\widetilde{A})) \subseteq cl(\widetilde{A})$.
Where $\widetilde{A}_\alpha = \{x : \widetilde{A}(x) \geq \alpha\}$ is the α-cut of $\widetilde{A}$ and $\alpha\widetilde{A}(x) = \alpha \wedge \widetilde{A}(x)$.

Definition 2.5. [11] For any $A \in P(X)$, the interior of A is denoted by $int(A) \in \Im(P(X))$ and defined as follows: $int(A)(x) = N_x(A)$.

Lemma 2.6. [6] Let (X, τ) be a fuzzifying topological space.If $[A \subseteq B] = 1$. Then (1) $\models int(A) \subseteq int(B)$; (2) $\models cl(A) \subseteq cl(B)$.

Definition 2.7. [14] Let (X, τ_1) and (X, τ_2) be two fuzzifying topological spaces. Then a system (X, τ_1, τ_2) consisting of a universe of discourse X with two fuzzifying topologies τ_1 and τ_2 on X is called a fuzzifying bitopological space.

3 (i,j)-semiopen Sets in Fuzzifying Bitopological Spaces

Definition 3.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) The family of fuzzifying $(i, j)-$semiopen sets, denoted by $s\tau_{(i,j)} \in \Im(P(X))$, is defined as follows:

$$A \in s\tau_{(i,j)} := \forall x\big(x \in A \to x \in cl_j(int_i(A))\big)$$

i.e., $s\tau_{(i,j)}(A) = \inf_{x \in A} cl_j\big(int_i(A)\big)(x)$.
(2) The family of fuzzifying $(i, j)-$semiclosed sets, denoted by $sF_{(i,j)} \in \Im(P(X))$, is defined as follows:

$$A \in sF_{(i,j)} := X \sim A \in s\tau_{(i,j)}$$

Lemma 3.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.If $[A \subseteq B] = 1$, then $\models cl_j(int_i(A)) \subseteq cl_j(int_i(B))$.

Proof. It is obtained from Lemma (2.6) (1) and (2) .

Lemma 3.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$. Then
(1) $\models X \sim \big(cl_j(int_i(A))\big) \equiv int_j(cl_i(X \sim A))$;
(2) $\models X \sim \big(int_j(cl_i(A))\big) \equiv cl_j(int_i(X \sim A))$.

Proof. From Theorem 2.2 (5) in [11], we have
(1) $\Big(X \sim \big(cl_j(int_i(A))\big)\Big)(x) = \big(int_j(X \sim int_i(A))\big)(x) = \big(int_j(cl_i(X \sim A))\big)(x)$.
(2) $\Big(X \sim \big(int_j(cl_i(A))\big)\Big)(x) = \big(cl_j(X \sim cl_i(A))\big)(x) = \big(cl_j(int_i(X \sim A))\big)(x)$.

Theorem 3.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $s\tau_{(i,j)}(X) = 1$, $s\tau_{(i,j)}(\phi) = 1$;
(2) For any $\{A_\lambda : \lambda \in \Lambda\}$, $s\tau_{(i,j)}(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} s\tau_{(i,j)}(A_\lambda)$.

Proof. The proof of (1) is straightforward.
(2) From Lemma (3.2), we have $\models cl_j(int_i(A_\lambda)) \subseteq cl_j(int_i(\bigcup_{\lambda\in\Lambda} A_\lambda))$. So

$$\begin{aligned} s\tau_{(i,j)}(\bigcup_{\lambda\in\Lambda} A_\lambda) &= \inf_{x\in(\bigcup_{\lambda\in\Lambda} A_\lambda)} cl_j(int_i(\bigcup_{\lambda\in\Lambda} A_\lambda))(x) \\ &= \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} cl_j(int_i(\bigcup_{\lambda\in\Lambda} A_\lambda))(x) \\ &\geq \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} cl_j(int_i(A_\lambda))(x) = \bigwedge_{\lambda\in\Lambda} s\tau_{(i,j)}(A_\lambda)(x). \end{aligned}$$

Theorem 3.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $sF_{(i,j)}(X) = 1$, $sF_{(i,j)}(\phi) = 1$;
(2) For any $\{A_\lambda : \lambda \in \Lambda\}$, $sF_{(i,j)}(\bigcap_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} sF_{(i,j)}(A_\lambda)$.

Proof. From Theorem (3.4) the proof is obtained.

Lemma 3.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $\models \tau_i \subseteq s\tau_{(i,j)}$; (2)$\models F_i \subseteq sF_{(i,j)}$.

Proof. (1) From Theorem 2.2 (3) in [11], we have

$$\begin{aligned} [A \in \tau_i] &= [A \equiv int_i(A)] \\ &= [A \subseteq int_i(A)] \wedge [int_i(A) \subseteq A] \\ &= [A \subseteq int_i(A)] \leq [A \subseteq cl_j(int_i(A))] = [A \in s\tau_{(i,j)}]. \end{aligned}$$

(2) From (1) above the proof is obtained.

Remark 3.7. The following example shows that
(1) $s\tau_i \subseteq s\tau_{(i,j)}$, (2) $s\tau_j \subseteq s\tau_{(i,j)}$, (3) $\tau_j \subseteq s\tau_{(i,j)}$ and (4) $s\tau_{(i,j)} \subseteq s\tau_{(j,i)}$ may not be true for any (X, τ_1, τ_2) fuzzifying bitopological space.

Example 3.8. Let $X = \{a, b, c\}$,$A = \{a, b\}$ and τ_1, τ_2 be two fuzzifying topologies on X defined as follow:

$$\tau_1(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X, \{a\}, \{a, c\}\}, \\ 1/4 & \text{if} \quad A \in \{\{c\}, \{b, c\}\}, \\ 0 & \text{if} \quad A \in \{\{b\}, \{a, b\}\}. \end{cases}$$

$$\tau_2(A) = \begin{cases} 1 & \text{if} \quad A \in \{\phi, X, \{b\}, \{a, c\}\}, \\ 1/4 & \text{if} \quad A \in \{\{a\}, \{a, b\}\}, \\ 0 & \text{if} \quad A \in \{\{c\}, \{b, c\}\}. \end{cases}$$

We have $int_1(A)(a) = 1$, $int_1(A)(b) = int_1(A)(c) = 0$,
$cl_1(int_1(A))(a) = 1$, $cl_1(int_1(A))(b) = cl_1(int_1(A))(c) = 3/4$;
$s\tau_1(A) = 3/4$ and $int_2(A)(a) = 1/4$, $int_2(A)(b) = 1$, $int_2(A)(c) = 0$,
$cl_2(int_2(A))(a) = cl_2(int_2(A))(c) = 1/4$, $cl_2(int_2(A))(b) = 1$; $s\tau_2(A) = 1/4$.
So $cl_2(int_1(A))(a) = cl_2(int_1(A))(c) = 1$, $cl_2(int_1(A))(b) = 0$, $s\tau_{(1,2)}(A) = 0$. Also $cl_1(int_2(A))(a) = 1/4 = cl_1(int_2(A))(c)$, $cl_1(int_2(A))(b) = 1$; $s\tau_{(2,1)}(A) = 1/4$.
Therefore $s\tau_2 \nsubseteq s\tau_{(1,2)}$, $s\tau_1 \nsubseteq s\tau_{(1,2)}$, $\tau_2 \nsubseteq s\tau_{(1,2)}$ and $s\tau_{(2,1)} \nsubseteq s\tau_{(1,2)}$.

