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SOME TOPOLOGICAL PROPERTIES OF SOFT DOUBLE TOPOLOGICAL SPACES

Osama Abd El-hamed El-Tantawy[1] <drosamat@yahoo.com>
Sobhy Ahmed Ali El-sheikh[2] <sobhyesheikh@yahoo.com>
Salama Hussien Ali Shaliel[2,*] <slamma-elarabi@yahoo.com>

[1]Mathematics Department, Faculty of Science, Zagazeg University, Zagazeg, Egypt.
[2]Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt.

Abstaract − In this paper, we introduce new separation axioms on soft double topological spaces and study some of their properties. Also, we define the soft double subspaces and study some related properties. Finally, we study the behaviour of the separation axioms under open (homeomorphism) mappings.

Keywords − *Soft double* T_i^*-*spaces* ($T_i^{**}-$*spaces*), $(i = 0, 1, 2, 3)$, SDT_0-*spaces*, $SDT_{\frac{1}{2}}-$*spaces*, SDT_1-*spaces, soft double Hausdorff spaces, soft double regular spaces, soft double* R_2-*spaces* (SDR_2-*spaces, for short*)*, soft double subspaces, soft double open mappings, soft double closed mappings, soft double homeomorphism mappings, soft double continuous functions and separation axioms.*

1 Introduction

Atanassov [1, 2, 3, 4] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [5] generalized topological structures in intuitionistic fuzzy case. The concept of intuitionistic sets and the topology on intuitionistic sets was first given by Coker [7, 6].

In 2005, the suggestion of J. G. Garcia et al. [8] that double set is a more appropriate name than flou (intuitionistic) set, and double topology for the flou (intuitionistic) topology. Kandil et al. [11, 12] introduced the concept of double sets, double topological spaces, continuous functions between these spaces and separation axioms on double topological spaces.

After presentation of the operations of soft sets [16], the properties and applications of soft set theory have been studied increasingly [1, 14, 16, 18].

* *Corresponding Author.*

Recently, in 2011, Shabir and Naz [19] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open(closed) soft sets, soft subspace, soft separation axioms and established their several properties. Hussain and Ahmad [9] investigated the properties of soft nbds and soft closure operator.

In [21] Tantawy, et al. introduced the concept of soft double sets (SD-sets, for short), soft double points (SD-points, for short), soft double topological space (SDTS, for short) and continuous functions between these spaces.

The purpose of this paper is to introduce some separation axioms on SDTS (SD-separation axioms, for short) and some of its basic properties, soft double subspace (SD-subspace, for short) and some properties related to it, continuous function and separation axioms on SDTS. Moreover, some basic properties of these notions have obtained.

2 Preliminary

In this section, we collect some definitions and theorems which will be needed in the sequel. For more details see [9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 22].

Definition 2.1. [12] Let X be a nonempty set.

1. A double set $\underline{A}$ is an ordered pair $(A_1, A_2) \in P(X) \times P(X)$ such that $A_1 \subseteq A_2$.
2. $D(X) = \{(A_1, A_2) \in P(X) \times P(X), A_1 \subseteq A_2\}$ is the family of all double sets on X.
3. Let $\eta_1, \eta_2 \subseteq P(X)$. The product of η_1 and η_2, denoted by $\eta_1 \hat{\times} \eta_2$, and defined by: $\eta_1 \hat{\times} \eta_2 = \{(A_1, A_2) \in \eta_1 \hat{\times} \eta_2 : A_1 \subseteq A_2\}$.
4. The double set $\underline{X} = (X, X)$ is called the universal double set.
5. The double set $\underline{\emptyset} = (\emptyset, \emptyset)$ is called the empty double set.
6. Let $x \in X$. Then, the double sets $\underline{x}_1 = (\{x\}, \{x\})$ and $\underline{x}_{\frac{1}{2}} = (\emptyset, \{x\})$ are said to be double points in X. The family of all double points in X, denoted by $DP(X)$ i.e, $DP(X) = \{x_t : x \in X, t \in \{\frac{1}{2}, 1\}\}$.
7. $\underline{x}_1 \underline{\in}\, \underline{A} \Leftrightarrow x \in A_1$ and $\underline{x}_{\frac{1}{2}} \underline{\in}\, \underline{A} \Leftrightarrow x \in A_2$.

Definition 2.2. [12] Let $\underline{A} = (A_1, A_2) \in D(X)$. $\underline{A}$ is called a finite double set if A_2 is a finite subset of X.

Definition 2.3. [12] Let $\underline{A} = (A_1, A_2)$, $\underline{B} = (B_1, B_2) \in D(X)$.

1. $\underline{A} \cup \underline{B} = (A_1 \cup B_1,\ A_2 \cup B_2)$.
2. $\underline{A} \cap \underline{B} = (A_1 \cap B_1,\ A_2 \cap B_2)$.

Definition 2.4. [11] Two double sets $\underline{A}$ and $\underline{B}$ are said to be a quasi-coincident, denoted by $\underline{A}q\underline{B}$, if $A_1 \cap B_2 \neq \emptyset$ or $A_2 \cap B_1 \neq \emptyset$. $\underline{A}$ is called a not quasi-coincident with $\underline{B}$, denoted by $\underline{A} \not{q} \underline{B}$, if $A_1 \cap B_2 = \emptyset$ and $A_2 \cap B_1 = \emptyset$.

Definition 2.5. [12] Let X be a non-empty set. The family η of double sets in X is called a double topology on X if it satisfies the following axioms:

1. $\underline{\emptyset}, \underline{X} \in \eta$,
2. If $\underline{A}, \underline{B} \in \eta$, then $\underline{A} \sqcap \underline{B} \in \eta$,
3. If $\{\underline{A}_s : s \in S\} \subseteq \eta$, then $\underline{\bigcup}_{s\in S}\underline{A}_s \in \eta$.

The pair (X, η) is called a double topological space. Each element of η is called an open double set in X. The complement of an open double set is called a closed double set.

Definition 2.6. [17] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a non-empty subset of E. A soft set F_A over the universal X is a mapping from the parameter set E to $P(X)$ with support A i.e., $F_A : E \longrightarrow P(X)$. In other words a soft set over X is a parameterized family of subsets of X, where $F_A(e) \neq \emptyset$ if $e \in A \subseteq E$ and $F_A(e) = \emptyset$ if $e \notin A$.
Note that, a soft set can be written in the following form, $F_A = \{(e, F_A(e)) : e \in A \subseteq E, F_A : E \longrightarrow P(X)\}$.

The family of all soft sets over X denoted by $S(X, E)$.

Definition 2.7. Let $F_E, G_E \in S(X, E)$.

1. F_E is said to be a null soft set, denoted by Φ, if $F_E(e) = \emptyset$, $\forall e \in E$. [16]
2. F_E is called absolute soft set, denoted by X_E, if $F_E(e) = X$, $\forall e \in E$. [16]

Definition 2.8. [19] Let τ be a collection of soft sets over a universal X with a fixed set of parameters E. τ is called a soft topology on X if it satisfies the following conditions:

1. $\Phi, X_E \in \tau$,
2. The union of any number of soft sets in τ belongs to τ,
3. The intersection of any two soft sets in τ belongs to τ.

The triple (X, τ, E) is called a soft topological space over X. Every element of τ is called an open soft set in X and its complement is called a closed soft set in X.

Definition 2.9. [21] Let X be an initial universe and E be a set of parameters. Let $D(X)$ denotes the family of all double sets over the universal X. A SD-set $\widetilde{F}_A$ over the universal X is a mapping from the parameter set E to $D(X)$ with support A i.e., $\widetilde{F}_A : E \longrightarrow D(X)$. In other words a SD-set over the universal X is a parameterized family of double subsets of X, where $\widetilde{F}_A(e) \neq \underline{\emptyset}$ if $e \in A \subseteq E$ and $\widetilde{F}_A(e) = \underline{\emptyset}$ if $e \notin A$.
Note that, a SD-set can be written in the following form, $\widetilde{F}_A = \{(e, \widetilde{F}_A(e)) : e \in A \subseteq E, \widetilde{F}_A : E \longrightarrow D(X)\}$.

The family of all SD-sets over X denoted by $SD(X)_E$.
In this paper we use the notation $\widetilde{F}_E$ for any SD-subset where, $\widetilde{F}_E(e) \neq \underline{\emptyset}$, $\forall e \in A$ *and* $\widetilde{F}_E(e) = \underline{\emptyset}$, $\forall e \notin A$.

