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NATURAL TRANSFORM AND SPECIAL FUNCTIONS

Anil Dhondiram Chindhe[1,*] <anilchindhe5@gmail.com>
Sakharam Kiwne[2] <sbkdcamath@gamil.com>

[1]Department of Mathematics, Balbhim College, Beed, Maharashtra, India
[2]Department of Mathematics, Deogiri College, Aurangabad, Maharashtra, India

Abstaract − The branch of integral transform attract many researcher in this field and hence various types of integral transforms are introduced. Natural transform is one of the newly defined transform which has wide range of applications in science and engineering field. In this paper we derived the Natural transform of some special functions.

Keywords − *Bessel's function, Hermite Polynomial, Hypergeometric function, Legendre Polynomials, Leguerre Polynomial,Natural Transform.*

1 Introduction

The Natural transform was established by Khan and Khan[1] as N - transform who studied its properties and application as unsteady fluid flow problem over a plane wall.Later on Belgacem [2, 3] defined the inverse Natural transform and studied some properties and applications of Natural transforms.In the literature survey we can see the further applications of Natural transform.[4, 5, 6, 7] The specialty of Natural transform is that it can converges to Laplace transform and Sumudo transform [8] just by changing the parameter.Natural transform is the theoretical dual of Laplace transform.We can derive Laplace, Sumudu, Fourier and Mellin transform from Natural transform.[9] Natural transform plays as a source for other transform and hence can be used to solve many complicated problems in engineering, fluid mechanics and other scientific discipline like Physics, Chemistry and Dynamics etc.

1.1 Preliminary Definition of Natural Transform

The Natural transform of the function $f(t) \in \Re^2$ is given by the following integral equation [3]

[*] *Corresponding Author.*

$$\mathbb{N}[f(t)] = R(s,u) = \int_0^\infty e^{-st} f(ut)dt \tag{1}$$

where $Re(s) > 0$, u $\in$ (τ_1, τ_2) provided the function $f(t)\in\Re^2$ is defined in the set

A=[$f(t)/\exists$ M,τ_1, $\tau_2 > 0$,|f(t)| < M $e^{\frac{|t|}{\tau_j}}$, if t $\in(-1)^j \times [0,\infty)$]

The inverse Natural transform related with Bromwich contour integral[2, 3] is defined by

$$\mathbb{N}^{-1}[R(s,u)] = f(t) = \lim_{T\to\infty} \frac{1}{2\Pi i} \int_{\gamma-iT}^{\gamma+iT} e^{\frac{st}{u}} R(s,u)ds \tag{2}$$

1.2 Some Standard Result of Natural Transform

In this section we assume that all the considered functions are such that their Natural tranform exists.[1],[3]

1. $\mathbb{N}[1] = \frac{1}{s}$
2. $\mathbb{N}[t] = \frac{u}{s^2}$
3. $\mathbb{N}[t^n]=\frac{u^n}{s^{n+1}}n!$
4. $\mathbb{N}[e^{at}] = \frac{1}{s-au}$
5. $\mathbb{N}[\frac{sin(at)}{a}] = \frac{u}{s^2+s^2u^2}$
6. $\mathbb{N}[cos(at)] = \frac{s}{s^2+s^2u^2}$
7. $\mathbb{N}[\frac{t^{n-1}e^{at}}{(n-1)!}] = \frac{u^{n-1}}{(s-au)^2}$
8. $\mathbb{N}[f^{(n)}(t)] = \frac{s^n}{u^n}.R(s,u) - \sum_{n=0}^{\infty} \frac{s^{n-(k+1)}}{u^{n-k}}.u^{(k)}(0)$
   where $f^{(n)}(t) = \frac{d^n f}{dt^n}$
9. The Convolution Theorem

   If F(s,u) and G(s,u) are the Natural transforms of respective functions $f(t)$ and $g(t)$ both defined in set A then ,
   $\mathbb{N}[(f * g)]=u.F(s,u)G(s,u)$

1.3 Pochhamber Symbol

The pochhamber symbol denoted by $(\alpha)_n$ is defined by[10] the equation

$$\begin{aligned}(\alpha)_n &= \alpha(\alpha+1)(\alpha+2)....(\alpha+n+1)\\ &= \prod_{m=1}^{n}(\alpha+m+1) \qquad for \quad n \geq 1\end{aligned}$$

In particular $(\alpha)_0 = 1 \quad for \quad \alpha \neq 0 \;, \quad (1)_n = n!$

1.4 Some Standard Results

1. if n is positive integer, then

$$\frac{\Gamma n}{\Gamma n+1} = (\alpha)_n$$

where α is neither zero nor a negative integer.