Theorem 3.9. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $\models cl_j(A) \equiv cl_j(int_i(A)) \longleftrightarrow A \in s\tau_{(i,j)}$;
(2) $\models int_j(A) \equiv int_j(cl_i(A)) \longleftrightarrow A \in sF(i,j)$.

Proof. (1) $[cl_j(A) \equiv cl_j(int_i(A))] = [cl_j(A) \subseteq cl_j(int_i(A))] \wedge [cl_j(int_i(A)) \subseteq cl_j(A)]$.
We know that $[int_i(A) \subseteq A] = 1$, so $[cl_j(int_i(A)) \subseteq cl_j(A)] = 1$. Then
$[cl_j(A) \equiv cl_j(int_i(A))] = [cl_j(A) \subseteq cl_j(int_i(A))] \leq [A \subseteq cl_j(int_i(A))] = [A \in s\tau_{(i,j)}]$.
Conversely, $[A \in s\tau_{(i,j)}] = [A \subseteq cl_j(int_i(A))] \leq [cl_j(A) \subseteq cl_j(cl_j(int_i(A)))]$.
From Definition (2.4) (4), we have $[cl_j(cl_j(int_i(A)))] \subseteq cl_j(int_i(A))] = 1$. Therefore

$$\begin{aligned}[A \in s\tau_{(i,j)}] &\leq [cl_j(A) \subseteq cl_j(int_i(A))] \\ &= [cl_j(A) \subseteq cl_j(int_i(A))] \wedge [cl_j(int_i(A)) \subseteq cl_j(A)] \\ &= [cl_j(A) \equiv cl_j(int_i(A))].\end{aligned}$$

(2) From (1) above and Lemma (3.3) (2), the proof is obtained.

Theorem 3.10. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then
(1) $\models A \in s\tau_{(i,j)} \leftrightarrow \forall x(x \in A \rightarrow \exists B(B \in s\tau_{(i,j)} \wedge x \in B \subseteq A))$;
(2) $\models A \in sF_{(i,j)} \leftrightarrow \forall x(x \in int_j(cl_i(A)) \rightarrow x \in A)$.

Proof. (1) $[\forall x(x \in A \rightarrow \exists B(B \in s\tau_{(i,j)} \wedge x \in B \subseteq A))] = \inf_{x \in A} \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B)$.
First, we have $\inf_{x \in A} \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B) \geq s\tau_{(i,j)}(A)$.
In the other hand, let $\beta_x = \{B : x \in B \subseteq A\}$.Then for any $f \in \prod_{x \in A} \beta_x$, we have $\bigcup_{x \in A} f(x) = A$, $s\tau_{(i,j)}(A) = s\tau_{(i,j)}(\bigcup_{x \in A} f(x)) \geq \inf_{x \in A} s\tau_{(i,j)}(f(x))$, and so

$$s\tau_{(i,j)}(A) \geq \sup_{f \in \prod_{x \in A} \beta_x} \inf_{x \in A} s\tau_{(i,j)}(f(x)) = \inf_{x \in A} \sup_{f \in \prod_{x \in A} \beta_x} s\tau_{(i,j)}(f(x)) = \inf_{x \in A} \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B).$$

(2) From Lemma (3.3) (2), we have

$$\begin{aligned}[\forall x(x \in int_j(cl_i(A)) \rightarrow x \in A)] &= [\forall x(x \in X \sim A \rightarrow x \in X \sim int_j(cl_i(A)))] \\ &= \inf_{x \in X \sim A} \big(X \sim int_j(cl_i(A))\big)(x) \\ &= \inf_{x \in X \sim A} \big(cl_j(int_i(X \sim A))\big)(x) \\ &= [X \sim A \in s\tau_{(i,j)}] = [A \in sF_{(i,j)}].\end{aligned}$$

Lemma 3.11. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.Then
(1) $\models B \doteqdot int_i(A) \longrightarrow B \subseteq A$;
(2) $\models B \doteqdot int_i(A) \wedge A \in s\tau_{(i,j)} \longrightarrow A \subseteq cl_j(B)$.

Proof. (1) $[B\dot{\equiv}int_i(A)] = [(B \subseteq int_i(A))\dot{\wedge}(int_i(A) \subseteq B)]$. If $[B \subseteq A] = 0$, then $[B \subseteq int_i(A)] = 0$. Therefor $[B\dot{\equiv}int_i(A)] = 0$.

$$\begin{aligned}(2)[(B\dot{\equiv}int_i(A)) \wedge A \in s\tau_{(i,j)}] &= [(B\dot{\equiv}int_i(A)) \wedge A \subseteq cl_j(int_i(A)] \\ &\leq [(int_i(A) \subseteq B) \wedge (A \subseteq cl_j(int_i(A)))] \\ &\leq [(cl_j(int_i(A)) \subseteq cl_j(B)) \wedge (A \subseteq cl_j(int_i(A)))] \\ &\leq [A \subseteq cl_j(B)].\end{aligned}$$

Theorem 3.12. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.Then
(1) $\models \exists U(U \in \tau_i \wedge U \subseteq A \subseteq cl_j(U)) \longrightarrow A \in s\tau_{(i,j)}$;
(2) $\models \exists V(V \in F_i \wedge int_j(V) \subseteq A \subseteq V) \longrightarrow A \in sF_{(i,j)}$.

Proof. (1) From Theorem 2.2 (3) [11],we have

$$\begin{aligned}[\exists U(U \in \tau_i \wedge U \subseteq A \subseteq cl_j(U))] &= \sup_{U\in P(X)}\big([U \in \tau_i] \wedge [U \subseteq A] \wedge [A \subseteq cl_j(U)]\big) \\ &= \sup_{U\subseteq A}\big([U \subseteq int_i(U)] \wedge [U \subseteq A] \wedge [A \subseteq cl_j(U)]\big) \\ &\leq \sup_{U\subseteq A}\big([U \subseteq int_i(U)] \wedge [int_i(U) \subseteq int_i(A)] \wedge [A \subseteq cl_j(U)]\big) \\ &\leq \sup_{U\subseteq A}\big([U \subseteq int_i(A)] \wedge [A \subseteq cl_j(U)]\big) \\ &\leq \sup_{U\subseteq A}\big([cl_j(U) \subseteq cl_j(int_i(A))] \wedge [A \subseteq cl_j(U))]\big) \\ &\leq \sup_{U\subseteq A}[A \subseteq cl_j(int_i(A))] = [A \in s\tau_{(i,j)}].\end{aligned}$$

(2) From (1) above and Theorem (2.2) (5) in [11], we have

$$\begin{aligned}[A \in sF_{(i,j)}] &= [X \sim A \in s\tau_{(i,j)}] \\ &\geq \big[\exists U(U \in \tau_i \wedge U \subseteq X \sim A \subseteq cl_j(U))\big] \\ &= \big[\exists U\big(U \in \tau_i \wedge X \sim cl_j(U) \subseteq A \subseteq X \sim U\big)\big] \\ &= \big[\exists U\big(U \in \tau_i \wedge int_j(X \sim U) \subseteq A \subseteq X \sim U\big)\big] \\ &= \big[\exists V\big(V \in F_i \wedge int_j(V) \subseteq A \subseteq V\big)\big].\end{aligned}$$

Remark 3.13. The proof of the inverse direction of Theorem (3.12) can be obtained by assuming that $[U\dot{\equiv}int_i(A)] = 1$, but the following example shows that even without the proposed requirement the proof is true. So the proof may be can obtained without the proposed requirement.