Definition 2.10. Let $\widetilde{F}_E, \widetilde{G}_E \in SD(X)_E$. Then,

1. $\widetilde{F}_E$ is called a null SD-set, denoted by $\widetilde{\Phi}$, where $\widetilde{F}_E(e) = \underline{\emptyset}$, $\forall e \in E$. [21]
2. $\widetilde{F}_E$ is called an absolute SD-set, denoted by $\widetilde{X}$, where $\widetilde{F}_E(e) = \underline{X}$, $\forall e \in E$. [21]
3. $\widetilde{F}_E$ is a SD-subset of $\widetilde{G}_E$, denoted by $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$, if $\widetilde{F}_E(e) \subseteq \widetilde{G}_E(e)$, $\forall e \in E$. [21]
4. $\widetilde{F}_E$ is equal to $\widetilde{G}_E$, denoted by $\widetilde{F}_E = \widetilde{G}_E$, if $\widetilde{F}_E(e) = \widetilde{G}_E(e)$, $\forall e \in E$. [21]
5. The union of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a SD-set $\widetilde{H}_E$ defined by: $\widetilde{H}_E(e) = (\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcup} \widetilde{G}_E(e)$, $\forall e \in E$. We write $\widetilde{F}_E \widetilde{\bigcup} \widetilde{G}_E = \widetilde{H}_E$. [21]
6. The intersection of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a SD-set $\widetilde{H}_E$ defined by: $\widetilde{H}_E(e) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E)(e) = \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E(e)$, $\forall e \in E$. We write $\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E = \widetilde{H}_E$. [21]
7. The difference of $\widetilde{F}_E$ and $\widetilde{G}_E$ is a SD-set $\widetilde{H}_E$ defined by: $\widetilde{H}_E(e) = \widetilde{F}_E(e) \setminus \widetilde{G}_E(e)$, $\forall e \in E$. We write $\widetilde{H}_E = \widetilde{F}_E \setminus \widetilde{G}_E$. [21]
8. The complement of $\widetilde{F}_E$, denoted by $\widetilde{F}^c_E$, defined by: $\widetilde{F}^c_E(e) = \underline{X} \setminus \widetilde{F}_E(e)$, $\forall e \in E$. and $(\widetilde{F}^c_E)^c = \widetilde{F}_E$. [21]

Definition 2.11. [21] Let $\widetilde{F}_E \in SD(X)_E$. $\widetilde{F}_E$ is called a SD-point for short over X if there exist $e \in E, x \in X$ and $t \in \{\frac{1}{2}, 1\}$ such that

$$\widetilde{F}_E(\alpha) = \begin{cases} \underline{x}_t, & \text{if } \alpha = e; \\ \underline{\emptyset}, & \text{if } \alpha \in E - \{e\}. \end{cases}$$

and we will denote $\widetilde{F}_E$ by $\widetilde{x}^e_t$.
The family of all SD-points over X will be denoted by $SDP(X)_E$.

Definition 2.12. [21] Two SD-sets $\widetilde{F}_E$ and $\widetilde{G}_E$ are said to be quasi- coincident, denoted by $\widetilde{F}_E \ \mathrm{q}\ \widetilde{G}_E$ if $\widetilde{F}_E(e) \ \mathrm{q}\ \widetilde{G}_E(e)$, for some $e \in E$. If $\widetilde{F}_E$ is not quasi- coincident with $\widetilde{G}_E$, we write $\widetilde{F}_E \not{q}\, \widetilde{G}_E$ or $\widetilde{F}_E(e) \not{q}\, \widetilde{G}_E(e)$, $\forall e \in E$.

Proposition 2.13. [21] Let $\widetilde{F}_E, \widetilde{G}_E, \widetilde{H}_E \in SD(X)_E$ *and* $\widetilde{x}^e_t \in SDP(X)_E$. Then,

1. $\widetilde{F}_E \not{q}\, \widetilde{G}_E \Leftrightarrow \widetilde{F}_E \widetilde{\subseteq} \widetilde{G}^c_E$.
2. $\widetilde{F}_E \not{q}\, \widetilde{G}_E, \widetilde{H}_E \widetilde{\subseteq} \widetilde{G}_E \Rightarrow \widetilde{F}_E \not{q}\, \widetilde{H}_E$.
3. $\widetilde{x}^e_t \not{q}\, (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \Leftrightarrow \widetilde{x}^e_t \not{q}\, \widetilde{F}_E$ *or* $\widetilde{x}^e_t \not{q}\, \widetilde{G}_E$.

Definition 2.14. [21] Let $SD(X)_E$ and $SD(Y)_K$ be the families of all SD-sets over X and Y, respectively.

1. The mapping $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ is called a soft double mapping, where $\beta : X \to Y$ and $\psi : E \to K$ are two mappings.

2. Let $\widetilde{F}_E \widetilde{\in} SD(X)_E$. Then, the image of $\widetilde{F}_E$ under the soft double mapping $f_{\beta\psi}$is the SD-set over Y, denoted by $f_{\beta\psi}(\widetilde{F}_E)$ and defined by:

$$f_{\beta\psi}(\widetilde{F}_E)(k) = \begin{cases} \beta(\underline{\bigcup}_{e\in\psi^{-1}(k)} \widetilde{F}_E(e)), & \text{if } \psi^{-1}(k) \neq \emptyset; \\ \underline{\emptyset}, & \text{otherwise.} \end{cases}$$

3. Let $\widetilde{G}_K \in SD(Y)_K$. The pre-image of $\widetilde{G}_K$ under the soft double mapping $f_{\beta\psi}$ is the SD-set over X, denoted by $f^{-1}_{\beta\psi}(\widetilde{G}_K)$ and defined by:

$$f^{-1}_{\beta\psi}(\widetilde{G}_K)(e) = \beta^{-1}(\widetilde{G}_K(\psi(e))).$$

Proposition 2.15. [21] Let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K, \widetilde{F}_E,\ \widetilde{G}_E \in SD(X)_E$ and $\widetilde{H}_K, \widetilde{L}_K \in SD(Y)_K$. Then,

1. If $\widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E$, then $f_{\beta\psi}(\widetilde{F}_E) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{G}_E)$.

2. If $\widetilde{H}_K \widetilde{\subseteq} \widetilde{L}_K$, then $f^{-1}_{\beta\psi}(\widetilde{H}_K) \widetilde{\subseteq} f^{-1}_{\beta\psi}(\widetilde{L}_K)$.

3. $\widetilde{F}_E \widetilde{\subseteq} f^{-1}_{\beta\psi}(f_{\beta\psi}(\widetilde{F}_E))$, the equality holds if $f_{\beta\psi}$ is an injective.

4. $f_{\beta\psi}(f^{-1}_{\beta\psi}(\widetilde{H}_K)) \widetilde{\subseteq} \widetilde{H}_K$, the equality holds if $f_{\beta\psi}$ is a surjective.

5. $f^{-1}_{\beta\psi}(\widetilde{H}^c_K) = (f^{-1}_{\beta\psi}(\widetilde{H}_K))^c$.

Definition 2.16. [21] Let $\widetilde{\tau}$ be a collection of SD-sets over X, i. e, $\widetilde{\tau} \subseteq SD(X)_E$. $\widetilde{\tau}$ is said to be a SD-topology over X if it satisfies the following conditions:

1. $\widetilde{\Phi}, \widetilde{X} \in \widetilde{\tau}$,

2. The union of any number of SD-sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,

3. The intersection of any two SD-sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triple $(X, \widetilde{\tau}, E)$ is called a SDTS. Every member of $\widetilde{\tau}$ is called an open SD-set and its complement is called a closed SD-set.

The family of all closed SD-sets we denoted by $\widetilde{\tau}^c$.

Definition 2.17. [21] Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in SD(X)_E$. $\widetilde{F}_E$ is called a quasi-neighborhood of a SD-point $\widetilde{x}^e_t$, if there exists $\widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{x}^e_t q \widetilde{G}_E \widetilde{\subseteq} \widetilde{F}_E$. The family of all quasi-neighborhoods of $\widetilde{x}^e_t$ denoted by $N^q_{(\widetilde{x}^e_t)_E}$.

Definition 2.18. [21] Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in SD(X)_E$. The soft double closure of $\widetilde{F}_E$, denoted by $cl_{\widetilde{\tau}}(\widetilde{F}_E)$, and defined by:
$cl_{\widetilde{\tau}}(\widetilde{F}_E) = \widetilde{\bigcap}\{\widetilde{G}_E \in \widetilde{\tau}^c : \widetilde{F}_E \widetilde{\subseteq} \widetilde{G}_E\}$.

Proposition 2.19. [21] Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in SD(X)_E$. Then, $cl_{\widetilde{\tau}}(\widetilde{F}_E)$ is the smallest closed SD-set containing $\widetilde{F}_E$.