2. If α is not an integer,

$$\frac{\Gamma 1-\alpha-n}{\Gamma 1-\alpha} = \frac{(-1)^n}{(\alpha)_n}$$

3.

$$(1-Z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n Z^n}{n!}$$

4.

$$(\alpha)_{n-k} = \frac{(\alpha)_n(-1)^k}{(1-\alpha-n)_k} \quad 0 \leq k \leq n$$

5. If $\alpha = 1$, then

$$(-1)_{n-k} = (n-k)! = \frac{n!(-1)^k}{(-n)_k} \quad 0 \leq k \leq n$$

6.

$$(\alpha)_{2n} = 2^{2n}(\frac{\alpha}{2})_n(\frac{\alpha+1}{2})_n$$

7. The function $f(a,b,c;Z)$ is written as $F\left[\begin{array}{cc} a,b & ; \\ c & ; \end{array} \mid Z\right]$ and is defined as

$$f(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}$$

8. The Hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q\left[\begin{array}{cc} a_1, a_2, \ldots\ldots., a_p & : \\ a_1, a_2, \ldots\ldots., a_p & : \end{array} \mid Z\right] = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p}(a_k)_n z^n}{\prod_{m=1}^{q}(b_m)_n n!}$$

2 Well known special functions

1. The Bessel's Function is defined by

$$J_n(t) = \sum_{n=0}^{\infty} \frac{(-1)^k t^{2k+1}}{2^{2k+n} k! \Gamma 1 + n - k} \tag{3}$$

2 The Legendre polynomial is defined by

$$P_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (\frac{1}{2})_{n-k} (2t)^{n-2k}}{(n-2k)!k!} \tag{4}$$

3 The Hermite polynomial is defined by

$$H_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n! (2t)^{n-2k}}{(n-2k)!k!} \tag{5}$$

4 The Leguerre polynomial is defined by

$$L_n(\alpha)_t = \sum_{k=0}^{\infty} \frac{(-1)^k (1+\alpha)_n t^k}{(n-k)!k!(1+\alpha)_k} \tag{6}$$

3 Main Result

3.1 The Natural transform of Hypergeometric function

$$\begin{aligned}
\mathbb{N}\{{}_pF_q \left[\begin{array}{ccc} a_1, a_2, ..., a_p & ; & \\ a_1, a_2, ..., a_q & ; & \end{array} \mid t \right] \} &= \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n (ut)^n}{\prod_{m=1}^{q} (b_m)_n n!} dt \\
&= \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{m=1}^{q} (b_m)_n n!} \int_0^{\infty} e^{-st} (ut)^n dt \\
&= \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{m=1}^{q} (b_m)_n n!} \mathbb{N}\{t^n\} \\
&= \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{m=1}^{q} (b_m)_n n!} \frac{u^n}{s^{n+1}} n! \\
\mathbb{N}\{{}_pF_q \left[\begin{array}{ccc} a_1, a_2, ..., a_p & ; & \\ a_1, a_2, ..., a_q & ; & \end{array} \mid t \right] \} &= \frac{n!}{s} {}_pF_q \left[\begin{array}{ccc} a_1, a_2, ..., a_p & ; & \\ a_1, a_2, ..., a_q & ; & \end{array} \mid \frac{u}{s} \right]
\end{aligned}$$

In particular,

$$\begin{aligned}
\mathbb{N}\{{}_2F_1\left[\begin{array}{cc} a,b & ; \\ 1 & ; \end{array} \mid t\right]\} &= \mathbb{N}\{\sum_{n=0}^{\infty}\frac{(a)_n(b)_n t^n}{(1)_n n!}\} \\
&= \int_0^{\infty} e^{-st}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(ut)^n}{(1)_n n!}dt \\
&= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1)_n n!}\int_0^{\infty} e^{-st}(ut)^n dt \\
&= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1)_n n!}\mathbb{N}\{t^n\} \\
&= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1)_n n!}\frac{u^n}{s^{n+1}}n! \\
&= {}_2F_0\left[\begin{array}{cc} a,b & ; \\ - & ; \end{array} \mid \frac{u}{s}\right]\frac{1}{s}
\end{aligned}$$