Example 3.14. From Example (3.8), $A = \{a, b\}$, $s\tau_{(2,1)}(A) = 1/4$ and
$int_2(A)(a) = 1/4$, $int_2(A)(b) = 1$, $int_2(A)(c) = 0$.
The family of all subsets of A is $\big\{\{a\}, \{b\}, \{a, b\}\big\}$ and $cl_1(\{a\})(a) = 1$,
$cl_1(\{a\})(b) = 3/4$, $cl_1(\{a\})(c) = 3/4$. Then $[A \subseteq cl_1(\{a\})] = \inf_{x\in A} cl_1(\{a\})(x) = 3/4$.
So $[\tau_2(\{a\}) \wedge A \subseteq cl_1(\{a\})] = \min(1/4, 3/4) = 1/4$.
By the same way, we have $[\tau_2(\{b\}) \wedge A \subseteq cl_1(\{b\})] = \min(1, 0) = 0$ and
$[\tau_2(\{a, b\}) \wedge A \subseteq cl_1(\{a, b\})] = \min(1/4, 1) = 1/4$.
Therefore $[\exists U(U \in \tau_2 \wedge U \subseteq A \subseteq cl_1(U))] = 1/4 = s\tau_{(2,1)}(A)$.

Note that $[U\dot{\equiv}int_2(A)] = [U \subseteq int_2(A)]\dot{\wedge}[int_2(A) \subseteq U]$ and
$[U \subseteq int_2(A)] = \inf_{x\in U} int_2(A)(x)$, $[int_2(A) \subseteq U] = \inf_{x\in X\sim U}\big(1 - int_2(A)(x)\big)$.
$[\{a\}\dot{\equiv}int_2(A)] = \max(0, 1/4+0-1) = 0$. $[\{b\}\dot{\equiv}int_2(A)] = \max(0, 1+3/4-1) = 3/4$
$[\{a, b\}\dot{\equiv}int_2(A)] = \max(0, 1 + 1/4 - 1) = 1/4$.

4 (i,j)-semi Neighborhood System in Fuzzifying Bitopological Spaces

Definition 4.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $x \in X$. Then the $(i, j)-$semi neighborhood system of x is denoted by $sN_x^{(i,j)} \in \Im(P(X))$ and defined as

$$A \in sN_x^{(i,j)} := \exists B(B \in s\tau_{(i,j)} \wedge x \in B \subseteq A)$$

i.e., $sN_x^{(i,j)}(A) = \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B)$.

Theorem 4.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \in P(X)$. Then
(1) $\models A \in s\tau_{(i,j)} \longleftrightarrow \forall x\big(x \in A \to \exists B(B \in sN_x^{(i,j)} \wedge B \subseteq A)\big)$;
(2) $N_x^i(A) \leq sN_x^{(i,j)}(A)$.

Proof. (1) From Theorem (3.10) (1), we have

$$\begin{aligned}[\forall x\big(x \in A \to \exists B(B \in sN_x^{(i,j)} \wedge B \subseteq A)\big)] &= \inf_{x \in A} \sup_{B \subseteq A} sN_x^{(i,j)}(B) \\ &= \inf_{x \in A} \sup_{B \subseteq A} \sup_{x \in C \subseteq B} s\tau_{(i,j)}(C) \\ &= \inf_{x \in A} \sup_{x \in C \subseteq A} s\tau_{(i,j)}(C) = s\tau_{(i,j)}(A).\end{aligned}$$

(2) From Lemma (3.6) (1), we have
$sN_x^{(i,j)}(A) = \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B) \geq \sup_{x \in B \subseteq A} \tau_i(B) = N_x^i(A)$.

Corollary 4.3. $s\tau_{(i,j)}(A) = \inf_{x \in A} sN_x^{(i,j)}(A)$.

Theorem 4.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The mapping $sN^{(i,j)} : X \to \Im^N(P(X))$, $x \mapsto sN_x^{(i,j)}$, where $\Im^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$, has the following properties:
(1) $\models A \in sN_x^{(i,j)} \to x \in A$;
(2) $\models A \subseteq B \to (A \in sN_x^{(i,j)} \to B \in sN_x^{(i,j)})$;
(3) $\models A \in sN_x^{(i,j)} \to \exists H(H \in sN_x^{(i,j)} \wedge H \subseteq A \wedge \forall y(y \in H \to H \in sN_y^{(i,j)}))$.

Proof. (1) If $[A \in sN_x^{(i,j)}] = 0$, then (1) is obtain.
If $[A \in sN_x^{(i,j)}] = \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B) > 0$, then there exists B_0 such that $x \in B_0 \subseteq A$. Now we have $[x \in A] = 1$. Therefore $[A \in sN_x^{(i,j)}] \leq [x \in A]$.
(2) Immediate.
(3) $[\exists H(H \in sN_x^{(i,j)} \wedge H \subseteq A \wedge \forall y(y \in H \to H \in sN_y^{(i,j)}))]$

$$\begin{aligned}&= \sup_{H \subseteq A} (sN_x^{(i,j)}(H) \wedge \inf_{y \in H} sN_y^{(i,j)}(H)) \\ &= \sup_{H \subseteq A} (sN_x^{(i,j)}(H) \wedge s\tau_{(i,j)}(H)) \\ &= \sup_{H \subseteq A} s\tau_{(i,j)}(H) \geq \sup_{x \in H \subseteq A} s\tau_{(i,j)}(H) = sN_x^{(i,j)}(A) = [A \in sN_x^{(i,j)}].\end{aligned}$$

5 (i,j)-semi Derived Sets and (i,j)-semi Closure Operator in Fuzzifying Bitopological Spaces

Definition 5.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The $(i, j)-$semi derived set $sd_{(i,j)}(A)$ of A is defined as follows:

$$x \in sd_{(i,j)}(A) := \forall B(B \in sN_x^{(i,j)} \to B \cap (A \sim \{x\}) \neq \phi)$$

i.e., $sd_{(i,j)}(A)(x) = \inf_{B\cap(A\sim\{x\})=\phi} (1 - sN_x^{(i,j)}(B))$.

Lemma 5.2. $sd_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\})$.

Proof.

$$\begin{aligned} sd_{(i,j)}(A)(x) &= 1 - \sup_{B\cap A\sim\{x\}=\phi} sN_x^{(i,j)}(B) = 1 - \sup_{B\subseteq(X\sim A)\cup\{x\}} \sup_{x\in C\subseteq B} s\tau_{(i,j)}(C) \\ &= 1 - \sup_{x\in C\subseteq(X\sim A)\cup\{x\}} s\tau_{(i,j)}(C) = 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}). \end{aligned}$$

Theorem 5.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A, B \in P(X)$. Then
(1) $\models sd_{(i,j)}(\phi) \equiv \phi$;
(2) $\models A \subseteq B \longrightarrow sd_{(i,j)}(A) \subseteq sd_{(i,j)}(B)$;
(3) $\models A \in sF_{(i,j)} \longleftrightarrow sd_{(i,j)}(A) \subseteq A$;
(4) $\models sd_{(i,j)}(A) \subseteq d_i(A)$.