Proposition 2.20. [21] Let $\widetilde{F}_E \in SD(X)_E$ and $\widetilde{x}_t^e \in SDP(X)_E$. Then,

$$\widetilde{x}_t^e \; q \; cl_{\widetilde{\tau}}(\widetilde{F}_E) \Leftrightarrow \; \forall \widetilde{G}_E \in \widetilde{\tau}, \widetilde{x}_t^e \widetilde{\in} \widetilde{G}_E, \widetilde{G}_E \; q \; \widetilde{F}_E.$$

Definition 2.21. [21] Let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$, where $\beta : X \to Y$ and $\psi : E \to K$. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\sigma}, K)$ be two SDT-spaces. $f_{\beta\psi}$ is called a soft double continuous mapping, denoted by SD-continuous, if $f_{\beta\psi}^{-1}(\widetilde{H}_K) \in \widetilde{\tau}$, whenever $\widetilde{H}_K \in \widetilde{\sigma}$.

Proposition 2.22. [21] Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\sigma}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping, $\widetilde{F}_E \in SD(X)_E$ and $\widetilde{H}_K \in SD(Y)_K$. Then, the following conditions are equivalent:

1. $f_{\beta\psi}$ is an SD-continuous,
2. $f_{\beta\psi}^{-1}(\widetilde{H}_K) \in \widetilde{\tau}^c, \; \forall \widetilde{H}_K \in \widetilde{\sigma}^c$,
3. $f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E)) \widetilde{\subseteq} cl_{\widetilde{\sigma}}(f_{\beta\psi}(\widetilde{F}_E)), \; \forall \widetilde{F}_E \in SD(X)_E$,
4. $cl_{\widetilde{\tau}}(f_{\beta\psi}^{-1}(\widetilde{H}_K)) \widetilde{\subseteq} f_{\beta\psi}^{-1}(cl_{\widetilde{\sigma}}(\widetilde{H}_K)), \; \forall \widetilde{H}_K \in SD(Y)_K$,

Definition 2.23. [10] A double topological space (X, η) is called $DT_{\frac{1}{2}}-$space iff for each $\underline{x}_t \in DP(X)$, either $\underline{x}_t$ is an open double set or $\underline{x}_t$ is a closed double set.

3 SD-separation axioms

Theorem 3.1. Let $\widetilde{F}_E, \widetilde{G}_E, \widetilde{H}_E \in SD(X)_E$. Then,

1. $\widetilde{F}_E \setminus \widetilde{G}_E = \widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c$.
2. $\widetilde{F}_E \setminus (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E) = (\widetilde{F}_E \setminus \widetilde{G}_E) \widetilde{\bigcap} (\widetilde{F}_E \setminus \widetilde{H}_E)$.
3. $\widetilde{F}_E \setminus (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E) = (\widetilde{F}_E \setminus \widetilde{G}_E) \widetilde{\bigcup} (\widetilde{F}_E \setminus \widetilde{H}_E)$.
4. $(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \setminus \widetilde{H}_E = (\widetilde{F}_E \setminus \widetilde{H}_E) \widetilde{\bigcap} (\widetilde{G}_E \setminus \widetilde{H}_E)$.

Proof.
1. $(\widetilde{F}_E \setminus \widetilde{G}_E)(e) = \widetilde{F}_E(e) \setminus \widetilde{G}_E(e) = \widetilde{F}_E(e) \underline{\bigcap} \widetilde{G}_E^c(e) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c)(e) \; \forall e \in E$. Hence $\widetilde{F}_E \setminus \widetilde{G}_E = \widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c$.
2. $\widetilde{F}_E \setminus (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E) = \widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcup} \widetilde{H}_E)^c = \widetilde{F}_E \widetilde{\bigcap} (\widetilde{G}_E^c \widetilde{\bigcap} \widetilde{H}_E^c) = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E^c) \widetilde{\bigcap} (\widetilde{F}_E \widetilde{\bigcap} \widetilde{H}_E^c) = (\widetilde{F}_E \setminus \widetilde{G}_E) \widetilde{\bigcap} (\widetilde{F}_E \setminus \widetilde{H}_E)$.
3. It is similar to (2).
4. $(\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \setminus \widetilde{H}_E = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{G}_E) \widetilde{\bigcap} \widetilde{H}_E^c = (\widetilde{F}_E \widetilde{\bigcap} \widetilde{H}_E^c) \widetilde{\bigcap} (\widetilde{G}_E \widetilde{\bigcap} \widetilde{H}_E^c) = (\widetilde{F}_E \setminus \widetilde{H}_E) \widetilde{\bigcap} (\widetilde{G}_E \setminus \widetilde{H}_E)$.

Proposition 3.2. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$. Then,

1. $x \neq y \Rightarrow \widetilde{x}_t^e \not{q}\, \widetilde{y}_r^{e'}$ for every $r, t \in \{\frac{1}{2}, 1\}, e, e' \in E$.

2. $\widetilde{x}_t^e \not{q}\, \widetilde{y}_r^e \Leftrightarrow x \neq y$ *or* $x = y, t = r = \frac{1}{2}$ *and*
   $\widetilde{x}_t^e q\, \widetilde{y}_r^{e'} \Leftrightarrow x = y$ *and* $t + r > 1$.

Proof. It is obvious.

Proposition 3.3. Let $(X, \widetilde{\tau}, E)$ be a SDTS and let $\widetilde{F}_E \in \tau, \widetilde{G}_E \in SD(X)_E$. Then, $\widetilde{F}_E\; q\; \widetilde{G}_E \Leftrightarrow \widetilde{F}_E\; q\; cl_{\widetilde{\tau}}(\widetilde{G}_E)$.

Proof. $\widetilde{F}_E \not{q}\, \widetilde{G}_E \Leftrightarrow \widetilde{G}_E \widetilde{\subseteq} \widetilde{F}_E^c \Leftrightarrow cl_{\widetilde{\tau}}(\widetilde{G}_E) \widetilde{\subseteq} \widetilde{F}_E^c[$ *by Proposition* $2.19] \Leftrightarrow \widetilde{F}_E \not{q}\, cl_{\widetilde{\tau}}(\widetilde{G}_E)$.

Definition 3.4. Let $\widetilde{\eta}$ be a collection of SD-sets over X, i. e, $\widetilde{\eta} \subseteq SD(X)_E$. Then, $\widetilde{\eta}$ is said to be a stratified soft double topology over X if it satisfies the following conditions:

1. $\widetilde{\Phi}, \widetilde{X}$ and $\widetilde{X}_\emptyset \in \widetilde{\eta}, \widetilde{X}_\emptyset(e) = (\emptyset, X), \; \forall\, e \in E$,

2. The union of any number of SD-sets in $\widetilde{\eta}$ belongs to $\widetilde{\eta}$,

3. The intersection of any two SD-sets in $\widetilde{\eta}$ belongs to $\widetilde{\eta}$.

The triple $(X, \widetilde{\eta}, E)$ is called a stratified soft double topological space (SSDTS). Each element of $\widetilde{\eta}$ is called an open SD-set in X. The complement of the open SD-set is called a closed SD-set.

Proposition 3.5. Let $f_{\beta\psi} : SD(X)_E \rightarrow SD(Y)_K, \widetilde{F}_E \in SD(X)_E$. Then, if $f_{\beta\psi}$ is one-one, onto, then $f_{\beta\psi}(\widetilde{F}_E^c) = (f_{\beta\psi}(\widetilde{F}_E))^c$.

Proof. Suppose that $f_{\beta\psi}$ is one-one, then $\widetilde{F}_E = f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E))$. Implies,

$$\widetilde{F}_E^c = (f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E)))^c = f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E))^c.$$

Since $f_{\beta\psi}$ is onto, then

$$f_{\beta\psi}(\widetilde{F}_E^c) = f_{\beta\psi}(f_{\beta\psi}^{-1}(f_{\beta\psi}(\widetilde{F}_E))^c) = (f_{\beta\psi}(\widetilde{F}_E))^c.$$

Hence, $f_{\beta\psi}(\widetilde{F}_E^c) = (f_{\beta\psi}(\widetilde{F}_E))^c$.

Definition 3.6. Let $(X, \widetilde{\tau}_1, E)$ and $(X, \widetilde{\tau}_2, E)$ be two SDTS over X.

1. If $\widetilde{\tau}_1 \subseteq \widetilde{\tau}_2$, then $\widetilde{\tau}_2$ is soft double finer than $\widetilde{\tau}_1$.

2. If $\widetilde{\tau}_1 \subset \widetilde{\tau}_2$, then $\widetilde{\tau}_2$ is soft double strictly finer than $\widetilde{\tau}_1$.

