SOME GENERALIZATIONS OF THE BANACH’S CONTRACTION PRINCIPLE ON A COMPLETE COMPLEX VALUED S-METRIC SPACE

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Abstract — In this paper we give some generalizations of the Banach’s contraction principle on a complete complex valued S-metric space. We verify our results with an example.

Keywords — Complex valued S-metric space, Fixed point, Banach’s contraction principle.

1 Introduction

Metric spaces and fixed-point theory have an important role in various areas of mathematics such as analysis, topology, differential equation etc. Fixed-point theory begin with the Banach’s contraction principle. Then the principle has been studied and generalized on some metric spaces (see [1], [2], [6], [7] and [8]). Recently, it has been introduced the notion of an S-metric space as a generalization of a metric space [8]. Some mathematicians proved new fixed-point theorems on an S-metric space (see [4], [5], [6], [8], [9] and [10]). Mlaiki presented the concept of a complex valued S-metric space and gave a common fixed-point theorem of two self-mappings on a complex valued S-metric space [3]. The present authors investigated new common fixed-point theorems using the notion of CS-compatibility on a complex valued S-metric space [7].

Let $X = \mathbb{C}$ and the function $S : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be defined by

$$S(x, y, z) = i (|x - z| + |y - z|),$$

**Edited by Oktay Muhtaroglu (Area Editor) and Naim Çağman (Editor-in-Chief).**

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for all \(x, y, z \in \mathbb{C}\). Then the function \(S\) is a complex valued \(S\)-metric space on \(\mathbb{C}\). Let us define the self-mapping \(T : \mathbb{C} \to \mathbb{C}\) as follows:

\[ Tx = 1 - x, \]

for all \(x \in \mathbb{C}\). Then \(T\) is a self-mapping on the complete complex valued \(S\)-metric space \((\mathbb{C}, S)\). \(T\) has a fixed point \(x = \frac{1}{2}\), but it does not satisfy the condition of Banach’s contraction principle. Therefore it is important to study new generalized fixed-point theorems.

Motivated by the above studies, in this paper, we investigate new fixed-point theorems as generalizations of the Banach’s contraction principle on a complete complex valued \(S\)-metric spaces. We expect that new generalized fixed-point theorems will be obtained using our main theorems.

In Section 2 we recall some known definitions, lemmas and a theorem. In Section 3 we generalize the Banach’s contraction principle on a complete complex valued \(S\)-metric space. Also we give an example which satisfies the conditions of our results, but does not satisfy the condition of Banach’s contraction principle.

## 2 Preliminary

In this section we recall some definitions, lemmas and a theorem which is called the Banach’s contraction principle.

Let \(\mathbb{C}\) be the set of complex numbers and \(z_1, z_2 \in \mathbb{C}\). The partial order \(\preceq\) is defined on \(\mathbb{C}\) as follows:

\[ z_1 \preceq z_2 \text{ if and only if } \text{Re}(z_1) \leq \text{Re}(z_2), \text{Im}(z_1) \leq \text{Im}(z_2) \]

and

\[ z_1 \prec z_2 \text{ if and only if } \text{Re}(z_1) < \text{Re}(z_2), \text{Im}(z_1) < \text{Im}(z_2). \]

Also we write \(z_1 \preceq z_2\) if one of the following conditions hold:

1. \(\text{Re}(z_1) = \text{Re}(z_2)\) and \(\text{Im}(z_1) < \text{Im}(z_2)\),
2. \(\text{Re}(z_1) < \text{Re}(z_2)\) and \(\text{Im}(z_1) = \text{Im}(z_2)\),
3. \(\text{Re}(z_1) = \text{Re}(z_2)\) and \(\text{Im}(z_1) = \text{Im}(z_2)\).

Note that

\[ 0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2| \]

and

\[ z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3. \]
Definition 2.1. [3] Let $X$ be a nonempty set. A complex valued $S$-metric on $X$ is a function $S : X \times X \times X \to \mathbb{C}$ that satisfies the following conditions for all $x, y, z, t \in X$:

- (CS1) $0 \preceq S(x, y, z)$,
- (CS2) $S(x, y, z) = 0$ if and only if $x = y = z$,
- (CS3) $S(x, y, z) \preceq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

The pair $(X, S)$ is called a complex valued $S$-metric space.

