ON SOME IDEALS OF INTUITIONISTIC FUZZY POINTS SEMIGROUPS

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Abstract – In this paper, the minimal ideal \( A \) of a semigroup \( S \) is characterized by the intuitionistic characteristic function \( \chi_A \). The existence of an intuitionistic fuzzy kernel in a semigroup is explored. Finally, we consider the semigroup \( \tilde{S} \) of the intuitionistic fuzzy points of a semigroup \( S \) and discuss some relations between some ideals \( A \) of \( S \) and the subset \( \subseteq_{\tilde{S}} \) of the semigroup \( \tilde{S} \).

Keywords – Semigroups; Intuitionistic fuzzy points; Intuitionistic fuzzy ideals.

1 Introduction

Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. Zadeh [16] introduced the concept of a fuzzy set for the first time and this concept was applied by Rosenfeld [14] to define fuzzy subgroups and fuzzy ideals. Based on this crucial work, Kuroki [9,10,11,12] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them. In [13], Kim considered the semigroup \( \tilde{S} \) of the fuzzy points of a semigroup \( S \), and discussed the relation between the fuzzy interior ideals and the subsets of \( \tilde{S} \), also see [6, 7]. Atanassov [4, 5] introduced the notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. Many concepts in fuzzy set theory were also extended to intuitionistic fuzzy set theory, such as intuitionistic fuzzy relations, intuitionistic \( L^\Gamma \)-fuzzy sets, intuitionistic fuzzy implications, intuitionistic fuzzy logics, intuitionistic fuzzy semigroups etc. Jun and Song [8] introduced the notion of intuitionistic fuzzy points. In [15] Sardar et al., defined some relations between the intuitionistic fuzzy ideals of a semigroup \( S \) and the set of all intuitionistic fuzzy points of \( S \). In [3] Akram characterized intuitionistic fuzzy ideals in ternary semigroups by intuitionistic fuzzy points. Also in [2] he analyzed some relations between the intuitionistic fuzzy \( \Gamma \)-ideals and the sets of intuitionistic fuzzy points of these \( \Gamma \)-ideals of a \( \Gamma \)-semigroup. In this paper, we consider the
semigroup \( \mathfrak{g} \) of the intuitionistic fuzzy points of a semigroup \( \mathfrak{s} \), and discuss some relations between some ideals \( A \) of \( \mathfrak{s} \) and the subset \( \mathfrak{c} \) of the semigroup \( \mathfrak{s} \).

2 Basic Definitions and Results

Let \( \mathfrak{s} \) be a semigroup. A nonempty subset \( A \) of \( \mathfrak{s} \) is called a left (resp., right) ideal of \( \mathfrak{s} \) if \( \mathfrak{s}A \subseteq A \) (resp., \( A \mathfrak{s} \subseteq A \)), and a two-sided ideal (or simply ideal) of \( \mathfrak{s} \) if \( \mathfrak{s}A \subseteq A \) is both a left and a right ideal of \( \mathfrak{s} \). A nonempty subset \( A \) of \( \mathfrak{s} \) is called an interior ideal of \( \mathfrak{s} \) if \( \mathfrak{s}A \subseteq A \). An ideal \( A \) of \( \mathfrak{s} \) is called minimal ideal of \( \mathfrak{s} \) if \( A \) does not properly contains any other ideal of \( \mathfrak{s} \). If the intersection \( \mathfrak{k} \) of all the ideals of a semigroup \( \mathfrak{s} \) is nonempty then we shall call \( \mathfrak{k} \) the kernel of \( \mathfrak{s} \). A sub-semigroup \( \mathfrak{a} \) of \( \mathfrak{s} \) is called a bi-ideal of \( \mathfrak{s} \) if \( \mathfrak{a} \mathfrak{a} \subseteq \mathfrak{a} \mathfrak{a} \). A function \( f \) from \( \mathfrak{s} \) to the closed interval \([0, 1]\) is called a fuzzy set in \( \mathfrak{s} \). The semigroup \( \mathfrak{s} \) itself is a fuzzy set in \( \mathfrak{s} \) such that \( s(x) = 1 \) for all \( x \in \mathfrak{s} \), denoted also by \( \mathfrak{s} \). Let \( A \) and \( B \) be two fuzzy sets in \( \mathfrak{s} \). Then the inclusion relation \( A \subseteq B \) is defined by \( A(x) \leq B(x) \) for all \( x \in \mathfrak{s} \). A sub-semigroup \( A \subseteq \mathfrak{s} \) and \( B \subseteq \mathfrak{s} \) are fuzzy sets in \( \mathfrak{s} \) defined by