3. If $\widetilde{\tau}_1 \subseteq \widetilde{\tau}_2$ or $\widetilde{\tau}_2 \subseteq \widetilde{\tau}_1$, then $\widetilde{\tau}_1$ is comparable with $\widetilde{\tau}_2$.

Example 3.7. Let X be the universal set, E be the set of parameters.

1. If $\widetilde{\tau}$ is the collection of all SD-sets which can be defined over X. Then, $\widetilde{\tau}$ is called the discrete SD-topology on X and $(X, \widetilde{\tau}, E)$ is said to be a discrete SDTS over X.

2. $\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}\}$ is called the indiscrete SD-topology on X and $(X, \widetilde{\tau}, E)$ is said to be a indiscrete SDTS over X.

Definition 3.8. Let $\widetilde{F}_E \in SD(X)_E$. $\widetilde{F}_E$ is a finite SD-set if $\widetilde{F}_E(e)$ is a finite double set, $\forall e \in E$.

Example 3.9. Let X be an infinite set. The family

$$\widetilde{\tau}_\infty = \{\widetilde{\Phi}\}\widetilde{\bigcup}\{\widetilde{F}_E \widetilde{\subseteq} \widetilde{X} : \widetilde{F}^c_E \; is \; finite \}$$

is called a co-finite SD-topology on X.

Definition 3.10. Let $(X, \widetilde{\tau}, E)$ be a SDTS and let Y be a non-empty subset of X. $\widetilde{Y}$ denotes the SD-set over X, such that $\widetilde{Y}(e) = \underline{Y}$, $\forall \; e \in E$.

Definition 3.11. Let $(X, \widetilde{\tau}, E)$ be a SDTS and let Y be a non-empty subset of $X, \widetilde{F}_E \in SD(X)_E$. The SD-subset over Y, will denote by $\widetilde{F}^Y_E$, and defined by:

$$\widetilde{F}^Y_E(e) = \underline{Y}\underline{\bigcap}\widetilde{F}_E(e), \; \forall e \in E.$$

We write $\widetilde{F}^Y_E = \widetilde{Y}\widetilde{\bigcap}\widetilde{F}_E$.

Definition 3.12. Let $(X, \widetilde{\tau}, E)$ be a SDTS and Y be a non-empty subset of X. The soft double topology over Y, will denoted by $\widetilde{\tau}_Y$, and defined by:

$$\widetilde{\tau}_Y = \{\widetilde{F}^Y_E : \widetilde{F}_E \in \widetilde{\tau}\}.$$

$(Y, \widetilde{\tau}_Y, E)$ is called a SD-subspace of a SDTS $(X, \widetilde{\tau}, E)$.

Example 3.13. Any SD-subspace of a SD-discrete topological space is a SD-discrete. Also, any SD-subspace of a SD-indiscrete topological space is a SD-indiscrete.

Definition 3.14. A SDTS $(X, \widetilde{\tau}, E)$ is said to be:

1. SDT_0-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ or $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E$.

2. $SDT_{\frac{1}{2}}-$space if each $\widetilde{x}^e_t \in SDP(X)_E$ is either open SD-set or closed SD-set.

3. SDT^*_0-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ or $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x \neq y, \; \forall x, y \in X$.

4. SDT^{**}_0-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ or $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x = y, \; \forall x, y \in X$.

5. SDT_1-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ and $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E$.

6. SDT^*_1-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ and $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x \neq y, \; \forall x, y \in X$.

7. SDT^{**}_1-space if $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r \Rightarrow cl_{\widetilde{\tau}}(\widetilde{x}^e_t) \notin \widetilde{y}^{e'}_r$ and $cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r) \notin \widetilde{x}^e_t$, $\forall \widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E, x = y, \; \forall x, y \in X$.

8. SDT_2-space or soft double Hausdorff space if $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'} \Rightarrow \exists\ \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{y}_r^{e'}},\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$.

9. SDT_2^*-space if $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'} \Rightarrow \exists \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{y}_r^{e'}},\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E, x \neq y,\ \forall x, y \in X$.

10. $SDT_2^{**}-$space if $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'} \Rightarrow \exists \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{y}_r^{e'}},\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E, x = y,\ \forall x, y \in X$.

11. SDR_2-space if $\widetilde{x}_t^e \notin \widetilde{F} \Rightarrow \exists\ \widetilde{O}_{\widetilde{x}_t^e}, \widetilde{O}_{\widetilde{F}} \in \widetilde{\tau}$ such that $\widetilde{O}_{\widetilde{x}_t^e} \notin \widetilde{O}_{\widetilde{F}},\ \forall \widetilde{x}_t^e \in SDP(X)_E,\ \forall \widetilde{F} \in \widetilde{\tau}^c$.

12. SDT_3-space or soft double regular space if it is SDR_2 and SDT_1-spaces.

13. SDT_3^*-space if it is SDR_2 and SDT_1^*-spaces.

14. $SDT_3^{**}-$space if it is SDR_2 and $SDT_1^{**}-$spaces.

Theorem 3.15. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1-space (SDT_1^*-space) iff $\forall \widetilde{x}_t^e\ \not q\ \exists\ \widetilde{O}_{\widetilde{x}_t^e}$ such that $\widetilde{y}_r^{e'}\ \not q\ \widetilde{O}_{\widetilde{x}_t^e}$ and $\exists\ \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{x}_t^e \notin \widetilde{O}_{\widetilde{y}_r^{e'}}$.

Proof. It follows from Proposition 2.20.

Theorem 3.16. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1^*-space iff $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'}, \widetilde{y}_r^{e'}, x \neq y,\ \forall x, y \in X \exists\ \widetilde{O}_{\widetilde{x}_t^e}$ such that $\widetilde{y}_r^{e'} \notin \widetilde{O}_{\widetilde{x}_t^e}$ and $\exists\ \widetilde{O}_{\widetilde{y}_r^{e'}}$ such that $\widetilde{x}_t^e \notin \widetilde{O}_{\widetilde{y}_r^{e'}}$.

Proof. It is obvious.

Theorem 3.17. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1-space iff $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e),\ \forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a SDT_1-space and let $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'}$. Then, $cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \notin \widetilde{y}_r^{e'}$. By Theorem 3.15, there exists $\widetilde{O}_{y_r^{e'}}$ such that $\widetilde{x}_t^e \not q\ \widetilde{O}_{y_r^{e'}}$. This implies that $\widetilde{O}_{y_r^{e'}} \widetilde{\subseteq} (\widetilde{x}_t^e)^c$, thus $(\widetilde{x}_t^e)^c$ is open SD-set, $\forall \widetilde{x}_t^e \in SDP(X)_E$, i.e, $\widetilde{x}_t^e$ is closed SD-set, $\forall \widetilde{x}_t^e \in SDP(X)_E$. Conversely, Suppose that $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e),\ \forall \widetilde{x}_t^e \in SDP(X)_E$ and let $\widetilde{x}_t^e \notin \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e$ and $\widetilde{y}_r^{e'}$ are closed SD-sets. So that, $cl_{\widetilde{\tau}}(\widetilde{x}_t^e)\ \not q\ \widetilde{y}_r^{e'}$ and $cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'})\ \not q\ \widetilde{x}_t^e,\ \forall \widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$. Hence, $(X, \widetilde{\tau}, E)$ is a SDT_1.

Theorem 3.18. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_1^*-space iff $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e),\ \forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. It is obvious.

Theorem 3.19. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is SDT_2-space iff $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e}),\ \forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. Suppose $(X,\widetilde{\tau},E)$ is a SDT_2-space and let $\widetilde{x}_t^e \not q\, \widetilde{y}_r^{e'}$.
Then, $\exists\ \widetilde{O}_{x_t^e} \in N_{(\widetilde{x}_t^e)_E^q}$, $\widetilde{O}_{y_r^{e'}} \in N_{(\widetilde{y}_r^{e'})_E^q}$ such that $\widetilde{O}_{x_t^e} \not q\, \widetilde{O}_{y_r^{e'}}$. So that $\widetilde{O}_{y_r^{e'}} \not q\, \widetilde{O}_{x_t^e}$, implies $\widetilde{O}_{y_r^{e'}} \not q\, \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$. Thus, $\widetilde{x}_t^e \widetilde{\supseteq} \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$. It is clear that, $\widetilde{x}_t^e \widetilde{\subseteq} \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$. Hence, $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$.

Conversely, let $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$, $\forall \widetilde{x}_t^e \in SDP(X)_E$ and let $\widetilde{x}_t^e \not q\, \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e \not q\, \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^{e'}} \in N^q_{(\widetilde{y}_r^{e'})_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}})$. This implies that, $\widetilde{x}_t^e \not q\, cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}})$, for some $\widetilde{O}_{y_r^{e'}} \in N_{(\widetilde{y}_r^{e'})_E^q}$. So, $\widetilde{x}_t^e \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}}))^c$ and $\widetilde{O}_{x_t^e} = (cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{y}_r^{e'}}))^c \not q\, \widetilde{O}_{\widetilde{y}_r^{e'}}$. Therefore, $(X,\widetilde{\tau},E)$ is a SDT_2.