Proof. (1) From Lemma (5.2), we have

$$\begin{aligned} sd_{(i,j)}(\phi)(x) &= 1 - sN_x^{(i,j)}((X \sim \phi) \cup \{x\}) \\ &= 1 - sN_x^{(i,j)}(X) = 1 - 1 = 0. \end{aligned}$$

(2) Let $A \subseteq B$, then From Lemma (5.2) and Theorem (4.4) (2), we have

$$\begin{aligned} sd_{(i,j)}(A)(x) &= 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}) \\ &\leq 1 - sN_x^{(i,j)}((X \sim B) \cup \{x\}) = sd_{(i,j)}(B)(x). \end{aligned}$$

(3) From Lemma (5.2) and Theorem (4.2) (1), we have

$$\begin{aligned} [sd_{(i,j)}(A) \subseteq A] &= \inf_{x\in X\sim A}(1 - sd_{(i,j)}(A)(x)) = \inf_{x\in X\sim A} sN_x^{(i,j)}((X \sim A) \cup \{x\}) \\ &= \inf_{x\in X\sim A} sN_x^{(i,j)}(X \sim A) = \inf_{x\in X\sim A} \sup_{x\in B\subseteq X\sim A} s\tau_{(i,j)}(B) \\ &= s\tau_{(i,j)}(X \sim A) = sF_{(i,j)}(A) = [A \in sF_{(i,j)}]. \end{aligned}$$

(4) From Theorem (4.2) (2) and Lemma (5.1) in [10], we have
$sd_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}) \leq 1 - N_x^i((X \sim A) \cup \{x\}) = d_i(A)(x)$.

Definition 5.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The Fuzzifying $(i, j)-$semi closure of A, is denoted and defined as follows:
$x \in scl_{(i,j)}(A) := \forall B((B \supseteq A) \wedge (B \in sF_{(i,j)}) \to x \in B)$,
i.e., $scl_{(i,j)}(A)(x) = \inf_{x\notin B\supseteq A}(1 - sF_{(i,j)}(B))$.

Lemma 5.5. [6] For any $A \in P(X)$ and $\widetilde{B} \in \Im(X)$, then $[\widetilde{B} \subseteq A] = [\widetilde{B} \cup A \subseteq A]$.

Theorem 5.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $A, B \in P(X)$ and $x \in X$. Then
(1) $scl_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}(X \sim A)$;
(2) $\models scl_{(i,j)}(\phi) = \phi$;
(3) $\models A \subseteq scl_{(i,j)}(A)$;
(4) $\models scl_{(i,j)}(A) \equiv sd_{(i,j)}(A) \cup A$;
(5) $\models x \in scl_{(i,j)}(A) \longleftrightarrow \forall B(B \in sN_x^{(i,j)} \longrightarrow A \cap B \neq \phi)$;
(6) $\models A \equiv scl_{(i,j)}(A) \longleftrightarrow A \in sF_{(i,j)}(A)$;
(7) $\models scl_{(i,j)}(A) \subseteq cl_i(A)$;
(8) $\models A \subseteq B \longrightarrow scl_{(i,j)}(A) \subseteq scl_{(i,j)}(B)$;
(9) $\models B \dot{\equiv} scl_{(i,j)}(A) \longrightarrow B \in sF_{(i,j)}$.

Proof.

$$\begin{aligned}(1)\ scl_{(i,j)}(A)(x) &= \inf_{x \notin B \supseteq A}(1 - sF_{(i,j)}(B)) \\ &= \inf_{x \notin B \supseteq A}(1 - s\tau_{(i,j)}(X \sim B)) \\ &= 1 - \sup_{x \in X \sim B \subseteq X \sim A} s\tau_{(i,j)}(X \sim B) = 1 - sN_x^{(i,j)}(X \sim A).\end{aligned}$$

(2) $scl_{(i,j)}(\phi)(x) = 1 - sN_x^{(i,j)}(X \sim \phi) = 1 - sN_x^{(i,j)}(X) = 0$.
(3) Let $A \in P(X)$ and for any $x \in X$. If $x \notin A$, then $[x \in A] \leq [x \in scl_{(i,j)}(A)]$. If $x \in A$, then $scl_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}(X \sim A) = 1 - 0 = 1$.
So $[x \in A] \leq [x \in scl_{(i,j)}(A)]$. Therefore $[A \subseteq scl_{(i,j)}(A)] = 1$.

(4) From Lemma (5.2) and (3) above, for any $x \in X$ we have
$[x \in (sd_{(i,j)}(A) \cup A)] = \max\big(1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}), A(x)\big)$.
If $x \in A$, then $[x \in (sd_{(i,j)}(A) \cup A)] = A(x) = 1 = [x \in scl_{(i,j)}(A)]$. If $x \notin A$, then $[x \in sd_{(i,j)}(A) \cup A] = 1 - sN_x^{(i,j)}(X \sim A) = [x \in scl_{(i,j)}(A)]$.
Therefore $[scl_{(i,j)}(A)] = [sd_{(i,j)}(A) \cup A]$.

$$\begin{aligned}(5)\qquad [\ \forall B\ (\ B \in sN_x^{(i,j)} \rightarrow A \cap B \neq \phi\ )\ ] &= \inf_{B \subseteq X \sim A}(\ 1 - sN_x^{(i,j)}(B)\ ) \\ &= 1 - sN_x^{(i,j)}(X \sim A) \\ &= [x \in scl_{(i,j)}(A)].\end{aligned}$$

(6) From Theorem (5.3) (3), Lemma (5.5), (4) above and since
$[A \subseteq sd_{(i,j)}(A) \cup A] = 1$, we have

$$\begin{aligned}sF_{(i,j)}(A) &= [sd_{(i,j)}(A) \subseteq A] = [sd_{(i,j)}(A) \cup A \subseteq A] \\ &= [sd_{(i,j)}(A) \cup A \subseteq A] \wedge [A \subseteq sd_{(i,j)}(A) \cup A] \\ &= [sd_{(i,j)}(A) \cup A \equiv A] = [A \equiv scl_{(i,j)}(A)].\end{aligned}$$

(7) From Lemma (3.6) (2), we have
$$scl_{(i,j)}(A)(x) = \inf_{x \notin B \supseteq A}(1 - sF_{(i,j)}(B)) \leq \inf_{x \notin B \supseteq A}(1 - F_i(B)) = cl_i(A).$$

(8) Let $A \subseteq B$, then $X \sim B \subseteq X \sim A$. From (1) above and Theorem (4.4) (2), we have $scl_{(i,j)}(A)(x) = 1 - sN_x^{(i,j)}(X \sim A) \leq 1 - sN_x^{(i,j)}(X \sim B) = scl_{(i,j)}(B)(x)$.