Definition 2.2. [3] Let $(X, S)$ be a complex valued $S$-metric space. Then

1. A sequence $\{a_n\}$ in $X$ converges to $x$ if and only if for all $\varepsilon$ such that $0 < \varepsilon \in \mathbb{C}$ there exists a natural number $n_0$ such that for all $n \geq n_0$, we have $S(a_n, a_n, x) \prec \varepsilon$ and it is denoted by

$$\lim_{n \to \infty} a_n = x.$$ 

2. A sequence $\{a_n\}$ in $X$ is called a Cauchy sequence if for all $\varepsilon$ such that $0 < \varepsilon \in \mathbb{C}$ there exists a natural number $n_0$ such that for all $n, m \geq n_0$, we have $S(a_n, a_n, a_m) \prec \varepsilon$.

3. A complex valued $S$-metric space $(X, S)$ is called complete if every Cauchy sequence is convergent.

Lemma 2.3. [3] Let $(X, S)$ be a complex valued $S$-metric space and $\{a_n\}$ be a sequence in $X$. Then $\{a_n\}$ converges to $x$ if and only if

$$|S(a_n, a_n, x)| \to 0,$$

as $n \to \infty$.

Lemma 2.4. [3] Let $(X, S)$ be a complex valued $S$-metric space and $\{a_n\}$ be a sequence in $X$. Then $\{a_n\}$ is a Cauchy sequence if and only if

$$|S(a_n, a_n, a_m)| \to 0,$$

as $n \to \infty$.

Lemma 2.5. [3] If $(X, S)$ be a complex valued $S$-metric space then

$$S(x, x, y) = S(y, y, x),$$

for all $x, y \in X$.

Lemma 2.6. [9] Let $(X, S), (Y, S')$ be two $S$-metric spaces and $f : X \to Y$ be a function. Then $f$ is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$. 
In the next section, we consider two complex valued $S$-metric spaces in Lemma 2.6.

Now we recall the following theorem which is called the Banach’s contraction principle.

**Theorem 2.7.** [7] Let $(X, S)$ be a complete complex valued $S$-metric space and $T$ be a self-mapping of $X$ satisfying

$$S(Tx, Tx, Ty) \lesssim hS(x, x, y)$$

for all $x, y \in X$ and some $0 \leq h < 1$. Then $f$ has a fixed point in $X$.

### 3 Main Results

In this section we prove new generalizations of the Banach’s contraction principle.

**Theorem 3.1.** Let $(X, S)$ be a complete complex valued $S$-metric space and $T$ be a self-mapping of $X$. If there exist nonnegative real numbers $c_1, c_2, c_3, c_4$ satisfying

$$\max\{c_1 + 3c_3 + 2c_4, c_1 + c_2 + c_3, c_2 + 2c_4\} < 1$$

then $T$ has a unique fixed point $x$ in $X$ and $T$ is continuous at $x$.

**Proof.** Let $a_0 \in X$ and the sequence $\{a_n\}$ be defined by

$$T^n a_0 = a_n.$$

Assume that $a_n \neq a_{n+1}$ for all $n$. Using the inequality 2 we obtain

$$S(a_n, a_n, a_{n+1}) = S(Ta_{n-1}, Ta_{n-1}, Ta_n) \leq c_1 S(a_{n-1}, a_{n-1}, a_n) + c_2 S(a_n, a_n, a_n) + c_3 S(Ty, Ty, x) + c_4 \max\{S(Tx, Tx, x), S(Ty, Ty, y)\}.$$

Using the condition $(CS3)$, we get

$$S(a_{n+1}, a_{n+1}, a_{n+1}) \leq 2S(a_{n+1}, a_{n+1}, a_n) + S(a_{n-1}, a_{n-1}, a_n).$$

Hence using the inequalities (3), (4) and Lemma 2.5, we have

$$S(a_n, a_n, a_{n+1}) \leq c_1 S(a_{n-1}, a_{n-1}, a_n) + 2c_3 S(a_{n+1}, a_{n+1}, a_n) + c_3 S(a_n, a_n, a_n).$$
we obtain
\[ (1 - 2c_3 - c_4)S(a_n, a_n, a_{n+1}) \leq (c_1 + c_3 + c_4)S(a_{n-1}, a_{n-1}, a_n) \]
and
\[ S(a_n, a_n, a_{n+1}) \leq \frac{c_1 + c_3 + c_4}{1 - 2c_3 - c_4}S(a_{n-1}, a_{n-1}, a_n). \] (5)