Theorem 3.20. Let $(X,\widetilde{\tau},E)$ be a SDTS. Then,
$(X,\widetilde{\tau},E)$ is SDT_2^*-space iff $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$, $\forall \widetilde{x}_t^e \in SDP(X)_E$.

Proof. It is obvious.

Theorem 3.21. Let $(X,\widetilde{\tau},E)$ be a SDTS. Then,
$(X,\widetilde{\tau},E)$ is a SDT_0-space $\rightarrow (X,\widetilde{\tau},E)$ is a SDT_0^*.

Proof. It is obvious.

Example 3.22. Let $X=\{h_1,h_2\}, E=\{e_1,e_2\}$ and let $\widetilde{\tau}=\{\widetilde{\Phi},\widetilde{X},\widetilde{F}_E^1,\widetilde{F}_E^2,\widetilde{F}_E^3,\widetilde{F}_E^4\}$, where
$\widetilde{F}_E^1(e_1)=\underline{\emptyset}, \widetilde{F}_E^1(e_2)=(\{h_2\},\{h_2\})$,
$\widetilde{F}_E^2(e_1)=\underline{\emptyset}, \widetilde{F}_E^2(e_2)=\underline{X}$,
$\widetilde{F}_E^3(e_1)=(\{h_1\},\{h_1\}), \widetilde{F}_E^3(e_2)=\underline{X}$,
$\widetilde{F}_E^4(e_1)=(\{h_2\},\{h_2\}), \widetilde{F}_E^4(e_2)=\underline{X}$.
Then, $(X,\widetilde{\tau},E)$ is a SDTS and SDT_0^*-space. But it is not SDT_0-space, for $\exists\ \widetilde{h}^{e_1}_{1_{\frac{1}{2}}} \in SDP(X)_E$ such that $\widetilde{h}^{e_1}_{1_{\frac{1}{2}}} \not q\, \widetilde{h}^{e_1}_{1_{\frac{1}{2}}}$, but $\widetilde{F}_E^{4c} = cl_{\widetilde{\tau}}(\widetilde{h}^{e_1}_{1_{\frac{1}{2}}})q\, \widetilde{h}^{e_1}_{1_{\frac{1}{2}}}$.

Theorem 3.23. Let $(X,\widetilde{\tau},E)$ be a SDTS. Then,
$(X,\widetilde{\tau},E)$ is a $SDT_{\frac{1}{2}}-$space $\rightarrow (X,\widetilde{\tau},E)$ is a SDT_0.

Proof. Suppose $(X,\widetilde{\tau},E)$ is a $SDT_{\frac{1}{2}}-$space and let $\widetilde{x}_t^e \not q\, \widetilde{y}_r^{e'}$. Now, if $\widetilde{x}_t^e$ is an open SD-point, then by Proposition 3.3 $cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'}) \not q\, \widetilde{x}_t^e$. On the other hand, if $\widetilde{x}_t^e$ is a closed SD-point, then $cl_{\widetilde{\tau}}(\widetilde{x}_t^e) = \widetilde{x}_t^e$. Implies, $cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \not q\, \widetilde{y}_r^{e'}$. Hence, $(X,\widetilde{\tau},E)$ is a SDT_0.

Example 3.24. Let $X=\{h_1,h_2\}, E=\{e_1,e_2\}$ and let $\widetilde{\tau}=\{\widetilde{\Phi},\widetilde{X},\widetilde{F}_E^1,\widetilde{F}_E^2,.......,\widetilde{F}_E^{37}\}$, where
$\widetilde{F}_E^1(e_1)=\underline{\emptyset}, \widetilde{F}_E^1(e_2)=(\{h_2\},\{h_2\})$,
$\widetilde{F}_E^2(e_1)=\underline{\emptyset}, \widetilde{F}_E^2(e_2)=\underline{X}$,
$\widetilde{F}_E^3(e_1)=(\{h_1\},\{h_1\}), \widetilde{F}_E^3(e_2)=\underline{X}$,
$\widetilde{F}_E^4(e_1)=(\{h_2\},\{h_2\}), \widetilde{F}_E^4(e_2)=\underline{X}$,
$\widetilde{F}_E^5(e_1)=\underline{X}, \widetilde{F}_E^5(e_2)=(\{h_2\},X)$,
$\widetilde{F}_E^6(e_1)=\underline{X}, \widetilde{F}_E^6(e_2)=(\{h_1\},X)$,

$\widetilde{F}_E^7(e_1) = (\{h_1\}, X), \widetilde{F}_E^7(e_2) = \underline{X},$
$\widetilde{F}_E^8(e_1) = (\{h_2\}, X), \widetilde{F}_E^8(e_2) = \underline{X},$
$\widetilde{F}_E^9(e_1) = \underline{X}, \widetilde{F}_E^9(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{10}(e_1) = (\{h_2\}, X), \widetilde{F}_E^{10}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{11}(e_1) = (\{h_1\}, X), \widetilde{F}_E^{11}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{12}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{12}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{13}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{13}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{14}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{14}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{15}(e_1) = (\{h_2\}, X), \widetilde{F}_E^{15}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{16}(e_1) = (\{h_1\}, X), \widetilde{F}_E^{16}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{17}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{17}(e_2) = (\emptyset, \{h_2\}),$
$\widetilde{F}_E^{18}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{18}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{19}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{19}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{20}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{20}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{21}(e_1) = (\{h_2\}, X), \widetilde{F}_E^{4}(e_{21}) = (\emptyset, X),$
$\widetilde{F}_E^{22}(e_1) = (\{h_1\}, X), \widetilde{F}_E^{4}(e_{22}) = (\emptyset, X),$
$\widetilde{F}_E^{23}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{23}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{24}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{24}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{25}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{25}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{26}(e_1) = (\emptyset, X), \widetilde{F}_E^{26}(e_2) = \underline{X},$
$\widetilde{F}_E^{27}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{27}(e_2) = \underline{X},$
$\widetilde{F}_E^{28}(e_1) = (\emptyset, X), \widetilde{F}_E^{28}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{29}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{29}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{30}(e_1) = (\emptyset, X), \widetilde{F}_E^{30}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{31}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{31}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{32}(e_1) = (\emptyset, X), \widetilde{F}_E^{32}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{33}(e_1) = (\emptyset, \{h_1\}), \widetilde{F}_E^{33}(e_2) = (\emptyset, X),$
$\widetilde{F}_E^{34}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{34}(e_2) = \underline{X},$
$\widetilde{F}_E^{35}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{35}(e_2) = (\{h_2\}, X),$
$\widetilde{F}_E^{36}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{36}(e_2) = (\{h_1\}, X),$
$\widetilde{F}_E^{37}(e_1) = (\emptyset, \{h_2\}), \widetilde{F}_E^{37}(e_2) = (\emptyset, X).$
Then, $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_0-space. But it is not $SDT_{\frac{1}{2}}-$space, for $\exists\, \widetilde{h}_{1_1}^{e_2} \in SDP(X)_E$, such that $\widetilde{h}_{1_1}^{e_2}$ is neither open nor closed SD-point.

Theorem 3.25. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_1-space $\rightarrow (X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}$.

Proof. Suppose $(X, \widetilde{\tau}, E)$ is a SDT_1-space, then every SD-point in X is a closed SD-point by Theorem 3.17. Hence, $(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}$.