\[
\begin{align*}
(A \cap B)(x) &= A(x) \land B(x) = \min\{A(x), B(x)\}, \\
(A \cup B)(x) &= A(x) \lor B(x) = \max\{A(x), B(x)\},
\end{align*}
\]

for all \( x \in \mathfrak{s} \).

For any \( \alpha \in (0, 1] \) and \( x \in \mathfrak{s} \), a fuzzy set \( x_\alpha \) in \( \mathfrak{s} \) is called a fuzzy point in \( \mathfrak{s} \) if

\[
x_\alpha(x) = \begin{cases} \alpha & \text{if } x = y, \\
0 & \text{otherwise,}
\end{cases}
\]

for all \( x \in \mathfrak{s} \). The fuzzy point \( x_\alpha \) is said to be contained in a fuzzy set \( A \), denoted by \( x_\alpha \in A \), iff \( \alpha \leq A(x) \).

**Definition 1.** [4, 5] The intuitionistic fuzzy sets (IFS, for short) defined on a non-empty set \( X \) as objects having the form

\[
A = \{ x : \mu_A(x), \nu_A(x) : x \in X \},
\]

where the functions \( \mu_A : X \to [0, 1] \) and \( \nu_A : X \to [0, 1] \) denote the degree of membership and the degree of non-membership of each element \( x \in X \) to the set \( A \) respectively, and \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \) for all \( x \in X \).

For the sake of simplicity, we shall use \( A = (\mu_A, \nu_A) \) for intuitionistic fuzzy set \( A = \{ x : \mu_A(x), \nu_A(x) : x \in X \} \).

**Definition 2.** [15] Let \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta \leq 1 \). An intuitionistic fuzzy point, written as \( x_{(\alpha, \beta)} \) is defined to be an intuitionistic fuzzy subset of \( \mathfrak{s} \), given by

\[
x_{(\alpha, \beta)}(x) = \begin{cases} (\alpha, \beta) & \text{if } x = y, \\
(0, 1) & \text{otherwise}
\end{cases}
\]
**Definition 3.** [15] A non-empty IFS \( A = (\mu_A, \nu_A) \) of a semigroup \( S \) is called an intuitionistic fuzzy subsemigroup of \( S \) if

\[(i) \quad \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \quad \forall x, y \in S, \]
\[(ii) \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y), \quad \forall x, y \in S. \]

**Definition 4.** [15] An intuitionistic fuzzy subsemigroup \( A = (\mu_A, \nu_A) \) of a semigroup \( S \) is called an intuitionistic fuzzy interior ideal of \( S \) if

\[(i) \quad \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \quad \forall x, w, y \in S, \]
\[(ii) \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y), \quad \forall x, w, y \in S. \]

**Definition 5.** [15] An intuitionistic fuzzy subsemigroup \( A = (\mu_A, \nu_A) \) of a semigroup \( S \) is called an intuitionistic fuzzy bi-ideal of \( S \) if

\[(i) \quad \mu_A(xw) \geq \mu_A(x) \wedge \mu_A(w), \quad \forall x, w, y \in S, \]
\[(ii) \quad \nu_A(xw) \leq \nu_A(x) \vee \nu_A(w), \quad \forall x, w, y \in S. \]

**Definition 6.** [15] A non-empty IFS \( A = (\mu_A, \nu_A) \) of a semigroup \( S \) is called an intuitionistic fuzzy left (right) ideal of \( S \) if

\[(i) \quad \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \quad \forall x, y \in S, \]
\[(ii) \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y), \quad \forall x, y \in S. \]

**Definition 7.** [15] A non-empty IFS \( A = (\mu_A, \nu_A) \) of a semigroup \( S \) is called an intuitionistic fuzzy two-sided ideal or an intuitionistic fuzzy ideal of \( S \) if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of \( S \).