Let \( c = \frac{c_1 + c_3 + c_4}{1 - 2c_3 - c_4} \). Then we find \( c < 1 \) since \( c_1 + 3c_3 + 2c_4 < 1 \). Using the inequality (6) and the condition (CS3), we have
\[ S(a_n, a_n, a_{n+1}) \leq c^n S(a_0, a_0, a_1). \] (6)

For all \( n, m \in \mathbb{N}, n < m \), using the inequality (6) and the condition (CS3), we have
\[ S(a_n, a_n, a_m) \leq 2S(a_n, a_n, a_{n+1}) + 2S(a_{n+1}, a_{n+1}, a_{n+2}) + \cdots + 2S(a_{m-1}, a_{m-1}, a_m) \leq 2(c^n + c^{n+1} + \cdots + c^{m-1})S(a_0, a_0, a_1) \leq 2c^n(1 + c + \cdots + c^{m-n})S(a_0, a_0, a_1) \leq 2c^n \frac{1 - c^{m-n}}{1 - c} S(a_0, a_0, a_1) \leq 2c^n S(a_0, a_0, a_1), \]
which implies
\[ |S(a_n, a_n, a_m)| \leq \frac{2c^n}{1 - c} |S(a_0, a_0, a_1)|. \]

Therefore \( |S(a_n, a_n, a_m)| \to 0 \) as \( n, m \to \infty \). Hence \( \{a_n\} \) is a Cauchy sequence. Since \((X, S)\) is complete, there exists \( x \in X \) such that \( \{a_n\} \) converges to \( x \).

Now we show that \( x \) is a fixed point of \( T \). Suppose that \( Tx \neq x \). Then we get
\[ S(a_n, a_n, Tx) = S(Ta_{n-1}, Ta_{n-1}, Tx) \leq c_1 S(a_{n-1}, a_{n-1}, x) + c_2 S(a_n, a_n, x) + c_3 S(Tx, Tx, a_{n-1}) + c_4 \max\{S(a_n, a_n, a_{n-1}), S(Tx, Tx, x)\} \]
and
\[ |S(a_n, a_n, Tx)| \leq c_1 |S(a_{n-1}, a_{n-1}, x)| + c_2 |S(a_n, a_n, x)| + c_3 |S(Tx, Tx, a_{n-1})| + c_4 \max\{S(a_n, a_n, a_{n-1}), S(Tx, Tx, x)\}. \]

If we take limit for \( n \to \infty \), then using the continuity of \( S \) and Lemma 2.5, we have
\[ |S(x, x, Tx)| = |S(Tx, Tx, x)| \leq (c_3 + c_4) |S(Tx, Tx, x)|, \]
which is a contradiction since \( 0 \leq c_3 + c_4 < 1 \). Hence we obtain \( Tx = x \).

Now we show that \( x \) is unique. Let \( y \) be another fixed point of \( T \) such that \( x \neq y \). Using the inequality (2) and Lemma 2.5, we have
\[ S(Tx, Tx, Ty) = S(x, x, y) \leq c_1 S(x, x, y) + c_2 S(x, x, y) + c_3 S(y, y, y) + c_4 \max\{S(x, x, x), S(y, y, y)\} \]
and
\[ |S(x, x, y)| \leq (c_1 + c_2 + c_3) |S(x, x, y)|, \]
which implies \( x = y \) since \( c_1 + c_2 + c_3 < 1 \).

Now we prove that \( T \) is continuous at \( x \). For \( n \in \mathbb{N} \), using the inequality (2), we get
\[
S(Ta_n, Ta_n, Tx) \leq c_1 S(a_n, a_n, x) + c_2 S(Ta_n, Ta_n, x) + c_3 S(Tx, Tx, a_n) + c_4 \max\{S(Ta_n, Ta_n, a_n), S(Tx, Tx, x)\}. \tag{7}
\]
Using the condition \((CS3)\), the inequality (7) and Lemma 2.5, we obtain
\[
S(Ta_n, Ta_n, Tx) \leq c_1 S(a_n, a_n, x) + c_2 S(Ta_n, Ta_n, x) + c_3 S(Tx, Tx, a_n) + 2c_4 S(a_n, a_n, x)
+ c_4 \max\{S(Ta_n, Ta_n, x), S(Tx, Tx, x)\},
\]
and
\[ (1 - c_2 - 2c_4)S(Ta_n, Ta_n, Tx) \leq (c_1 + c_3 + c_4)S(a_n, a_n, x), \]
which implies
\[ |S(Ta_n, Ta_n, Tx)| \leq \frac{c_1 + c_3 + c_4}{1 - c_2 - 2c_4} |S(a_n, a_n, x)|. \]
If we take limit for \( n \to \infty \), then we have
\[ |S(Ta_n, Ta_n, Tx)| \to 0. \]
Therefore \( \{Ta_n\} \) is convergent to \( Tx = x \). Consequently, \( T \) is continuous at \( x \) by Lemma 2.6. \( \square \)