(9) If $[A \subseteq B] = 0$, then $[B \dot{\equiv} scl_{(i,j)}(A)] = 0$. Now suppose that $[A \subseteq B] = 1$. We have $[B \subseteq scl_{(i,j)}(A)] = 1 - \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A)$ and
$[scl_{(i,j)}(A) \subseteq B] = \inf_{x\in X\sim B} sN_x^{(i,j)}(X \sim A)$. Therefore
$[B \dot{\equiv} scl_{(i,j)}(A)] = \max(0, \inf_{x\in X\sim B} sN_x^{(i,j)}(X \sim A) - \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A))$.
Let $[B \dot{\equiv} scl_{(i,j)}(A)] > t$. Then $\inf_{x\in X\sim B} sN_x^{(i,j)}(X \sim A) > t + \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A)$. For any $x \in X \sim B$, we have $sN_x^{(i,j)}(X \sim A) > t + \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A)$. Therefore $\sup_{x\in C\subseteq X\sim A} s\tau_{(i,j)}(C) > t + \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A)$, i.e., there exists C_x such that $x \in C_x \subseteq X \sim A$ and $s\tau_{(i,j)}(C_x) > t + \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A)$. Now we want to prove $C_x \subseteq X \sim B$. If not, then there exists $x' \in C_x$ and $x' \in B \sim A$. Hence we obtain $\sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A) \geq sN_{x'}^{(i,j)}(X \sim A) \geq s\tau_{(i,j)}(C_x) > t + \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A)$, a contradiction. Therefore $sF_{(i,j)}(B) = s\tau_{(i,j)}(X \sim B) = \inf_{x\in X\sim B} sN_x^{(i,j)}(X \sim B) \geq \inf_{x\in X\sim B} s\tau_{(i,j)}(C_x) \geq s\tau_{(i,j)}(C_x) > t + \sup_{x\in B\sim A} sN_x^{(i,j)}(X \sim A) > t$. Since t is arbitrary, it holds that $[B \dot{\equiv} scl_{(i,j)}(A)] \leq [B \in sF_{(i,j)}]$.

6 (i,j)-semi Interior, (i,j)-semi Exterior and (i,j)-semi Boundary Operators in Fuzzifying Bitopological Spaces

Definition 6.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \in P(X)$, the $(i, j)-$semi interior of A is defined as follows:

$$sint_{(i,j)}(A)(x) = sN_x^{(i,j)}(A)$$

Theorem 6.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $A, B \in P(X)$ and $x \in X$.Then
(1) $\models sint_{(i,j)}(X) \equiv X$;
(2) $\models sint_{(i,j)}(A) \subseteq A$;
(3) $\models int_i(A) \subseteq sint_{(i,j)}(A)$;
(4) $\models B \in s\tau_{(i,j)} \wedge B \subseteq A \longrightarrow B \subseteq sint_{(i,j)}(A)$;
(5) $\models A \equiv sint_{(i,j)}(A) \longleftrightarrow A \in s\tau_{(i,j)}$;
(6) $\models A \subseteq B \longrightarrow sint_{(i,j)}(A) \subseteq sint_{(i,j)}(B)$;
(7) $\models sint_{(i,j)}(A) \equiv X \sim scl_{(i,j)}(X \sim A)$;
(8) $\models sint_{(i,j)}(A) \equiv A \cap (X \sim sd_{(i,j)}(X \sim A))$;
(9) $\models B \dot{\equiv} sint_{(i,j)}(A) \longrightarrow B \in s\tau_{(i,j)}$.

Proof. (1) $sint_{(i,j)}(X)(x) = sN_x^{(i,j)}(X) = 1$. Therefore $sint_{(i,j)}(X) \equiv X$.
(2) Let $A \in P(X)$ and $x \in X$. If $x \notin A$, then $sint_{(i,j)}(A)(x) = sN_x^{(i,j)}(A) = 0$. Therefore $sint_{(i,j)}(A) \subseteq A$.
(3) From Theorem (4.2) (2), we have $int_i(A)(x) = N_x^i(A) \leq sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.

(4) If $B \not\subseteq A$, then $[(B \in s\tau_{(i,j)}) \wedge (B \subseteq A)] = 0$. If $B \subseteq A$, then

$$\begin{aligned}[B \subseteq sint_{(i,j)}(A)] &= \inf_{x\in B} sint_{(i,j)}(A)(x) \\ &= \inf_{x\in B} sN_x^{(i,j)}(A) \\ &\geq \inf_{x\in B} sN_x^{(i,j)}(B) = s\tau_{(i,j)}(B) = [(B \in s\tau_{(i,j)}) \wedge (B \subseteq A)].\end{aligned}$$

(5) $$\begin{aligned}[\ A \equiv sint_{(i,j)}(A)\ ] &= \min\Big(\inf_{x\in A} sint_{(i,j)}(A)(x)\ ,\ \inf_{x\in X\sim A}(1 - sint_{(i,j)}(A)(x))\Big) \\ &= \min\big(\inf_{x\in A} sN_x^{(i,j)}(A)\ ,\ \inf_{x\in X\sim A}(1 - sN_x^{(i,j)}(A))\big) \\ &= \inf_{x\in A} sN_x^{(i,j)}(A) = s\tau_{(i,j)}(A) = [A \in s\tau_{(i,j)}].\end{aligned}$$

(6) From Definition (6.1) and Theorem (4.4) (2), the proof is straightforward.

(7) From Theorem (5.6) (1), we have
$\big(X \sim scl_{(i,j)}(X \sim A)\big)(x) = 1 - (1 - sN_x^{(i,j)}(A)) = sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.

(8) From Lemma (5.2), we have
$[A \cap (X \sim sd_{(i,j)}(X \sim A))] = \min(A(x), sN_x^{(i,j)}(A \cup \{x\}))$
If $x \notin A$, then $[A \cap (X \sim sd_{(i,j)}(X \sim A))] = 0 = sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.
If $x \in A$, then $[A \cap (X \sim sd_{(i,j)}(X \sim A))] = sN_x^{(i,j)}(A) = sint_{(i,j)}(A)(x)$.

(9) From Theorem (5.6) (9) and (7) above, we have
$[B \dot{\equiv} sint_{(i,j)}(A)] = [X \sim B \dot{\equiv} scl_{(i,j)}(X \sim A)] \leq [X \sim B \in sF_{(i,j)}] = [B \in s\tau_{(i,j)}]$.

Definition 6.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$. The $(i,j)-$semi exterior of A is defined as follows:
$x \in sext_{(i,j)}(A) :=\ x \in sint_{(i,j)}(X \sim A)$,
i.e., $sext_{(i,j)}(A)(x) = sint_{(i,j)}(X \sim A)(x)$.

Theorem 6.4. For any A
(1) $\models sext_{(i,j)}(\phi) \equiv X$;
(2) $\models sext_{(i,j)}(A) \subseteq X \sim A$;
(3) $\models ext_i(A) \subseteq sext_{(i,j)}(A)$;
(4) $\models A \in sF_{(i,j)} \longleftrightarrow sext_{(i,j)}(A) \equiv X \sim A$;
(5) $\models B \in sF_{(i,j)} \wedge A \subseteq B \longrightarrow X \sim B \subseteq sext_{(i,j)}(A)$;
(6) $\models B \subseteq A \longrightarrow sext_{(i,j)}(B) \subseteq sext_{(i,j)}(A)$;
(7) $\models sext_{(i,j)}(A) \equiv (X \sim A) \cap (X \sim sd_{(i,j)}(A))$;
(8) $\models sext_{(i,j)}(A) \equiv X \sim scl_{(i,j)}(A)$;
(9) $\models x \in sext_{(i,j)}(A) \longleftrightarrow \exists B(x \in B \in s\tau_{(i,j)} \wedge B \cap A = \phi)$.