Example 3.26. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and let
$\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}_E^1, \widetilde{F}_E^2, \widetilde{F}_E^3, \widetilde{F}_E^4, \widetilde{F}_E^5, \widetilde{F}_E^6,$
$\widetilde{F}_E^7, \widetilde{F}_E^8, \widetilde{F}_E^9, \widetilde{F}_E^{10}, \widetilde{F}_E^{11}, \widetilde{F}_E^{12}\}, \widetilde{F}_E^{13}, \widetilde{F}_E^{14}, \widetilde{F}_E^{15}\},$

where $\widetilde{F}^1_E(e_1)=\underline{\emptyset},\widetilde{F}^1_E(e_2)=(\{h_1\},\{h_1\})$,
$\widetilde{F}^2_E(e_1)=\underline{\emptyset},\widetilde{F}^2_E(e_2)=(\emptyset,\{h_1\})$,
$\widetilde{F}^3_E(e_1)=(\{h_1\},\{h_1\}),\widetilde{F}^3_E(e_2)=\underline{\emptyset}$,
$\widetilde{F}^4_E(e_1)=(\emptyset,\{h_1\}),\widetilde{F}^4_E(e_2)=\underline{\emptyset}$,
$\widetilde{F}^5_E(e_1)=\underline{X},\widetilde{F}^5_E(e_2)=(\{h_1\},\{h_1\})$,
$\widetilde{F}^6_E(e_1)=\underline{X},\widetilde{F}^6_E(e_2)=(\{h_1\},X)$,
$\widetilde{F}^7_E(e_1)=(\{h_1\},\{h_1\}),\widetilde{F}^7_E(e_2)=\underline{X}$,
$\widetilde{F}^8_E(e_1)=(\{h_1\},X),\widetilde{F}^8_E(e_2)=\underline{X}$,
$\widetilde{F}^9_E(e_1)=(\{h_1\},\{h_1\}),\widetilde{F}^9_E(e_2)=(\{h_1\},\{h_1\})$,
$\widetilde{F}^{10}_E(e_1)=(\emptyset,\{h_1\}),\widetilde{F}^{10}_E(e_2)=(\{h_1\},\{h_1\})$,
$\widetilde{F}^{11}_E(e_1)=(\{h_1\},\{h_1\}),\widetilde{F}^{11}_E(e_2)=(\emptyset,\{h_1\})$,
$\widetilde{F}^{12}_E(e_1)=(\emptyset,\{h_1\}),\widetilde{F}^{12}_E(e_2)=(\emptyset,\{h_1\})$,
$\widetilde{F}^{13}_E(e_1)=(\{h_1\},X),\widetilde{F}^{13}_E(e_2)=(\{h_1\},\{h_1\})$,
$\widetilde{F}^{14}_E(e_1)=(\{h_1\},\{h_1\}),\widetilde{F}^{14}_E(e_2)=(\{h_1\},X)$,
$\widetilde{F}^{15}_E(e_1)=(\{h_1\},X),\widetilde{F}^{15}_E(e_2)=(\{h_1\},X)$.

Then, $(X,\widetilde{\tau},E)$ is a SDTS and $SDT_{\frac{1}{2}}-$space. But it is not SDT_1-space for the SD-point $\widetilde{h}^{e_1}_{1_{\frac{1}{2}}}$ is not a closed SD-point.

Theorem 3.27. Let $(X,\widetilde{\tau},E)$ be a SDTS. Then,
$(X,\widetilde{\tau},E)$ is a SDT_2-space $\rightarrow (X,\widetilde{\tau},E)$ is a SDT_1.

Proof. Suppose $(X,\widetilde{\tau},E)$ is a SDT_2-space, then $\widetilde{x}^e_t=\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}^e_t}\in N^q_{(\widetilde{x}^e_t)_E}} cl_{\widetilde{\tau}}\widetilde{O}_{\widetilde{x}^e_t},\ \forall \widetilde{x}^e_t\in SDP(X)$. It follows that, every SD-point in X is a closed SD-point. Hence by Theorem 3.17, $(X,\widetilde{\tau},E)$ is a SDT_1.

Example 3.28. Let N be the set of all natural numbers. Then, the family $\widetilde{\tau}_N=\{\widetilde{\Phi}\}\widetilde{\bigcup}\{\widetilde{F}_E\widetilde{\subseteq}\widetilde{N}:\widetilde{F}^c_E\ is\ finite\ \}$ is a co-finite SD-topology over X, $(N,\widetilde{\tau},E)$ is a co-finite SDTS and SDT_1-space. But it is not SDT_2-space for, $\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{n}^e_t}\in N^q_{(\widetilde{n}^e_t)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{n}^e_t})=\widetilde{N}\neq\widetilde{n}^e_t$.

Theorem 3.29. Let $(X,\widetilde{\tau},E)$ be a SDTS. Then,
$(X,\widetilde{\tau},E)$ is a SDT_3-space $\rightarrow (X,\widetilde{\tau},E)$ is a SDT_2-space.

Proof. Suppose $(X,\widetilde{\tau},E)$ is a SDT_3-space and let $\widetilde{x}^e_t \not q\, \widetilde{y}^{e'}_r$. Then, $\widetilde{x}^e_t=cl_{\widetilde{\tau}}(\widetilde{x}^e_t),\ \forall\widetilde{x}^e_t\in SDP(X)$ [by hypothesis]. It follows that $\exists\ \widetilde{O}_{\widetilde{y}^{e'}_r}\in N^q_{(\widetilde{y}^{e'}_r)_E},\widetilde{O}_{\widetilde{x}^e_t}\in N^q_{(\widetilde{x}^e_t)_E}$ such that $\widetilde{O}_{\widetilde{y}^{e'}_r}\not q\,\widetilde{O}_{\widetilde{x}^e_t}$. Hence, $(X,\widetilde{\tau},E)$ is a SDT_2-space.

Theorem 3.30. Let $(X,\widetilde{\tau},E)$ be a SDTS. Then,
$(X,\widetilde{\tau},E)$ is a SDT^*_1-space $\rightarrow (X,\widetilde{\tau},E)$ is a SDT^*_0.

Proof. It is obvious.

Example 3.31. From example 3.22, we have $(X,\widetilde{\tau},E)$ is a SDT^*_0-space. But it is not SDT^*_1-space for, $\widetilde{h}^{e_1}_{1_1}\not q\,\widetilde{h}^{e_2}_{2_1}$, but $cl(\widetilde{h}^{e_2}_{2_1})=\widetilde{X}\ q\ \widetilde{h}^{e_1}_{1_1}$.

Theorem 3.32. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_1-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_1^*.

Proof. It is obvious.

Example 3.33. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and let $\widetilde{\tau} = \{\widetilde{\Phi}, \widetilde{X}, \widetilde{F}_E^1, \widetilde{F}_E^2, \widetilde{F}_E^3, \widetilde{F}_E^4, \widetilde{F}_E^5, \widetilde{F}_E^6, \widetilde{F}_E^7, \widetilde{F}_E^8, \widetilde{F}_E^9, \widetilde{F}_E^{10}, \widetilde{F}_E^{11}, \widetilde{F}_E^{12}\}, \widetilde{F}_E^{13}, \widetilde{F}_E^{14}\}$, where
$\widetilde{F}_E^1(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^1(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^2(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^2(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^3(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^3(e_2) = \underline{\emptyset}$,
$\widetilde{F}_E^4(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^4(e_2) = \underline{\emptyset}$,
$\widetilde{F}_E^5(e_1) = \underline{\emptyset}, \widetilde{F}_E^5(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^6(e_1) = \underline{\emptyset}, \widetilde{F}_E^6(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^7(e_1) = \widetilde{X}, \widetilde{F}_E^7(e_2) = \underline{\emptyset}$,
$\widetilde{F}_E^8(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^8(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^9(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^9(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^{10}(e_1) = \widetilde{X}, \widetilde{F}_E^{10}(e_2) = (\{h_1\}, \{h_1\})$,
$\widetilde{F}_E^{11}(e_1) = \widetilde{X}, \widetilde{F}_E^{11}(e_2) = (\{h_2\}, \{h_2\})$,
$\widetilde{F}_E^{12}(e_1) = \underline{\emptyset}, \widetilde{F}_E^{12}(e_2) = \underline{X}$,
$\widetilde{F}_E^{13}(e_1) = (\{h_1\}, \{h_1\}), \widetilde{F}_E^{13}(e_2) = \underline{X}$,
$\widetilde{F}_E^{14}(e_1) = (\{h_2\}, \{h_2\}), \widetilde{F}_E^{14}(e_2) = \underline{X}$.
Then, $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_1^*-space. But it is not SDT_1-space for, $\widetilde{h}_{1\frac{1}{2}}^{e_1} \neq cl(\widetilde{h}_{1\frac{1}{2}}^{e_1})$.

Theorem 3.34. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_2^*-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_1^*.

Proof. It follows from Theorem 3.16, 3.18.

Example 3.35. From example 3.28, we have $(N, \widetilde{\tau}, E)$ is a co-finite SDTS and SDT_1^*-space. But it is not SDT_2^*-space for, $\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{n}_t^e} \in N^q_{(\widetilde{n}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{n}_t^e}) = \widetilde{N} \neq \widetilde{n}_t^e$.

Theorem 3.36. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_2-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_2^*.

Proof. It is obvious.

Example 3.37. From example 3.33, we have $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_2^*-space. But it is not SDT_2-space, for $\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{h}_1^{e_1}} \in N^q_{(\widetilde{h}_1^{e_1})_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{h}_1^{e_1}}) = (\widetilde{F}_E^{14})^c \neq \widetilde{h}_1^{e_1}$.