**Remark 3.2.** (1) Theorem 3.1 is a generalization of the Banach’s contraction principle on complete complex valued \( S \)-metric spaces. Indeed, if we take \( c_1 = h \) and \( c_2 = c_3 = c_4 = 0 \) in Theorem 3.1, then we obtain the Banach’s contraction condition in Theorem 2.7.

(2) If we take the function \( S : X \times X \times X \to [0, \infty) \) in Theorem 3.1, Then we have Theorem 3 in [6].

**Corollary 3.3.** Let \( (X, S) \) be a complete complex valued \( S \)-metric space and \( T \) be a self-mapping of \( X \). If there exist nonnegative real numbers \( c_1, c_2, c_3, c_4 \) satisfying
\[ \max\{c_1 + 3c_3 + 2c_4, c_1 + c_2 + c_3, c_2 + 2c_4\} < 1 \]
such that
\[
S(T^p x, T^p x, T^p y) \leq c_1 S(x, x, y) + c_2 S(T^p x, T^p x, y) + c_3 S(T^p y, T^p y, x) + c_4 \max\{S(T^p x, T^p x, x), S(T^p y, T^p y, y)\},
\]
for all \( x, y \in X \) and some \( p \in \mathbb{N} \), then \( T \) has a unique fixed point \( x \) in \( X \) and \( T^p \) is continuous at \( x \).
Proof. Using the similar arguments in Theorem 3.1, we can easily see that $T^p$ has a unique fixed point $x$ in $X$ and $T^p$ is continuous at $x$. Also we obtain

$$Tx = T^p x = T^{p+1} x = T^pTx,$$

which implies that $Tx$ is a fixed point of $T^p$. Consequently we have $Tx = x$ since $x$ is a unique fixed point. \qed

**Theorem 3.4.** Let $(X, S)$ be a complete complex valued $S$-metric space and $T$ be a self-mapping of $X$. If there exist nonnegative real numbers $c_1, c_2, c_3, c_4, c_5, c_6$ satisfying $\max\{c_1 + c_2 + 3c_4 + c_5 + 3c_6, c_1 + c_3 + c_4 + c_6, 2c_2 + c_3 + 2c_6\} < 1$ such that

$$S(Tx, Tx, Ty) \leq c_1 S(x, x, y) + c_2 S(Tx, Tx, x) + c_3 S(Tx, Tx, y)$$

(8)

$$+ c_4 S(Ty, Ty, x) + c_5 S(Ty, Ty, y) + c_6 \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, x), S(Ty, Ty, y)\},$$

for all $x, y \in X$, then $T$ has a unique fixed point $x$ in $X$ and $T$ is continuous at $x$.

**Proof.** Let $a_0 \in X$ and the sequence $\{a_n\}$ be defined by

$$T^n a_0 = a_n.$$

Assume that $a_n \neq a_{n+1}$ for all $n$. Using the inequality 8, the condition $(CS3)$ and Lemma 2.5, we obtain

$$S(a_n, a_n, a_{n+1}) = S(Ta_{n-1}, Ta_{n-1}, Ta_n) \leq c_1 S(a_{n-1}, a_{n-1}, a_n) + c_2 S(a_n, a_n, a_{n-1}) + c_3 S(a_{n-1}, a_{n-1}, a_n) + c_4 S(Ta_{n-1}, Ta_{n-1}, a_n) + c_5 S(Ta_{n-1}, Ta_{n-1}, a_n) + c_6 \max\{S(a_{n-1}, a_{n-1}, a_n), S(a_{n-1}, a_{n-1}, a_n), S(a_n, a_n, a_{n-1}), S(a_n, a_n, a_{n-1}), S(a_{n-1}, a_{n-1}, a_n), S(a_{n-1}, a_{n-1}, a_n)\}$$

(9)

and

$$S(a_n, a_n, a_{n+1}) \leq \frac{c_1 + c_2 + c_4 + c_6}{2c_4 + c_5 + 2c_6} S(a_{n-1}, a_{n-1}, a_n).$$

