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NORMS OVER FUZZY LIE ALGEBRA

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Abstaract − In this paper we introduce the concept of fuzzy Lie ideal and anti fuzzy Lie ideal by using a t-norm T and a t-conorm C, respectively. Next we introduce the concept of quotient fuzzy Lie ideal with respect to t-norm T. We investigate some their properties and obtain new results.

Keywords − *Lie algebra, ideals, fuzzy set theory, t-norm.*

1 Introduction

Lie algebras were first discovered by Sophus Lie (1842-1899) when he attempted to classify certain "smooth" subgroups of general linear groups. Lie algebra is applied in different domains of physics and mathematics, such as spectroscopy of molecules, atoms, nuclei, hadrons, hyperbolic and stochastic differential equations. The notion of fuzzy sets was first introduced by Zadeh[4]. Fuzzy and anti fuzzy Lie ideals in Lie algebras have been studied in[1, 2, 3]. In this paper we have tried apply the concepts of norms to fuzzy Lie algebras and fuzzy Lie ideals.

2 Preliminary

In this section, we first review some elementary aspects that are necessary for this paper.
A Lie algebra is a vector space L over a field F (equal to R or C) on which $L \times L \to L$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:
(1) $[x, y]$ is bilinear,
(2) $[x, x] = 0$ for all $x \in L$,
(3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity),
for all $x, y, z \in L$.

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In this paper by L will be denoted a Lie algebra. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is anti commutative, i.e., $[x, y] = -[y, x]$. A subspace H of L closed under $[,]$ will be called a Lie subalgebra. A subspace I of L with the property $[I, L] \subseteq I$ will be called a Lie ideal of L. Obviously, any Lie ideal is a subalgebra.
A t-norm T is a function $T : [0, 1]\times[0, 1] \to [0, 1]$ having the following four properties:
(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$. Replacing 1 by 0 in condition (T1), we obtain the concept of t-conorm C. If T be a t-norm, then

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$ and we can replace t-conorm C by t-norm T. Recall that $T(C)$ is idempotent if for all $x \in [0, 1]$, $T(x, x) = x(C(x, x) = x)$.
Let L_1 and L_2 be Lie algebras over a field F. A linear transformation $f : L_1 \to L_2$ is called a Lie homomorphism if $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L_1$.
Let T and C be t-norm and t-conorm, respectively. For all $x, y \in [0, 1]$, we say T and C are dual when

$$T(x, y) = 1 - C(1 - x, 1 - y),$$
$$C(x, y) = 1 - T(1 - x, 1 - y).$$

Let $\mu : L \to [0, 1]$. The complement of μ, denoted by μ^c is the fuzzy set in L given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in L$.

3 Fuzzy Lie Subalgebra with Respect to a t-norm

In this section, we define the notion of fuzzy Lie subalgebra of L with respect to a t-norm T and investigate some related properties.

Definition 3.1. Let μ be a fuzzy set on L, i.e., a map $\mu : L \to [0, 1]$. A fuzzy set $\mu : L \to [0, 1]$ is called a fuzzy Lie subalgebra of L with respect to a t-norm T if
(1) $\mu(x + y) \geq T(\mu(x), \mu(y))$,
(2) $\mu(\alpha x) \geq \mu(x)$,
(3) $\mu([x, y]) \geq T(\mu(x), \mu(y))$
hold for all $x, y \in L$ and $\alpha \in F$.
A fuzzy subset $\mu : L \to [0, 1]$ satisfying (1), (2) and
(4) $\mu([x, y]) \geq \mu(x)$
is called a fuzzy Lie ideal of L with respect to a t-norm T.

Example 3.2. Let $L = R^3$ and $[x, y] = x \times y$, where $\times$ is cross product, for all $x, y \in L$. By routine calculations, it is clear that L is a Lie algebra over a field R.

Define $\mu : L \to [0,1]$ by

$$\mu(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 = x_2 = x_3 = 0 \\ 0.50 & \text{if } x_1 = x_2 = 0 \; and \; x_3 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

if $T(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$, then μ is a fuzzy Lie subalgebra of L with respect to a t-norm T.