Theorem 3.38. Let $(X, \widetilde{\tau}, E)$ be a SDTS. Then,
$(X, \widetilde{\tau}, E)$ is a SDT_3-space $\rightarrow (X, \widetilde{\tau}, E)$ is a SDT_3^*.

Proof. It is obvious.

Example 3.39. From example 3.33, we have $(X, \widetilde{\tau}, E)$ is a SDTS and SDT_3^*-space. But it is not SDT_3-space, since $(X, \widetilde{\tau}, E)$ is not a SDT_1-space.

Remark 3.40. 1. From example 3.26 $(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}-$space, but it is not SDT_1^*. and from example 3.33 $(X, \widetilde{\tau}, E)$ is a SDT_1^*-space, but it is not $SDT_{\frac{1}{2}}$.

2. From example 3.28 $(X, \widetilde{\tau}, E)$ is a SDT_1-space, but it is not SDT_2^*. and from example 3.33 $(X, \widetilde{\tau}, E)$ is a SDT_2^*-space, but it is not SDT_1.

3. From example 3.33 $(X, \widetilde{\tau}, E)$ is a SDT_3^*-space, but it is not SDT_2.

Remark 3.41. Theorems 3.16, 3.18, 3.20, 3.21, 3.30, 3.32, 3.34, 3.36, 3.38 are satisfied if we replace SDT_i^* by SDT_i^{**}, $(i = 0, 1, 2, 3)$.

Remark 3.42. Let $(X, \widetilde{\tau}, E)$ be a $SDTS$. Then,

1. SDT_i^* is SDT_i, $(i = 0, 1, 3)$ iff $\forall\, x \in X$, $\widetilde{x}^e_{\frac{1}{2}} \not\in cl_{\widetilde{\tau}}(\widetilde{x}^e_{\frac{1}{2}})$.

2. SDT_2^* is SDT_2 iff $\forall\, x \in X$, $\exists\, \widetilde{O}_{\widetilde{x}^e_{\frac{1}{2}}} \not\in \widetilde{O}_{\widetilde{x}^e_{\frac{1}{2}}}$.

Corollary 3.43. For a $SDTS$ $(X, \widetilde{\tau}, E)$ we have the following implication:

$$\begin{array}{ccccccccc} SDT_3^* & \to & SDT_2^* & \to & SDT_1^* & & \to & & SDT_0^*. \\ \uparrow & & \uparrow & & \uparrow & & & \nwarrow & \\ SDT_3 & \to & SDT_2 & \to & SDT_1 & \to & SDT_{\frac{1}{2}} & \to & SDT_0. \\ \downarrow & & \downarrow & & \downarrow & & & \swarrow & \\ SDT_3^{**} & \to & SDT_2^{**} & \to & SDT_1^{**} & & \to & & SDT_0^{**}. \end{array}$$

4 SD-subspaces

Theorem 4.1. Let $(Y, \widetilde{\tau}_Y, E)$ be a SD-subspace of a SD-space $(X, \widetilde{\tau}, E)$ and $\widetilde{F}_E \in SD(X)_E$. Then,

1. If $\widetilde{F}_E \in \widetilde{\tau}_Y$ and $\widetilde{Y}_E \in \widetilde{\tau}$, then $\widetilde{F}_E \in \widetilde{\tau}$.

2. $\widetilde{F}_E \in \widetilde{\tau}^c_Y$ iff $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$ for some $\widetilde{G}_E \in \widetilde{\tau}^c$.

Proof. 1. Let $\widetilde{F}_E \in \widetilde{\tau}_Y$. Then, $\exists \widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$. Now, if $\widetilde{Y}_E \in \widetilde{\tau}$, then $\widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E \in \widetilde{\tau}$. Hence, $\widetilde{F}_E \in \widetilde{\tau}$.

2. Let $\widetilde{F}_E \in \widetilde{\tau}^c_Y$. Then, $\widetilde{F}_E = \widetilde{Y}_E \setminus \widetilde{G}_E, \widetilde{G}_E \in \widetilde{\tau}_Y$ and $\widetilde{G}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E$ for some $\widetilde{H}_E \in \widetilde{\tau}$.
Now,$\widetilde{F}_E = \widetilde{Y}_E \setminus (\widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E) = \widetilde{Y}_E \setminus \widetilde{H}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}^c_E$, where $\widetilde{H}^c_E \in \widetilde{\tau}^c$. Therefore, $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$ for some $\widetilde{G}_E \in \widetilde{\tau}^c$.
Conversely, suppose that $\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$ for some $\widetilde{G}_E \in \widetilde{\tau}^c$, then
$\widetilde{F}_E = \widetilde{Y}_E \widetilde{\bigcap} \widetilde{G}_E$
$= \widetilde{Y}_E \widetilde{\bigcap} (\widetilde{X} \setminus \widetilde{H}_E), (\widetilde{G}_E = \widetilde{X} \setminus \widetilde{H}_E, \widetilde{H}_E \in \widetilde{\tau})$
$= \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}^c_E$
$= \widetilde{Y}_E \setminus \widetilde{H}_E$
$= \widetilde{Y}_E \setminus (\widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E), \widetilde{Y}_E \widetilde{\bigcap} \widetilde{H}_E \in \widetilde{\tau}_Y$.
Therefore, $\widetilde{F}_E \in \widetilde{\tau}^c_Y$. Hence, the result.

Theorem 4.2. Let $\widetilde{F}_E \in SD(X)_E, \widetilde{x}_t^e \in SDP(X)_E$ and $Y \subseteq X$. Then,
$\widetilde{x}_t^e \; q \; \widetilde{F}_E$ and $\widetilde{x}_t^e \widetilde{\in} \widetilde{Y} \Leftrightarrow \widetilde{x}_t^e \; q \; (\widetilde{F}_E \widetilde{\cap} \widetilde{Y})$.

Proof. If $t = 1$
$\widetilde{x}_1^e \; q \; \widetilde{F}_E$ and $\widetilde{x}_1^e \widetilde{\in} \widetilde{Y}$
$\Leftrightarrow \widetilde{x}_1^e(e) \; q \; \widetilde{F}_E(e)$ and $\widetilde{x}_1^e(e) \underline{\in} \widetilde{Y}(e), e \in E$
$\Leftrightarrow \underline{x}_1 \; q \; \widetilde{F}_E(e) = (A_1, A_2)$ and $\underline{x}_1 \underline{\in} \widetilde{Y}(e) = \underline{Y} = (Y, Y), \; e \in E$
$\Leftrightarrow (x \in A_1 \; or \; x \in A_2)$ and $x \in Y$
$\Leftrightarrow x \in (A_1 \cap Y) \; or \; x \in (A_2 \cap Y)$
$\Leftrightarrow \underline{x}_1 \; q \; (\widetilde{F}_E(e) \underline{\sqcap} \; \underline{Y})$
$\Leftrightarrow \widetilde{x}_1^e \; q \; (\widetilde{F}_E \widetilde{\cap} \widetilde{Y})$.
If $t = \frac{1}{2}$
$\widetilde{x}_{\frac{1}{2}}^e \; q \; \widetilde{F}_E$ and $\widetilde{x}_{\frac{1}{2}}^e \widetilde{\in} \widetilde{Y}$
$\Leftrightarrow \widetilde{x}_{\frac{1}{2}}^e(e) \; q \; \widetilde{F}_E(e)$ and $\widetilde{x}_{\frac{1}{2}}^e(e) \underline{\in} \widetilde{Y}(e), e \in E$
$\Leftrightarrow \underline{x}_{\frac{1}{2}} \; q \; \widetilde{F}_E(e) = (A_1, A_2)$ and $\underline{x}_{\frac{1}{2}} \underline{\in} \widetilde{Y}(e) = \underline{Y} = (Y, Y), \; e \in E$
$\Leftrightarrow (x \in A_1)$ and $x \in Y$
$\Leftrightarrow x \in (A_1 \cap Y)$
$\Leftrightarrow \underline{x}_{\frac{1}{2}} \; q \; (\widetilde{F}_E(e) \underline{\sqcap} \; \underline{Y})$
$\Leftrightarrow \widetilde{x}_{\frac{1}{2}}^e \; q \; (\widetilde{F}_E \widetilde{\cap} \widetilde{Y})$.
Hence, the result.

Theorem 4.3. Let $(Y, \widetilde{\tau}_Y, E)$ be a SD-subspace of a SD-space $(X, \widetilde{\tau}, E)$ and let $\widetilde{N}_E^Y \in SD(Y)_E, \widetilde{y}_r^e \in SDP(Y)_E$. Then, if $\widetilde{N}_E^Y = \widetilde{Y} \widetilde{\cap} \widetilde{N}_E$ for some $\widetilde{N}_E \in \widetilde{N}^q(\widetilde{y}_r^e)_E$, then $\widetilde{N}_E^Y \in \widetilde{N}_Y^q(\widetilde{y}_r^e)_E$ ($nbd.w.r.t(Y, \widetilde{\tau}_Y, E)$).