3.2 The Natural transform of Bessel's function

$$\begin{aligned}
\mathbb{N}\{J_n(t)\} &= \mathbb{N}\{\sum_{n=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{2k+n}k!\Gamma 1+n-k}\} \\
&= \int_0^{\infty} e^{-st}\sum_{n=0}^{\infty}\frac{(-1)^k(ut)^{2k+n}}{2^{2k+n}k!\Gamma 1+n-k}dt \\
&= \sum_{n=0}^{\infty}\frac{(-1)^k}{2^{2k+n}k!\Gamma 1+n-k}\int_0^{\infty} e^{-st}(ut)^{2k+n}dt \\
&= \sum_{n=0}^{\infty}\frac{(-1)^k}{2^{2k+n}k!\Gamma 1+n-k}\mathbb{N}\{t^{2k+n}\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^k}{2^{2k+n}k!\Gamma 1+n-k}\frac{u^{2k+n}}{s^{2k+n+1}}(2k+n)! \\
&= \{\sum_{n=0}^{\infty}\frac{(-1)^k\Gamma n+1}{2^{2k}k!\Gamma 1+n-k\Gamma n+1}\frac{u^{2k}}{s^{2k}}(2k+n)!\}\frac{u^n}{2^n s^{n+1}} \\
&= \{\sum_{n=0}^{\infty}\frac{(-1)^k(1+n)_{2k}}{2^{2k}k!(1+n)_k}\frac{u^{2k}}{s^{2k}}\}\frac{u^n}{2^n s^{n+1}} \\
&= \{\sum_{n=0}^{\infty}\frac{(-1)^k(\frac{1+n}{2})_k(1+\frac{n}{2})_k}{k!(1+n)_k}\frac{u^{2k}}{s^{2k}}\}\frac{u^n}{2^n s^{n+1}} \\
\therefore \mathbb{N}\{J_n(t)\} &= {}_2F_1\left[\begin{array}{cc} \frac{n+1}{2}, 1+\frac{n}{2} & ; \\ 1+n & ; \end{array} \mid -\frac{u^2}{s^2}\right]\frac{u^n}{2^n s^{n+1}}
\end{aligned}$$

3.3 The Natural transform of Legendre Polynomial

$$
\begin{aligned}
\mathbb{N}\{P_n(t)\} &= \mathbb{N}\{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k(\frac{1}{2})_{n-k}(2t)^{n-2k}}{(n-2k)!k!}\} \\
&= \int_0^\infty e^{-st}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k(\frac{1}{2})_{n-k}(2ut)^{n-2k}}{(n-2k)!k!}dt \\
&= 2^n\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k(\frac{1}{2})_n(-n)_{2k}}{k!(1-1/2-n)_k(-1)^{2k}n!2^{2k}}\int_0^\infty e^{-st}(ut)^{n-2k}dt \\
&= 2^n\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k(\frac{1}{2})_n(-n)_{2k}}{k!(1-1/2-n)_k(-1)^{2k}n!2^{2k}}\mathbb{N}\{t^{n-2k}\} \\
&= 2^n\frac{(\frac{1}{2})_n}{n!}\{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-n)_{2k}}{(1/2-n)_k k!2^{2k}}\frac{u^{n-2k}}{s^{n-2k+1}}\Gamma n-2k+1\} \\
&= 2^n\frac{(\frac{1}{2})_n u^n}{s^{n+1}n!}\{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-n)_{2k}}{(1/2-n)_k k!2^{2k}}\frac{s^{2k}}{u^{2k}}\Gamma n-2k+1\} \\
&= 2^n\Gamma n+1\frac{(\frac{1}{2})_n u^n}{s^{n+1}n!}\{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\frac{n}{2})_k(\frac{-n+1}{2})_k 2^{2k}}{(1/2-n)_k 2^{2k}k!}\frac{s^{2k}}{u^{2k}}\frac{\Gamma 1-(-n)-2k}{\Gamma 1-(-n)}\} \\
&= 2^n\frac{(\frac{1}{2})_n u^n}{s^{n+1}}\{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\frac{n}{2})_k(\frac{-n+1}{2})_k(-1)^{2k}}{(1/2-n)_k(-n)_{2k}k!}\frac{s^{2k}}{u^{2k}}\} \\
&= 2^n\frac{(\frac{1}{2})_n u^n}{s^{n+1}}\{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\frac{n}{2})_k(\frac{-n+1}{2})_k(-1)^{2k}}{(1/2-n)_k(-\frac{n}{2})_k(\frac{-n+1}{2})_k k!2^{2k}}\frac{s^{2k}}{u^{2k}}\} \\
&= 2^n\frac{(\frac{1}{2})_n u^n}{s^{n+1}}\{\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^{2k}}{(1/2-n)_k k!}\left(\frac{s^2}{4u^2}\right)^k\}
\end{aligned}
$$