Lemma 3.3. Let μ be a fuzzy Lie subalgebra of L with respect to a t-norm T.
(1) If T be idempotent, then for all $x \in L$ we have that $\mu(0) \geq \mu(x)$.
(2) $\mu([x,y]) = \mu([y,x])$.

Proof. (1) Let μ be a fuzzy Lie subalgebra of L with respect to a t-norm T and $x \in L$. Then $\mu(0) = \mu(x + (-x)) \geq T(\mu(x), \mu(-x)) \geq T(\mu(x), \mu(x)) = \mu(x)$.
(2) $\mu([x,y]) = \mu(-[y,x]) \geq \mu([y,x]) = \mu(-[x,y]) \geq \mu([x,y])$.

Proposition 3.4. Let μ be a fuzzy Lie ideal in a Lie algebra L with respect to a t-norm T such that T be idempotent. Then for all $t \in [0,1]$ the set $L(\mu, t) = \{x \in L \mid \mu(x) \geq t\}$ is a Lie ideal of L.

Proof. Let $x, y \in L(\mu, t)$ and $\alpha \in F$. Then $\mu(x+y) \geq T(\mu(x), \mu(y) \geq T(t,t) = t$ and $\mu(\alpha x) \geq \mu(x) \geq t$. Therefore $x + y, \alpha x \in L(\mu, t)$. Also if $x \in L(\mu, t)$ and $y \in L$, then from $\mu([x,y]) \geq \mu(x) \geq t$ we have that $[x,y] \in L(\mu, t)$. This completes the proof.

Definition 3.5. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. Let $\mu : L_1 \to [0,1]$ and $\nu : L_2 \to [0,1]$ be two fuzzy sets of L_1 and L_2 respectively. For all $x \in L_1$ and $y \in L_2$ define

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in L_1, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$.

Proposition 3.6. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If μ is a fuzzy Lie ideal of L_1 with respect to a t-norm T, then $f(\mu)$ is a fuzzy Lie ideal of L_2 with respect to a t-norm T.

Proof. Let $x_1, x_2 \in L_1$ and $y_1, y_2 \in L_2$. If $y_1 = f(x_1)$ and $y_2 = f(x_2)$, then
(1)

$$\begin{aligned} f(\mu)(y_1 + y_2) &= \sup\{\mu(x_1 + x_2) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &= T(\sup\{\mu(x_1) \mid y_1 = f(x_1)\}, \sup\{\mu(x_2) \mid y_2 = f(x_2)\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)). \end{aligned}$$

(2) $f(\mu)(\alpha y_1) = \sup\{\mu(\alpha x_1) \mid \alpha y_1 = f(\alpha x_1) = \alpha f(x_1)\} \geq \sup\{\mu(x_1) \mid x_1 = f(y_1)\} = f(\mu)(y_1)$.
(3) $f(\mu)([y_1, y_2]) = \sup\{\mu([x_1, x_2]) \mid y_1 = f(x_1), y_2 = f(x_2)\} \geq \sup\{\mu(x_1) \mid y_1 = f(x_1)\} = f(\mu)(y_1)$.

Proposition 3.7. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If ν is a fuzzy Lie ideal of L_2 with respect to a t-norm T, then $f^{-1}(\nu)$ is a fuzzy Lie ideal of L_1 with respect to a t-norm T.

Proof. Let $x, y \in L_1$ and $\alpha \in F$. Then

$$\begin{aligned} f^{-1}(\nu)(x+y) &= \nu(f(x+y)) = \nu(f(x)+f(y)) \\ &\geq T(\nu(f(x)), \nu(f(y))) = T(f^{-1}(\nu)(x), f^{-1}(\nu)(y)), \end{aligned}$$

$f^{-1}(\nu)(\alpha x) = \nu(f(\alpha x)) = \nu(\alpha f(x)) \geq \nu(f(x)) = f^{-1}(\nu)(x)$, and
$f^{-1}(\nu)([x,y]) = \nu(f([x,y]) = \nu([f(x), f(y)]) \geq \nu(f(x)) = f^{-1}(\nu)(x)$.
Thus $f^{-1}(\nu)$ is a fuzzy Lie ideal of L_1 with respect to a t-norm T.

Definition 3.8. let μ and ν be fuzzy Lie ideals of a Lie algebra L with respect to a t-norm T. Define the intersection of μ and ν the function $\mu \cap \nu : L \to [0,1]$ such that $(\mu \cap \nu)(x) = T(\mu(x), \nu(x))$ for all $x \in L$.