Let $c = \frac{c_1 + c_2 + c_4 + c_6}{2c_4 + c_5 + 2c_6}$. Then we find $c < 1$ since $c_1 + c_2 + 3c_4 + c_5 + 3c_6 < 1$. Using the inequality (9), we obtain

$$S(a_n, a_n, a_{n+1}) \leq c^n S(a_0, a_0, a_1).$$

(10)

For all $n, m \in \mathbb{N}$, $n < m$, using the inequality (10) and the condition $(CS3)$, we have

$$S(a_n, a_n, a_m) \leq \frac{2c^n}{1-c} S(a_0, a_0, a_1).$$
which implies

\[ |S(a_n, a_m)| \leq \frac{2c^n}{1-c} |S(a_0, a_1)|. \]

Therefore \( |S(a_n, a_m)| \to 0 \) as \( n, m \to \infty \). Hence \( \{a_n\} \) is a Cauchy sequence. Since \((X, S)\) is complete, there exists \( x \in X \) such that \( \{a_n\} \) converges to \( x \).

Now we show that \( x \) is a fixed point of \( T \). Suppose that \( Tx \neq x \). Then we get

\[
S(a_n, a_n, Tx) = S(Ta_{n-1}, Ta_{n-1}, Tx) \leq c_1 S(a_{n-1}, a_{n-1}, x) + c_2 S(a_n, a_n, a_{n-1}) + c_3 S(a_n, a_n, x) + c_4 S(Tx, Tx, a_{n-1}) + c_5 S(Tx, Tx, x) + c_6 \max\{S(a_{n-1}, a_{n-1}, x), S(a_n, a_n, a_{n-1}), S(a_n, a_n, x), S(Tx, Tx, a_{n-1}), S(Tx, Tx, x)\}
\]

and

\[
|S(a_n, a_n, Tx)| \leq c_1 |S(a_{n-1}, a_{n-1}, x)| + c_2 |S(a_n, a_n, a_{n-1})| + c_3 |S(a_n, a_n, x)| + c_4 |S(Tx, Tx, a_{n-1})| + c_5 |S(Tx, Tx, x)| + c_6 \max\{S(a_{n-1}, a_{n-1}, x), S(a_n, a_n, a_{n-1}), S(a_n, a_n, x), S(Tx, Tx, a_{n-1}), S(Tx, Tx, x)\}.
\]

If we take limit for \( n \to \infty \), then using the continuity of \( S \) and Lemma 2.5, we have

\[
|S(Tx, Tx, x)| \leq (c_4 + c_5 + c_6) |S(Tx, Tx, x)|,
\]

which is a contradiction since \( 0 \leq c_4 + c_5 + c_6 < 1 \). Hence we obtain \( Tx = x \).

Now we show that \( x \) is unique. Let \( y \) be another fixed point of \( T \) such that \( x \neq y \).

Using the inequality (8) and Lemma 2.5, we have

\[
S(Tx, Tx, Ty) = S(x, x, y) \leq c_1 S(x, x, y) + c_2 S(x, x, x) + c_3 S(x, x, y) + c_4 S(y, y, x) + c_5 S(y, y, y) + c_6 \max\{S(x, x, y), S(x, x, x), S(y, y, y)\}
\]

and

\[
|S(x, x, y)| \leq (c_1 + c_3 + c_4 + c_6) |S(x, x, y)|,
\]

which implies \( x = y \) since \( c_1 + c_3 + c_4 + c_6 < 1 \).