Let \( A \) be a subset of a semigroup \( S \) and \( A^c \) be the complement of \( A = (\mu_A, \nu_A) \) is defined as:

\[
C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise}, \end{cases} \quad C_{A^c}(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise}, \end{cases}
\]

for all \( x \in S \).

Let \( IFS(S) \) be the set of all intuitionistic fuzzy sets in a semigroup \( S \). For each \( A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in IFS(S) \), the product of \( A \) and \( B \) is an intuitionistic fuzzy set \( A \times B \) defined as follows:

\[
A \times B = \{ (x, \mu_{A \times B}(x), \nu_{A \times B}(x)) : \forall x \in S \},
\]

where

\[
\mu_{A \times B}(x) = \bigvee_{xuv} \mu_A(u) \wedge \mu_B(v) \quad \text{if } uv = x,
\]
\[\text{otherwise}.
\]
Lemma 1. [1] For any nonempty subsets $A$ and $B$ of a semigroup $S$, we have $A \subseteq B$ if and only if $x_A \subseteq x_B$.

Lemma 2. [1] Let $A$ be a nonempty subset of a semigroup $S$, then $A$ is an ideal of $S$ if and only if $x_A$ is an intuitionistic fuzzy ideal of $S$.

Theorem 1. A nonempty subset $A$ of a semigroup $S$ is a minimal ideal if and only if $x_A$ is a minimal intuitionistic fuzzy ideal of $S$.

Proof. Let $A$ be a minimal ideal of $S$, then by lemma 2, $x_A$ is an intuitionistic fuzzy ideal of $S$. Suppose that $x_A$ is not minimal intuitionistic fuzzy ideal of $S$, then there exists some intuitionistic fuzzy ideal $B$ of $S$ such that $x_B \subseteq x_A$, hence, lemma 1 implies that $B \subseteq A$, where $B$ is an ideal of $S$. This is a contradiction to the fact that $A$ is minimal ideal of $S$. Thus $x_A$ is minimal intuitionistic fuzzy ideal of $S$. Conversely, let $A$ be a minimal intuitionistic fuzzy ideal of $S$, then $x_A$ is an ideal of $S$. Suppose that $A$ is not minimal ideal of $S$, then there exists some ideal $B$ of $S$ such that $B \subseteq A$. Now by lemma 1, $x_B \subseteq x_A$, where $x_B$ is an intuitionistic fuzzy ideal of $S$. This contradicts that $x_A$ is a minimal intuitionistic fuzzy ideal of $S$. Thus $A$ is minimal ideal of $S$. □

Lemma 3. If $A = (x_A, x_B)$ is a minimal intuitionistic fuzzy ideal of a semigroup $S$, then $A$ is the intuitionistic fuzzy kernel of $S$.

Proof. Let $B = (x_B, x_B)$ be any intuitionistic fuzzy ideal of $S$, then $B \cdot A \subseteq B \subseteq A \cdot B$. Since $B \subseteq A$ is an intuitionistic fuzzy ideal of $S$ and $B \cap A = A$, it follows that $B \cap A = A$. But then $A = B \cap A = A$, so $A$ is contained in every intuitionistic fuzzy ideal of $S$ and hence is an intuitionistic fuzzy kernel of $S$. □

Lemma 4. If $A = (x_A, x_A)$ is an intuitionistic fuzzy kernel of a semigroup $S$, then $A$ is a simple intuitionistic fuzzy subsemigroup of $S$.

Proof. Since $A$ is an intuitionistic fuzzy ideal of $S$, so $A$ is an intuitionistic fuzzy subsemigroup of $S$. To show that $A$ is simple, let $B$ be any intuitionistic fuzzy ideal of $A$, then $A \cdot B \cdot A$ is an intuitionistic fuzzy ideal of $S$, since

$$S \cdot (A \cdot B \cdot A) = (S \cdot A) = B \cdot A \subseteq A \cdot B = A$$

and

$$(A \cdot B \cdot A) \cdot S = (A \cdot S) = A \cdot B \subseteq A \cdot A \subseteq A$$

Also, $A \cdot B \subseteq A \subseteq A$, but by lemma 3, $A$ is minimal intuitionistic fuzzy ideal of $S$. Hence $A \cdot B \cdot A = A$. Also, $A \cdot B = A$ implies that $A \subseteq B$. Thus $A = B$, that is $A$ is simple subsemigroup of $S$. □