Proof. From Theorem (6.2), we obtain (1),(2),(3),(4),(5),(6),(7) and (8).

(9) $$\begin{aligned}[\exists B(\ x \in B \in s\tau_{(i,j)} \wedge B \cap A = \phi\ )] &= \sup_{x\in B\subseteq X\sim A} s\tau_{(i,j)}(B) = sN_x^{(i,j)}(X \sim A) \\ &= sint_{(i,j)}(X \sim A)(x).\end{aligned}$$

Definition 6.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \subseteq X$. The $(i,j)-$semi boundary of A is defined as follows:

$x \in sb_{(i,j)}(A) := \big(x \notin sint_{(i,j)}(A)\big) \wedge \big(x \notin sint_{(i,j)}(X \sim A)\big)$,
i.e., $sb_{(i,j)}(A)(x) = \min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big)$.

Lemma 6.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $A \in P(X)$ and $x \in X$. Then $\models x \in sb_{(i,j)}(A) \longleftrightarrow \forall B\big(B \in sN_x^{(i,j)} \to (B \cap A \neq \phi) \wedge (B \cap (X \sim A) \neq \phi)\big)$.

Proof. $[\forall B\big(B \in sN_x^{(i,j)} \to (B \cap A \neq \phi) \wedge (B \cap (X \sim A) \neq \phi)\big)]$

$$\begin{aligned}
&= min\big(\inf_{B\subseteq A}(1 - sN_x^{(i,j)}(B)), \inf_{B\subseteq X\sim A}(1 - sN_x^{(i,j)}(B))\big)\\
&= min\big(1 - sN_x^{(i,j)}(A), 1 - sN_x^{(i,j)}(X \sim A)\big)\\
&= min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big) = [x \in sb_{(i,j)}(A)].
\end{aligned}$$

Theorem 6.7. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $A \in P(X)$. Then
(1) $\models sb_{(i,j)}(A) \equiv scl_{(i,j)}(A) \cap scl_{(i,j)}(X \sim A)$;
(2) $\models sb_{(i,j)}(A) \equiv sb_{(i,j)}(X \sim A)$;
(3) $\models X \sim sb_{(i,j)}(A) \equiv sint_{(i,j)}(A) \cup sint_{(i,j)}(X \sim A)$;
(4) $\models scl_{(i,j)}(A) \equiv A \cup sb_{(i,j)}(A)$;
(5) $\models sb_{(i,j)}(A) \subseteq A \longleftrightarrow A \in sF_{(i,j)}$;
(6) $\models sint_{(i,j)}(A) \equiv A \cap (X \sim sb_{(i,j)}(A))$;
(7) $\models (sb_{(i,j)}(A) \cap A \equiv \phi) \longleftrightarrow A \in s\tau_{(i,j)}$;
(8) $\models sb_{(i,j)}(A) \subseteq b_i(A)$;
(9) $\models X \sim sb_{(i,j)}(A) \equiv sint_{(i,j)}(A) \cup sext_{(i,j)}(A)$.

Proof. (1) From Theorem (6.2) (7), we obtain

$$\begin{aligned}
\big(scl_{(i,j)}(A) \cap scl_{(i,j)}(X \sim A)\big)(x) &= \min\big(scl_{(i,j)}(A)(x), scl_{(i,j)}(X \sim A)(x)\big)\\
&= \min\big(1 - sint_{(i,j)}(X \sim A)(x), 1 - sint_{(i,j)}(A)(x)\big)\\
&= sb_{(i,j)}(A)(x).
\end{aligned}$$

(2) Straightforward.
(3) From (1) above and Theorem (6.2) (7), we obtain

$$\begin{aligned}
X \sim sb_{(i,j)}(A) &\equiv X \sim \big(scl_{(i,j)}(A) \cap scl_{(i,j)}(X \sim A)\big)\\
&\equiv \big(X \sim scl_{(i,j)}(A)\big) \cup \big(X \sim scl_{(i,j)}(X \sim A)\big)\\
&\equiv sint_{(i,j)}(X \sim A) \cup sint_{(i,j)}(A).
\end{aligned}$$

(4) If $x \in A$, then $scl_{(i,j)}(A)(x) = 1 = (A \cup sb_{(i,j)}(A))(x)$.
If $x \notin A$, then

$$\begin{aligned}
(A \cup sb_{(i,j)}(A))(x) &= sb_{(i,j)}(A)(x)\\
&= min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big)\\
&= 1 - sint_{(i,j)}(X \sim A)(x) = scl_{(i,j)}(A)(x).
\end{aligned}$$

(5) From Theorem (5.3) (3), Theorem (5.6) (4), Lemma (5.5) and (4) above, we obtain

$$\begin{aligned}
A \in sF_{(i,j)} &\longleftrightarrow sd_{(i,j)}(A) \subseteq A\\
&\longleftrightarrow A \cup sd_{(i,j)}(A) \subseteq A\\
&\longleftrightarrow scl_{(i,j)}(A) \subseteq A\\
&\longleftrightarrow A \cup sb_{(i,j)}(A) \subseteq A\\
&\longleftrightarrow sb_{(i,j)}(A) \subseteq A.
\end{aligned}$$

(6) From Theorem (6.2) (7) and (4) above, we obtain

$$\begin{aligned} sint_{(i,j)}(A) &\equiv X \sim scl_{(i,j)}(X \sim A) \\ &\equiv X \sim \big(X \sim A \cup sb_{(i,j)}(X \sim A)\big) \\ &\equiv A \cap \big(X \sim sb_{(i,j)}(X \sim A)\big) \equiv A \cap \big(X \sim sb_{(i,j)}(A)\big). \end{aligned}$$

(7) From Theorem (6.2) (5) and (6) above, we obtain

$$\begin{aligned} sb_{(i,j)}(A) \cap A \equiv \phi &\longleftrightarrow (X \sim sb_{(i,j)}(A)) \cup (X \sim A) \equiv X \\ &\longleftrightarrow A \subseteq X \sim sb_{(i,j)}(A) \\ &\longleftrightarrow A \cap (X \sim sb_{(i,j)}(A)) \equiv A \\ &\longleftrightarrow sint_{(i,j)}(A) \equiv A \longleftrightarrow A \in s\tau_{(i,j)}. \end{aligned}$$

(8) From Theorem (6.2) (3), we have

$$\begin{aligned} sb_{(i,j)}(A)(x) &= \min\big(1 - sint_{(i,j)}(A)(x), 1 - sint_{(i,j)}(X \sim A)(x)\big) \\ &\le \min\big(1 - int_i(A)(x), 1 - int_i(X \sim A)(x)\big) = b_i(A)(x). \end{aligned}$$

(9) From (3) above, we have
$X \sim sb_{(i,j)}(A) \equiv sint_{(i,j)}(A) \cup sint_{(i,j)}(X \sim A) \equiv sint_{(i,j)}(A) \cup sext_{(i,j)}(A)$.

7 (i,j)-semi Convergence of Nets in Fuzzifying Bitopological Spaces

Definition 7.1. :Let (X, τ_1, τ_2) be a fuzzifying bitopological space. The class of all nets in X is denoted by $N(X) = \{S | S : D \to X, \text{where } (D, \geq) \text{ is a directed set}\}$.