Proposition 3.9. let μ and ν be two fuzzy Lie ideals of a Lie algebra L with respect to a t-norm T such that T be idempotent. Then $\mu \cap \nu$ be a fuzzy Lie ideal in a Lie algebra L with respect to a t-norm T.

Proof. Let $x, y \in L$ and $\alpha \in F$. Then
(1)

$$(\mu \cap \nu)(x+y) = T(\mu(x+y), \nu(x+y)) \geq T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y)))$$

$$= T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) = T((\mu \cap \nu)(x), (\mu \cap \nu)(y)).$$

(2) $(\mu \cap \nu)(\alpha x) = T(\mu(\alpha x), \nu(\alpha x)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x)$.
(3) $(\mu \cap \nu)([x,y]) = T(\mu([x,y]), \nu([x,y])) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x)$.
Hence $\mu \cap \nu$ be a fuzzy Lie ideal in a Lie algebra L with respect to a t-norm T.

Next we will introduce the concept of quotient fuzzy Lie ideal.

Definition 3.10. Let L be a Lie algebra, $\mu : L \to [0,1]$ and I be an ideal of L. Define $\mu_{L/I} : L/I \to [0,1]$ by

$$\mu_{L/I}(x+I) = \begin{cases} T(\mu(x), \mu(i)) & \text{if } x \neq i \\ 1 & \text{if } x = i \end{cases}$$

for all $x \in L$ and $i \in I$.

Proposition 3.11. Let μ be a fuzzy Lie ideal of L with respect to a t-norm T. If T be idempotent, then $\mu_{L/I}$ will be a fuzzy Lie ideal of L/I with respect to a t-norm T.

Proof. Let $x+I, y+I \in L/I$ and $i \in I$ such that $x \neq i \neq y$.
(1)

$$\mu_{L/I}((x+I)+(y+I)) = \mu_{L/I}((x+y)+I) = T(\mu(x+y), \mu(i))$$
$$\geq T(T(\mu(x), \mu(y)), \mu(i)) = T(T(\mu(x), \mu(y)), T(\mu(i), \mu(i)))$$
$$= T(T(\mu(x), \mu(i)), T(\mu(y), \mu(i))) = T(\mu_{L/I}(x+I), \mu_{L/I}(y+I)).$$

(2) $\mu_{L/I}(\alpha(x+I)) = \mu_{L/I}(\alpha x+I) = T(\mu(\alpha x), \mu(i)) \geq T(\mu(x), \mu(i)) = \mu_{L/I}(x+I)$.
(3) $\mu_{L/I}([x,y]+I) = T(\mu([x,y]), \mu(i)) \geq T(\mu(x), \mu(i)) = \mu_{L/I}(x+I)$.

4 Anti Fuzzy Lie Subalgebra with Respect to a t-conorm

Definition 4.1. A fuzzy set $\mu : L \to [0,1]$ is called an anti fuzzy Lie subalgebra of L with respect to a t-conorm C if
(1) $\mu(x+y) \leq C(\mu(x), \mu(y))$,
(2) $\mu(\alpha x) \leq \mu(x)$,
(3) $\mu([x,y]) \leq T(\mu(x), \mu(y))$
hold for all $x, y \in L$ and $\alpha \in F$.
A fuzzy subset $\mu : L \to [0,1]$ satisfying (1), (2) and
(4) $\mu([x,y]) \leq \mu(x)$
is called an anti fuzzy Lie ideal of L with respect to a t-conorm C.

Proposition 4.2. Let μ be an anti fuzzy Lie ideal in a Lie algebra L with respect to a t-conorm C such that C be idempotent. Then for all $t \in [0,1]$ the set $L(\mu, t) = \{x \in L \mid \mu(x) \leq t\}$ is a Lie ideal of L.

Proof. Let $x, y \in L(\mu, t)$ and $\alpha \in F$. Then $\mu(x+y) \leq C(\mu(x), \mu(y) \leq C(t,t) = t$ and $\mu(\alpha x) \leq \mu(x) = t$. Therefore $x+y, \alpha x \in L(\mu, t)$. Also if $x \in L(\mu, t)$ and $y \in L$, then from $\mu([x,y]) \leq \mu(x) \leq t$ we have that $[x,y] \in L(\mu, t)$. Thus $L(\mu, t)$ will be a Lie ideal of L.