Now we prove that \( T \) is continuous at \( x \). For \( n \in \mathbb{N} \), using the inequality (8), the
condition (CS3) and Lemma 2.5, we obtain
\[
S(Ta_n, Ta_n, Tx) \leq c_1 S(a_n, a_n, x) + c_2 S(Ta_n, Ta_n, x) + c_3 S(Ta_n, Ta_n, x)
\]
\[+ c_4 S(Tx, Tx, a_n) + c_5 S(Tx, Tx, x)
\]
\[+ c_6 \max \{S(a_n, a_n, x), S(Ta_n, Ta_n, a_n), S(Ta_n, Ta_n, x), S(Tx, Tx, a_n), S(Tx, Tx, x)\}\]
\[\leq c_1 S(a_n, a_n, x) + 2c_2 S(Ta_n, Ta_n, x) + c_2 S(a_n, a_n, x)
\]
\[+ c_3 S(Ta_n, Ta_n, x) + c_4 S(Tx, Tx, a_n)
\]
\[+ c_5 \max \{S(a_n, a_n, x), S(Ta_n, Ta_n, x) + S(a_n, a_n, x), S(Ta_n, Ta_n, x)\}\]
\[= (c_1 + c_2 + c_4 + c_6) S(a_n, a_n, x) + (2c_2 + c_3 + 2c_5) S(Tx, Tx, Ta_n)\]
and
\[
(1 - 2c_2 - c_3 - 2c_6) S(Ta_n, Ta_n, Tx) \leq (c_1 + c_2 + c_4 + c_6) S(a_n, a_n, x),
\]
which implies
\[
|S(Ta_n, Ta_n, Tx)| \leq \frac{c_1 + c_2 + c_4 + c_6}{1 - 2c_2 - c_3 - 2c_6} |S(a_n, a_n, x)|.
\]

If we take limit for \(n \to \infty\), then we have
\[
|S(Ta_n, Ta_n, Tx)| \to 0.
\]
Therefore \(\{Ta_n\}\) is convergent to \(Tx = x\). Consequently, \(T\) is continuous at \(x\) by Lemma 2.6.

**Remark 3.5.** (1) Theorem 3.4 is a generalization of Banach’s contraction principle
on complete complex valued S-metric spaces. Indeed, if we take \(c_1 = h\) and \(c_2 = c_3 = c_4 = c_5 = c_6 = 0\) in Theorem 3.4, then we obtain the Banach’s contraction
condition in Theorem 2.7.

(2) If we take the function \(S : X \times X \times X \to [0, \infty)\) in Theorem 3.4, Then we have Theorem 4 in [6].

**Corollary 3.6.** Let \((X, S)\) be a complete complex valued S-metric space and \(T\)
be a self-mapping of \(X\). If there exist nonnegative real numbers \(c_1, c_2, c_3, c_4, c_5, c_6\)
satisfying \(\max \{c_1 + c_2 + 3c_4 + c_5 + 3c_6, c_1 + c_3 + c_4 + c_6, 2c_2 + c_3 + 2c_6\} < 1\) such that
\[
S(T^p x, T^p x, T^p y) \leq c_1 S(x, x, y) + c_2 S(T^p x, T^p x, x) + c_3 S(T^p x, T^p y, x)
\]
\[+ c_4 S(T^p y, T^p y, x) + c_5 S(T^p y, T^p y, y) + c_6 \max \{S(x, x, y), S(T^p x, T^p x, x), S(T^p x, T^p x, y), S(T^p y, T^p y, x), S(T^p y, T^p y, y)\},
\]
for all \(x, y \in X\) and some \(p \in \mathbb{N}\), then \(T\) has a unique fixed point \(x\) in \(X\) and \(T^p\) is continuous at \(x\).
Proof. It follows from Theorem 3.4 by the same argument used in the proof of Corollary 3.3.

In the following example we give a self-mapping satisfying the conditions of our results, but does not satisfy the condition of the Banach’s contraction principle.

Example 3.7. Let $X = \mathbb{R}$ and the function $S : X \times X \times X \to \mathbb{C}$ be defined as

$$S(x, y, z) = e^{it}(|x - z| + |x + z - 2y|),$$

for all $x, y, z, t \in \mathbb{R}$. Then $(\mathbb{R}, S)$ is a complete complex valued $S$-metric space. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} x + 70 & \text{if } x \in \{0, 6\} \\ 65 & \text{if otherwise} \end{cases},$$

for all $x \in \mathbb{R}$. Therefore $T$ satisfies the inequality (2) in Theorem 3.1 for $c_1 = c_2 = c_3 = 0$, $c_4 = \frac{1}{4}$ and the inequality (8) in Theorem 3.4 for $c_1 = c_3 = c_4 = c_5 = 0$, $c_2 = c_6 = \frac{1}{5}$. So $T$ has a unique fixed point $x = 65$. But $T$ does not satisfy the Banach’s contraction condition in Theorem 2.7. Indeed, for $x = 6$, $y = 2$, we obtain

$$S(Tx, Tx, Ty) = S(76, 76, 65) = 22e^{it} \leq hS(x, x, y) = hS(6, 6, 2) = 8he^{it}$$

and

$$|22e^{it}| = 22 \leq |8he^{it}| = 8h,$$

which is a contradiction $h < 1$.

References