Definition 7.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.The binary fuzzy predicates $\rhd^s_{(i,j)}$, $\propto^s_{(i,j)} \in \Im(N(X) \times X)$, are defined as follows:
$S \rhd^s_{(i,j)} x := \forall A(A \in sN_x^{(i,j)} \to S \subsetneqq A)$,
$S \propto^s_{(i,j)} x := \forall A(A \in sN_x^{(i,j)} \to S \subsetneqq A)$, $S \in N(X)$,
where $S \rhd^s_{(i,j)} x$, $S \propto^s_{(i,j)} x$ stand for "S is $(i,j)-$semi converges to x", "x is $(i,j)-$semi accumulation point of S". Also, $\subsetneqq$ and $\subsetneqq$ are the binary crisp predicates "almost in" and "often in", respectively.

Definition 7.3. The fuzzy sets,
$lim^s_{(i,j)} T(x) = [T \rhd^s_{(i,j)} x]$;
$adh^s_{(i,j)} T(x) = [T \propto^s_{(i,j)} x]$,
where $T \in N(X)$, are called $(i,j)-$semi limit and $(i,j)-$semi adherence of T, respectively.

Theorem 7.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, $x \in X$, $A \in P(X)$ and $S \in N(X)$.Then
(1) $\models \exists S((S \subseteq A \sim \{x\}) \wedge (S \rhd^s_{(i,j)} x)) \longrightarrow x \in sd_{(i,j)}(A)$;
(2) $\models \exists S((S \subseteq A) \wedge (S \rhd^s_{(i,j)} x)) \longrightarrow x \in scl_{(i,j)}(A)$;
(3) $\models A \in sF_{(i,j)} \longrightarrow \forall S(S \subseteq A \to lim^s_{(i,j)} S \subseteq A)$;
(4) $\models \exists T((T < S) \wedge (T \rhd^s_{(i,j)} x)) \longrightarrow S \propto^s_{(i,j)} x$,
where $S \subseteq A$ and $T < S$ stand for "S is all in A", "T is a subnet of S", respectively.

Proof. (1) We know that, $[S \rhd^s_{(i,j)} x] = \inf_{S \not\subsetneq A}(1 - sN_x^{(i,j)}(A))$.Also,
$[\exists S((S \subseteq A \sim \{x\}) \wedge (S \rhd^s_{(i,j)} x))] = \sup_{S \subseteq A \sim \{x\}} \inf_{S \not\subsetneq B}(1 - sN_x^{(i,j)}(B))$.
First, for any $S \in N(X)$ such that $S \subseteq A \sim \{x\}$, we have $S \not\subsetneq (X \sim A) \cup \{x\}$. Therefore, $\inf_{S \not\subsetneq B}(1 - sN_x^{(i,j)}(B)) \leq 1 - sN_x^{(i,j)}((X \sim A) \cup \{x\}) = [x \in sd_{(i,j)}(A)]$.

(2) If $x \in A$, then from Theorem (5.6) (1) we can prove this similar (1) above. If $x \notin A$, then $A \sim \{x\} = A$ from Theorem (5.6) (1) and (1) above we have,

$$\begin{aligned}[\exists S((S \subseteq A) \wedge (S \rhd^s_{(i,j)} x))] &= [\exists S((S \subseteq A \sim \{x\}) \wedge (S \rhd^s_{(i,j)} x))] \\ &\leq 1 - sN_x^{(i,j)}(X \sim A) = scl_{(i,j)}(A)(x) = [x \in scl_{(i,j)}(A)].\end{aligned}$$

(3) $$\begin{aligned}[\ \forall S(S \subseteq A \to lim^s_{(i,j)} S \subseteq A)\ ] &= \inf_{S \subseteq A} \inf_{x \notin A} \Big(\ 1 - \inf_{S \not\subsetneq B}(\ 1 - sN_x^{(i,j)}(B)\ )\ \Big) \\ &= \inf_{S \subseteq A} \inf_{x \notin A} \sup_{S \not\subsetneq B} sN_x^{(i,j)}(B).\end{aligned}$$

In the other hand, from Theorem (5.6) (6) and (2) above, we have

$$\begin{aligned}[A \in sF_{(i,j)}] &= [A \equiv scl_{(i,j)}(A)] = [scl_{(i,j)}(A) \subseteq A] \wedge [A \subseteq scl_{(i,j)}(A)] \\ &= [scl_{(i,j)}(A) \subseteq A] = [X \sim A \subseteq X \sim scl_{(i,j)}(A)] \\ &= \inf_{x \in X \sim A}(1 - scl_{(i,j)}(A)(x)) \\ &\leq \inf_{x \in X \sim A}(1 - \sup_{S \subseteq A} \inf_{S \not\subsetneq B}(1 - sN_x^{(i,j)}(B))) \\ &= \inf_{x \notin A} \inf_{S \subseteq A} \sup_{S \not\subsetneq B} sN_x^{(i,j)}(B) = [\forall S(S \subseteq A \to lim^s_{(i,j)} S \subseteq A)].\end{aligned}$$

(4) $[S \propto^s_{(i,j)} x] = \inf_{S \not\subsetneq A}(1 - sN_x^{(i,j)}(A))$,
$[\exists T((T < S) \wedge (T \rhd^s_{(i,j)} x))] = \sup_{T < S} \inf_{T \not\subsetneq A}(1 - sN_x^{(i,j)}(A))$.
Set $\mathcal{A}_S = \{A | S \not\subsetneq A\}$, $\mathcal{B}_T = \{A | T \not\subsetneq A\}$. Then for any $T < S$, we have $\mathcal{A}_S \subseteq \mathcal{B}_T$. In fact, suppose $T = S \circ K$. If $S \not\subsetneq A$, then there exists $\sigma_0 \in \mathcal{D}_S$ such that $S(\sigma) \notin A$ when $\sigma \geq \sigma_0$. Now, we will show that $T \not\subsetneq A$. If not, then there exists $\mu_0 \in \mathcal{D}_T$ such that $T(\mu) \in A$, when $\mu \geq \mu_0$. Moreover, there exists $\mu_1 \in \mathcal{D}_T$ such that $K(\mu_1) \geq \sigma_0$ because $T < S$, and there exists $\mu_2 \in \mathcal{D}_T$ such that $\mu_2 \geq \mu_0, \mu_1$ because $\mathcal{D}_T$ is directed. In this way, $K(\mu_2) \geq \sigma_0$, $S(K(\mu_2)) \notin A$ and $S(K(\mu_2)) = T(\mu_2) \in A$, a contradiction. Therefore,

$$\begin{aligned}[\exists T((T < S) \wedge (T \rhd^s_{(i,j)} x))] &= \sup_{T < S} \inf_{A \in \mathcal{B}_T}(1 - sN_x^{(i,j)}(A)) \\ &\leq \inf_{A \in \mathcal{A}_S}(1 - sN_x^{(i,j)}(A)) = [S \propto^s_{(i,j)} x].\end{aligned}$$

Theorem 7.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. If T is a universal net, then $\models lim^s_{(i,j)} T = adh^s_{(i,j)} T$.