Definition 4.3. let μ and ν be anti fuzzy Lie ideals of a Lie algebra L with respect to a t-conorm C. Define the union of μ and ν the function $\mu \cup \nu : L \to [0,1]$ such that $(\mu \cup \nu)(x) = C(\mu(x), \nu(x))$ for all $x \in L$.

Proposition 4.4. let μ and ν be two anti fuzzy Lie ideals of a Lie algebra L with respect to a t-conorm C such that C be idempotent. Then $\mu \cup \nu$ be an anti fuzzy Lie ideal in a Lie algebra L with respect to a t-conorm C.

Proof. Let $x, y \in L$ and $\alpha \in F$. Then
(1)

$$(\mu \cup \nu)(x+y) = C(\mu(x+y), \nu(x+y)) \leq C(C(\mu(x), \mu(y)), C(\nu(x), \nu(y)))$$
$$= C(C(\mu(x), \nu(x)), C(\mu(y), \nu(y))) = C((\mu \cup \nu)(x), (\mu \cup \nu)(y)).$$

(2) $(\mu \cup \nu)(\alpha x) = C(\mu(\alpha x), \nu(\alpha x)) \leq C(\mu(x), \nu(x)) = (\mu \cup \nu)(x)$.
(3) $(\mu \cup \nu)([x, y]) = C(\mu([x, y]), \nu([x, y])) \leq C(\mu(x), \nu(x)) = (\mu \cup \nu)(x)$.
Hence $\mu \cup \nu$ be an anti fuzzy Lie ideal in a Lie algebra L with respect to a t-conorm C.

Proposition 4.5. Let $f : L_1 \to L_2$ be an epimorphism of Lie algebras. If ν be an anti fuzzy Lie ideal of L_2 with respect to a t-conorm C, then $f^{-1}(\nu)$ will be an anti fuzzy Lie ideal of L_1 with respect to a t-conorm C.

Proof. Let $x, y \in L_1$ and $\alpha \in F$. Then

$$\begin{aligned} f^{-1}(\nu)(x+y) &= \nu(f(x+y)) = \nu(f(x) + f(y)) \\ &\leq C(\nu(f(x)), \nu(f(y))) = C(f^{-1}(\nu)(x), f^{-1}(\nu)(y)), \end{aligned}$$

$f^{-1}(\nu)(\alpha x) = \nu(f(\alpha x)) = \nu(\alpha f(x)) \leq \nu(f(x)) = f^{-1}(\nu)(x)$, and
$f^{-1}(\nu)([x, y]) = \nu(f([x, y]) = \nu([f(x), f(y)]) \leq \nu(f(x)) = f^{-1}(\nu)(x)$.
Therefore $f^{-1}(\nu)$ is an anti fuzzy Lie ideal of L_1 with respect to a t-conorm C.

Proposition 4.6. Let L be a Lie algebra and $\mu : L \to [0, 1]$. Then μ be a fuzzy Lie ideal of L with respect to a t-norm T if and only if μ^c be an anti fuzzy Lie ideal of L with respect to a t-conorm C.

Proof. Let μ be a fuzzy Lie ideal of L with respect to a t-norm T and $x, y \in L$ and $\alpha \in F$.
(1) From $\mu(x + y) \geq T(\mu(x), \mu(y))$ we have

$$1 - \mu^c(x + y) \geq T(1 - \mu^c(x), 1 - \mu^c(y)),$$

which implies that

$$\begin{aligned} \mu^c(x + y) &\leq 1 - T(1 - \mu^c(x), 1 - \mu^c(y)) \\ &= C(\mu^c(x), \mu^c(y)). \end{aligned}$$

(2) $\mu^c(\alpha x) = 1 - \mu(\alpha x) \leq 1 - \mu(x) = \mu^c(x)$.
(3) $\mu^c([x, y]) = 1 - \mu([x, y]) \leq 1 - \mu(x) = \mu^c(x)$.
Hence μ^c will be an anti fuzzy Lie ideal of L with respect to a t-conorm C.
Converse also can be proved similarly.

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