Lemma 5. Let $A = (x_A, x_B)$ be an intuitionistic fuzzy left ideal of a semigroup $S$ and $x_{(A \cdot B)}$ be any intuitionistic fuzzy point of $S$, then $A \cdot x_{(A \cdot B)}$ is a minimal intuitionistic fuzzy left ideal of $S$. 

\[x_{A \cdot x_{(A \cdot B)}}(c) = \bigwedge_{x_{A \cdot (x_{(A \cdot B)}(c))} = x} x_{A \cdot y_{(A \cdot B)}}(d) = x_{A \cdot y_{(A \cdot B)}}(c) \text{ if and only if } x_{A \cdot y_{(A \cdot B)}}(c) = x \]
Proof. $A \circ x_{(a,b)}$ is an intuitionistic fuzzy left ideal of $S$. Suppose $B$ is an intuitionistic fuzzy left ideal of $S$ and let $D = \{y \in Y : y \circ x_{(a,b)} \in B\}$. Let $s_{(a_1,b_1)} \in A$ and $s_{(a_2,b_2)} \in A$, then $s_{(a_1,b_1)} \circ x_{(a_2,b_2)} = s_{(a_1,b_1)} \circ s_{(a_2,b_2)} \in D$. Hence $A \circ x_{(a,b)} \subseteq D$, which implies that $A \circ x_{(a,b)} \subseteq A = D$. Thus $A \circ x_{(a,b)}$ is an intuitionistic fuzzy left ideal of $S$, and because of minimality of $A$, we get $D = A$. Hence $A \circ x_{(a,b)} = B$ and therefore, $A \circ x_{(a,b)}$ is a minimal intuitionistic fuzzy left ideal of $S$.

3 Main Results

If $S$ is a semigroup, then $\mathcal{SF}(S)$ is a semigroup with the product "$\cdot$"[15]. Let $S$ be the set of all intuitionistic fuzzy points in a semigroup $S$. Then $x_{(a,b)} \circ y_{(c,d)} = (xy)_{(a+c,a+c)} \in S$, for $x_{(a,b)}, y_{(c,d)} \in S$ and $(xy)_{(a+c,a+c)} = (x_{(a,b)} \circ y_{(c,d)}) \circ (xy)_{(a+c,a+c)}$. Thus $S$ is a subsemigroup of $\mathcal{SF}(S)[15]$. For any $A \subseteq \mathcal{SF}(S)$, $A$ denotes the set of all intuitionistic fuzzy points contained in $A$, that is, $A = \{x_{(a,b)} : x_{(a,b)} \in A, (x_{(a,b)}) \leq 1\}$. For any $A \subseteq \mathcal{SF}(S)$, we define the product of $A$ and $B$ as $A \cdot B = \{x_{(a,b)} \circ y_{(c,d)} : x_{(a,b)}, y_{(c,d)} \in A, B\}$.

**Lemma 6.** [15] Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two intuitionistic fuzzy subsets of a semigroup $S$, then

1. $A \cup B = A \cup B$
2. $A \cap B = A \cap B$
3. $A \cdot B = A \cdot B$

**Lemma 7.** Let $A$ be nonempty subset of a semigroup $S$, we have $x_{(a,b)} \in X_A$ if and only if $x \in A$.

**Proof.** Suppose that $x_{(a,b)} \in X_A$ for any $x \in A$, then $C_A(x) \geq a$. Hence $C_A(x) = 1$ for any $a > 0$, which implies that $x \in A$. Conversely, Let $x \in A$, then $C_A(x) = 1 \geq a$ and $C_A(x) = 0 < b$ for any $a, b > 0$. This means that $x_{(a,b)} \in X_A$.

**Lemma 8.** For any nonempty subsets $A$ and $B$ of a semigroup $S$, we have

1. $A \subseteq B$ if and only if $X_A \subseteq X_B$
2. $A \subseteq B$ if and only if $X_A \subseteq X_B$

**Proof.** (1) Assume that $A \subseteq B$, and let $x_{(a,b)} \in X_A$. By lemma 7, $x \in A \subseteq B$ and $x_{(a,b)} \in X_B$, this implies that $X_A \subseteq X_B$. Conversely, suppose that $X_A \subseteq X_B$. Let $x \in A$, then by lemma 7, for any $a, b > 0$, $x_{(a,b)} \in X_A$ and $x_{(a,b)} \in X_B$, which implies that $x \in B$. (2) it is obvious that if $A \subseteq B$, then $X_A \subseteq X_B$. Now assume that $X_A \subseteq X_B$ and let $x_{(a,b)} \in X_A$, then $X_B \subseteq X_B$. Then $A \subseteq B$ and consequently, we have $X_A \subseteq X_B$. This completes the proof.