Proof. For any net $T \in N(X)$ and any $A \subseteq X$ one can obtain that if $T \not\subsetneq A$, then $T \not\subsetneq A$. Suppose T is a universal net in X and $T \not\subsetneq A$. Then, $T \subsetneq X \sim A$. So $T \not\subsetneq A$ (Indeed, $T \subsetneq X \sim A$ if and only if there exists $m \in D$ such that for every $n \in D$, $n \geq m$, $T(n) \in X \sim A$ if and only if there exists $m \in D$ such that for every $n \in D$,

$n \geq m$, $T(n) \notin A$ if and only if $T \not\subsetneq A$.). Hence for any universal net T in X, we have
$lim^s_{(i,j)}T(x) = \inf_{T \not\subsetneq A}(1 - sN_x^{(i,j)}(A)) = \inf_{T \not\subsetneq A}(1 - sN_x^{(i,j)}(A)) = adh^s_{(i,j)}T(x)$.

Lemma 7.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.
$\models (T \rhd^s_{(i,j)} x)) \longleftrightarrow \forall A(x \in A \in s\tau_{(i,j)} \to T \subsetneq A)$.

Proof. If $B \subseteq A$ and $T \not\subsetneq A$, then $T \not\subsetneq B$

$$\begin{aligned}
[T \rhd^s_{(i,j)} x)] &= \inf_{T \not\subsetneq A}(1 - sN_x^{(i,j)}(A)) \\
&= 1 - \sup_{T \not\subsetneq A} \sup_{x \in B \subseteq A} s\tau_{(i,j)}(B) \\
&\geq 1 - \sup_{T \not\subsetneq B, x \in B} s\tau_{(i,j)}(B) \\
&= \inf_{T \not\subsetneq B, x \in B}(1 - s\tau_{(i,j)}(B)) = [\forall A(x \in A \in s\tau_{(i,j)} \to T \subsetneq A)].
\end{aligned}$$

Conversely, since

$$\begin{aligned}
[\forall A(x \in A \in s\tau_{(i,j)} \to T \subsetneq A)] &= \inf_{T \not\subsetneq A, x \in A}(1 - s\tau_{(i,j)}(A)) \\
&= \inf_{T \not\subsetneq A, x \in A}(1 - \inf_{x \in A} \sup_{B \subseteq A} sN_x^{(i,j)}(B)) \\
&\geq 1 - \sup_{T \not\subsetneq B, x \in B} sN_x^{(i,j)}(B) \\
&= \inf_{T \not\subsetneq B, x \in B}(1 - sN_x^{(i,j)}(B)) = [T \rhd^s_{(i,j)} x].
\end{aligned}$$

8 (i,j)-semi Convergence of Filters in Fuzzifying Bitopological Spaces

Definition 8.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space and $F(X)$ be the set of all filters on X. The binary fuzzy predicates $\rhd^s_{(i,j)}$, $\propto^s_{(i,j)} \in \Im(F(X) \times X)$ are defined as follows:
$K \rhd^s_{(i,j)} x := \forall A(A \in sN_x^{(i,j)} \to A \in K)$,
$K \propto^s_{(i,j)} x := \forall A(A \in K \to x \in scl_{(i,j)}(A))$, where $K \in F(X)$.

Definition 8.2. The fuzzy sets,
$lim^s_{(i,j)}K(x) = [K \rhd^s_{(i,j)} x]$;
$adh^s_{(i,j)}K(x) = [K \propto^s_{(i,j)} x]$,
are called $(i,j)-$semi limit and $(i,j)-$semi adherence sets K, respectively.

Theorem 8.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.
(1) If $T \in N(X)$ and K^T is the filter corresponding to T, i.e., $K^T = \{A | T \subsetneq A\}$, then
(a) $\models lim^s_{(i,j)}K^T = lim^s_{(i,j)}T$;
(b) $\models adh^s_{(i,j)}K^T = adh^s_{(i,j)}T$.
(2) If $K \in F(X)$ and T^K is the net corresponding to K, i.e., $T^K : D \to X$, $(x, A) \mapsto x$,$(x, A) \in D$, where $D = \{(x, A)| x \in A \in K\}$, $(x, A) \geq (y, B)$ iff $A \subseteq B$, then
(a) $\models lim^s_{(i,j)}T^K = lim^s_{(i,j)}K$;
(b) $\models adh^s_{(i,j)}T^K = adh^s_{(i,j)}K$.

Proof. (1) For any $x \in X$, we have

(a) $lim^s_{(i,j)}K^T(x) = \inf_{A \notin K^T}(1 - sN_x^{(i,j)}(A)) = \inf_{T \not\subsetneq A}(1 - sN_x^{(i,j)}(A)) = lim^s_{(i,j)}T(x)$.

$$(b)\ adh^s_{(i,j)}K^T(x) = \inf_{A \in K^T} scl_{(i,j)}(A)(x) = \inf_{T \subsetneq A}(1 - sN_x^{(i,j)}(X \sim A))$$
$$= \inf_{T \not\subsetneq X \sim A}(1 - sN_x^{(i,j)}(X \sim A)) = adh^s_{(i,j)}T(x).$$

(2) (a) First we prove that $T^K \subsetneq A$ if and only if $A \in K$. If $A \in K$, then $A \neq \phi$ and there exists at least an element $x \in A$. So for $(x, A) \in D$ and $(y, B) \in D$ such that $(y, B) \geq (x, A)$, then $B \subseteq A$ and so $T^K(y, B) = y \in B \subseteq A$. Thus $T^K \subsetneq A$.

Conversely, suppose $T^K \subsetneq A$, then there exists $(y, B) \in D$, for all $(z, C) \in D$, such that $(z, C) \geq (y, B)$ and we have $T^K(z, C) \in A$. So for every $z \in B$, $(z, B) \geq (y, B)$ and $T^K(z, B) = z \in A$ implies $B \subseteq A$. Then $A \in K$. Thus $T^K \not\subsetneq A$ if and only if $A \notin K$. Now,

$$lim^s_{(i,j)}T^K(x) = [T^K \rhd^s_{(i,j)} x] = \inf_{T^K \not\subsetneq A}(1 - sN_x^{(i,j)}(A))$$
$$= \inf_{A \notin K}(1 - sN_x^{(i,j)}(A)) = lim^s_{(i,j)}K(x).$$

(b) First we prove that $X \sim A \in K$ if and only if $T^K \not\subsetneq A$. Suppose $T^K \not\subsetneq A$, then there exists $(z, B) \in D$ such that for every $(y, C) \in D$ with $(y, C) \geq (z, B), T^K(y, C) \notin A$. Now for every $x \in B, (x, B) \geq (z, B)$ and $T^K(x, B) = x \notin A$, i.e., $B \cap A = \phi$ so $B \subseteq X \sim A$ and then $X \sim A \in K$.

Conversely, suppose $X \sim A \in K$, then $X \sim A \neq \phi$ and thus it contains at least an element x. Now, for any $(z, C) \in D$ such that $(z, C) \geq (x, X \sim A)$, one can have that $T^K(z, C) = z \notin A$. Hence, $T^K \not\subsetneq A$. Now,

$$adh^s_{(i,j)}T^K(x) = [T^K \propto^s_{(i,j)} x] = \inf_{T^K \not\subsetneq A}(1 - sN_x^{(i,j)}(A))$$
$$= \inf_{X \sim A \in K} scl_{(i,j)}(X \sim A) = \inf_{B \in K} scl_{(i,j)}(B) = adh^s_{(i,j)}K(x).$$

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