$$
\therefore \mathbb{N}\{P_n(t)\} = {}_0F_1\left[\begin{array}{cc} - & ; \\ (\frac{1}{2}-n) & ; \end{array} \mid \frac{s^2}{2u^2}\right]2^n\frac{(\frac{1}{2})_n u^n}{s^{n+1}}
$$

3.4 The Natural transform of Hermite Polynomial

$$
\begin{aligned}
\mathbb{N}\{H_n(t)\} &= \mathbb{N}\{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2t)^{n-2k}}{(n-2k)!k!}\} \\
&= \int_0^\infty e^{-st} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2ut)^{n-2k}}{(n-2k)!k!} dt \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2)^n}{(n-2k)!k!2^{2k}} \int_0^\infty e^{-st}(ut)^{n-2k} dt \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2)^n}{(n-2k)!k!2^{2k}} \mathbb{N}\{t^{n-2k}\} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2)^n}{(n-2k)!k!2^{2k}} \frac{u^{n-2k}}{s^{n-2k+1}} \Gamma n-2k+1 \\
&= 2^n \frac{u^n}{s^{n+1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(-n)_{2k}}{(-1)^{2k} k! 2^{2k}} \frac{s^{2k}}{u^{2k}} \Gamma n-2k+1 \\
&= 2^n \frac{u^n}{s^{n+1}} \Gamma n+1 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(-n)_{2k}}{(-1)^{2k} k!} \left(\frac{s^2}{4u^2}\right)^k \frac{\Gamma n-2k+1}{\Gamma n+1} \\
&= 2^n \frac{u^n}{s^{n+1}} n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!} \left(\frac{s^2}{4u^2}\right)^k \\
\therefore \mathbb{N}\{H_n(t)\} &= {}_0F_0 \left[\begin{array}{cc} - & ; \\ - & ; \end{array} \mid -\frac{s^2}{4u^2} \right] 2^n \frac{u^n}{s^{n+1}} n!
\end{aligned}
$$

3.5 The Natural transform of Leguerre Polynomial

$$\begin{aligned}
\mathbb{N}\{L_n(\alpha)_t\} &= \mathbb{N}\{\sum_{k=0}^{\infty}\frac{(-1)^k(1+\alpha)_n t^k}{(n-k)!k!(1+\alpha)_k}\} \\
&= \int_0^{\infty} e^{-st}\sum_{k=0}^{\infty}\frac{(-1)^k(1+\alpha)_n t^k}{(n-k)!k!(1+\alpha)_k}dt \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n-k)!k!(1+\alpha)_k}\int_0^{\infty} e^{-st}(ut)^k dt \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n-k)!k!(1+\alpha)_k}\mathbb{N}\{t^k\} \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n-k)!k!(1+\alpha)_k}\frac{u^k}{s^{k+1}}\Gamma k+1 \\
&= (1+\alpha)_n\sum_{k=0}^{\infty}\frac{(-1)^k(-n)_k}{(-1)^k k!(1+\alpha)_k n!}\frac{u^k}{s^{k+1}}\Gamma k+1 \\
&= \{\sum_{k=0}^{\infty}\frac{(1)_k(-n)_k}{k!(1+\alpha)_k}\left(\frac{u^k}{s^k}\right)^k\}\frac{(1+\alpha)_n}{sn!} \\
\therefore \mathbb{N}\{L_n(\alpha)_t\} &= {}_2F_1\left[\begin{matrix}(-n),1 & ; & \\ (1+\alpha) & ; & \end{matrix} \mid \frac{u}{s}\right]\frac{(1+\alpha)_n}{sn!}
\end{aligned}$$

Note that throughout the discussion, we assume that the validity of integration term by term in the summation.

4 Conclusion

In this paper we have find the Natural transform of some special well known functions in terms of Hypergeometric function.These special functions are useful in solving differential equations.

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