**Lemma 9.** Let $A$ be a nonempty subset of a semigroup $S$. Then $A$ is an ideal of $S$ if and only if $X_A$ is an ideal of $S$. 
Proof. By lemma 2, $A$ is an ideal of $S$ if and only if $\mathcal{A}$ is a fuzzy ideal of $S$. And from theorem 3.5[13], $\mathcal{A}$ is a fuzzy ideal of $S$ if and only if $\mathcal{A}$ is an ideal of $S$. $\blacksquare$

**Theorem 2.** A nonempty subset $A$ of a semigroup $S$ is minimal ideal if and only if $\mathcal{A}$ is a minimal intuitionistic fuzzy ideal of $S$.

Proof. Let $A$ be a minimal ideal of $S$, then by lemma 2, $\mathcal{A}$ is an intuitionistic fuzzy ideal of $S$. Suppose that $\mathcal{A}$ is not minimal intuitionistic fuzzy ideal of $S$, then there exists some intuitionistic fuzzy ideal $\mathcal{B}$ of $S$ such that $\mathcal{B} \subseteq \mathcal{A}$. Hence, lemma 1 implies that $B \subseteq A$, where $B$ is an ideal of $S$. This is a contradiction to the fact that $A$ is minimal ideal of $S$. Thus $\mathcal{A}$ is minimal intuitionistic fuzzy ideal of $S$. Conversely, let $\mathcal{A}$ be a minimal intuitionistic fuzzy ideal of $S$, then by lemma, $A$ is an ideal of $S$. Suppose that $A$ is not minimal ideal of $S$, then there exists some ideal $B$ of $S$ such that $B \subseteq A$. Now by lemma, $\mathcal{B} = \mathcal{A}$, where $\mathcal{B}$ is an intuitionistic fuzzy ideal of $S$. This contradicts that $\mathcal{A}$ is a minimal intuitionistic fuzzy ideal of $S$. Thus $A$ is minimal ideal of $S$. $\blacksquare$

**Theorem 3.** Let $A$ be a nonempty subset of a semigroup $S$. Then $A$ is a minimal ideal of $S$ if and only if $\mathcal{A}$ is a minimal ideal of $S$.

Proof. By theorem 1, $A$ is a minimal ideal of $S$ if and only if $\mathcal{A}$ is a fuzzy ideal of $S$. We only need to prove that, $\mathcal{A}$ is a minimal intuitionistic fuzzy ideal of $S$ if and only if $\mathcal{A}$ is a minimal ideal of $S$. Let $\mathcal{A}$ be a minimal intuitionistic fuzzy ideal of $S$, then $\mathcal{A}$ is an ideal of $S$. Suppose that $\mathcal{A}$ is not minimal, then there exists some ideals $\mathcal{B}$ of $S$ such that $\mathcal{A} \subseteq \mathcal{B}$, which implies that $\mathcal{B} \subseteq \mathcal{A}$, where $\mathcal{B}$ is an intuitionistic fuzzy ideal of $S$. This is a contradiction to $\mathcal{A}$ is a minimal intuitionistic fuzzy ideal of $S$. Thus $\mathcal{A}$ is a minimal ideal of $S$. Conversely, assume that $\mathcal{A}$ is a minimal ideal of $S$ and that $\mathcal{A}$ is not a minimal intuitionistic fuzzy ideal of $S$. Then there exists an intuitionistic fuzzy ideal $\mathcal{B}$ of $S$ such that $\mathcal{A} \subseteq \mathcal{B}$, hence $\mathcal{A} \subseteq \mathcal{B}$, where $\mathcal{B}$ is an ideal of $S$. This contradicts that $\mathcal{A}$ is a minimal ideal of $S$. This completes the proof of the theorem. $\blacksquare$

**Theorem 4.** Let $A$ be a nonempty subset of a semigroup $S$. Then $A$ is the kernel of $S$ if and only if $\mathcal{A}$ is the kernel of $S$.

Proof. Suppose that $A$ is the kernel of $S$, then $A = \bigcap_{i}I_{i}$ where $I_{i}$ is an ideal of $S$. Let $\mathcal{A}$ be an ideal of $S$, then $B$ is an ideal of $S$. Now we need to show that $\mathcal{A} = \mathcal{B}$, let $x \in A$ and also $x \in B$, since $A$ is the kernel of $S$. This implies that $\mathcal{A} = \mathcal{B}$ and hence, $\mathcal{A}$ is the kernel of $S$. Conversely, let $\mathcal{A}$ be the kernel of $S$, then $\mathcal{A} \subseteq \mathcal{B}$ for every ideal $\mathcal{B}$ of $S$. Thus $A \subseteq B$ and therefore, $A$ is the kernel of $S$. $\blacksquare$

The following lemma weakens the condition of theorem 4.

**Lemma 10.** Let $A$ be a minimal ideal of a semigroup $S$, then $\mathcal{A}$ is the kernel of $S$.

Proof. Since $A$ is a minimal ideal of $S$, then $\mathcal{A}$ is a minimal intuitionistic fuzzy ideal of $S$. Also lemma 3 implies that $\mathcal{A}$ is the fuzzy kernel of $S$. Now, let $\mathcal{B}$ be an intuitionistic fuzzy
ideal of $\mathcal{S}$, then we have $\mathcal{I}_A \subseteq \mathcal{I}_B$ and hence $\mathcal{I}_A \subseteq \mathcal{I}_B$. So $\mathcal{I}_A$ is a minimal ideal contained in every ideal of $\mathcal{S}$. Thus $\mathcal{I}_A$ is the kernel of $\mathcal{S}$. ■

**Theorem 5.** Let $\mathcal{A}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then $\mathcal{A}$ is an interior ideal of $\mathcal{S}$ if and only if $\mathcal{I}_A$ is an interior ideal of $\mathcal{S}$.

**Proof.** Let $\mathcal{A}$ be an interior ideal of $\mathcal{S}$, and let $\gamma_{(\gamma \alpha \beta)} = (\gamma \alpha \beta) \in \mathcal{S}$ and $\gamma_{(\gamma \alpha \beta)} \in \mathcal{I}_A$. Since $\gamma \in \mathcal{A}$, then $\gamma_{(\gamma \alpha \beta)} \in \mathcal{I}_A$. This implies that $\mathcal{I}_A = \mathcal{I}_A \subseteq \mathcal{I}_B$. Conversely, suppose that $\mathcal{I}_A$ is an interior ideal of $\mathcal{S}$. Let $\gamma \in \mathcal{S}$ and $\gamma \in \mathcal{A}$, then $\gamma_{(\gamma \alpha \beta)} \in \mathcal{I}_A$. Assume that $\gamma_{(\gamma \alpha \beta)} = (\gamma \alpha \beta) \in \mathcal{I}_A$. Then $\gamma_{(\gamma \alpha \beta)} \in \mathcal{I}_A$. This implies that $\mathcal{I}_A \subseteq \mathcal{I}_B$, and hence $\mathcal{A}$ is an interior ideal of $\mathcal{S}$. ■

**Theorem 6.** Let $\mathcal{A}$ be a nonempty subset of a semigroup $\mathcal{S}$. Then $\mathcal{A}$ is a bi-ideal of $\mathcal{S}$ if and only if $\mathcal{I}_A$ is a bi-ideal of $\mathcal{S}$.

**Proof.** Let $\mathcal{A}$ be a bi-ideal of $\mathcal{S}$, and let $\gamma_{(\gamma \alpha \beta)} = (\gamma \alpha \beta) \in \mathcal{S}$ and $\gamma_{(\gamma \alpha \beta)} \in \mathcal{S}$. Since $\gamma \in \mathcal{A}$ and $\gamma \in \mathcal{A}$, then $\gamma_{(\gamma \alpha \beta)} \in \mathcal{I}_A$. This implies that $\mathcal{I}_A \subseteq \mathcal{I}_B$, and hence $\mathcal{A}$ is a bi-ideal of $\mathcal{S}$. ■

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**References**