Proof. Let $\widetilde{N}_E^Y = \widetilde{Y} \widetilde{\cap} \widetilde{N}_E, \widetilde{N}_E \in \widetilde{N}^q(\widetilde{y}_r^e)_E$. Then, $\exists \widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{y}_r^e \; q \; \widetilde{G}_E \widetilde{\subseteq} \widetilde{N}_E$. Thus, $\widetilde{y}_r^e \; q \; \widetilde{G}_E \widetilde{\cap} \widetilde{Y} \widetilde{\subseteq} \widetilde{N}_E \widetilde{\cap} \widetilde{Y} = \widetilde{N}_E^Y$. Therefore, $\widetilde{y}_r^e \; q \; \widetilde{G}_E^Y \widetilde{\subseteq} \widetilde{N}_E^Y$. Hence, $\widetilde{N}_E^Y \in \widetilde{N}_Y^q(\widetilde{y}_r^e)_E$.

Theorem 4.4. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_0^*-space $(X, \widetilde{\tau}, E)$ is a SDT_0^*.

Proof. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(Y)_E, \; x \neq y$ such that $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E, \; x \neq y$ and $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Implies, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e)$. Thus, $\widetilde{x}_t^e \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'}) \widetilde{\cap} \widetilde{Y}$ or $\widetilde{y}_r^{e'} \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \widetilde{\cap} \widetilde{Y}$. Therefore, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{x}_t^e)$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_0^*.

Theorem 4.5. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a SDT_0-space $(X, \widetilde{\tau}, E)$ is a SDT_0.

Proof. Let $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(Y)_E$ such that $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Then, $\widetilde{x}_t^e, \widetilde{y}_r^{e'} \in SDP(X)_E$ and $\widetilde{x}_t^e \not{q} \; \widetilde{y}_r^{e'}$. Implies, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e)$. Thus, $\widetilde{x}_t^e \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{y}_r^{e'}) \widetilde{\cap} \widetilde{Y}$ or $\widetilde{y}_r^{e'} \widetilde{\cap} \widetilde{Y} \not{q} \; cl_{\widetilde{\tau}}(\widetilde{x}_t^e) \widetilde{\cap} \widetilde{Y}$. Therefore, $\widetilde{x}_t^e \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{y}_r^{e'})$ or $\widetilde{y}_r^{e'} \not{q} \; cl_{\widetilde{\tau_Y}}(\widetilde{x}_t^e)$. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a SDT_0.

Theorem 4.6. A SD-subspace $(Y, \widetilde{\tau}_Y, E)$ of a $SDT_{\frac{1}{2}}-$space $(X, \widetilde{\tau}, E)$ is a $SDT_{\frac{1}{2}}$.

Proof. Let $\widetilde{y}_r^e \in SDP(Y)_E$. Then, $\widetilde{y}_r^e \in SDP(X)_E$. This implies that, $\widetilde{y}_r^e$ is an open or closed SD-set in X. Therefore, $\widetilde{y}_r^e = \widetilde{y}_r^e \widetilde{\cap} \widetilde{Y}$ is an open or closed SD-set in Y. Hence, $(Y, \widetilde{\tau}_Y, E)$ is a $SDT_{\frac{1}{2}}$.

Theorem 4.7. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a SDT_1-space $(X,\widetilde{\tau},E)$ is a SDT_1.

Proof. Let $\widetilde{y}^e_r \in SDP(Y)_E$. Then, $\widetilde{y}^e_r \in SDP(X)_E$. This implies that, $\widetilde{y}^e_r = cl_{\widetilde{\tau}}(\widetilde{y}^e_r)$. It follows that, $\widetilde{y}^e_r \widetilde{\cap} \widetilde{Y} = cl_{\widetilde{\tau}}(\widetilde{y}^e_r)\widetilde{\cap}\widetilde{Y}$. Therefore, $\widetilde{y}^e_r = cl_{\widetilde{\tau}_Y}(\underline{y}_r)$. Hence, $(Y,\widetilde{\tau}_Y,E)$ is a SDT_1.

Theorem 4.8. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a SDT_1^*-space $(X,\widetilde{\tau},E)$ is a SDT_1^*.

Proof. Let $\widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(Y)_E$ such that $\widetilde{x}^e_t \not\Uparrow \widetilde{y}^{e'}_r$. Then, $\widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E$ and $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r$. This implies that, $\widetilde{x}^e_t \notin cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r)$ and $\widetilde{y}^{e'}_r \notin cl_{\widetilde{\tau}}(\widetilde{x}^e_t)$. Thus, $\widetilde{x}^e_t\widetilde{\cap}\widetilde{Y} \notin cl_{\widetilde{\tau}}(\widetilde{y}^{e'}_r)\widetilde{\cap}\widetilde{Y}$ and $\widetilde{y}^{e'}_r\widetilde{\cap}\widetilde{Y} \notin cl_{\widetilde{\tau}}(\widetilde{x}^e_t)\widetilde{\cap}\widetilde{Y}$. Therefore, $\widetilde{x}^e_t \notin cl_{\widetilde{\tau_Y}}(\widetilde{y}^{e'}_r)$ and $\widetilde{y}^{e'}_r \notin cl_{\widetilde{\tau_Y}}(\widetilde{x}^e_t)$. Hence, $(Y,\widetilde{\tau}_Y,E)$ is a SDT_1^*.

Theorem 4.9. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a SDT_2-space $(X,\widetilde{\tau},E)$ is a SDT_2.

Proof. Let $\widetilde{y}^e_r \in SDP(Y)_E$. Then, $\widetilde{y}^e_r \in SDP(X)_E$. Implies, $\widetilde{y}^e_r = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}^e_r}\in N^q_{(\widetilde{y}^e_r)_E}} cl_{\widetilde{\tau}}\widetilde{O}_{\widetilde{y}^e_r}$. It follows that, $\widetilde{y}^e_r\widetilde{\cap}\widetilde{Y} = [\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}^e_r}\in N^q_{(\widetilde{y}^e_r)_E}} cl_{\widetilde{\tau}}\widetilde{O}_{\widetilde{y}^e_r}]\widetilde{\cap}\widetilde{Y}$. Therefore, $\widetilde{y}^e_r = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}^e_r}\in N^q_Y(\widetilde{y}^e_r)_E} cl_{\widetilde{\tau}_Y}\widetilde{O}_{\widetilde{y}^e_r}$. Hence, $(Y,\widetilde{\tau}_Y,E)$ is a SDT_2.

Theorem 4.10. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a SDT_2^*-space $(X,\widetilde{\tau},E)$ is a SDT_2^*.

Proof. Let $\widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(Y)_E$ such that $\widetilde{x}^e_t \not\Uparrow \widetilde{y}^{e'}_r$. Then, $\widetilde{x}^e_t, \widetilde{y}^{e'}_r \in SDP(X)_E$ and $\widetilde{x}^e_t \notin \widetilde{y}^{e'}_r$. This implies that, there exist $\widetilde{O}_{\widetilde{x}^e_t}, \widetilde{O}_{\widetilde{y}^{e'}_r} \widetilde{\in} \widetilde{\tau}$ such that $\widetilde{O}_{\widetilde{x}^e_t} \not\Uparrow \widetilde{O}_{\widetilde{y}^{e'}_r}$. It follows that, $\widetilde{O}^*_{\widetilde{x}^e_t} = \widetilde{O}_{\widetilde{x}^e_t}\widetilde{\cap}\widetilde{Y} \not\Uparrow \widetilde{O}_{\widetilde{y}^{e'}_r}\widetilde{\cap}\widetilde{Y} = \widetilde{O}^*_{\widetilde{y}^{e'}_r}$ and $\widetilde{O}^*_{\widetilde{x}^e_t}, \widetilde{O}^*_{\widetilde{y}^{e'}_r} \in \widetilde{\tau}_Y$. Hence, $(Y,\widetilde{\tau}_Y,E)$ is a SDT_2^*.

Theorem 4.11. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a SDR_2-space $(X,\widetilde{\tau},E)$ is a SDR_2.

Proof. Let $\widetilde{y}^e_r \in SDP(Y)_E$ and $\widetilde{y}^e_r \not\Uparrow \widetilde{F}\widetilde{\cap}\widetilde{Y}, \widetilde{F} \in \widetilde{\tau}^c$. Then, $\widetilde{y}^e_r \not\Uparrow \widetilde{F}$ [by Proposition 2.13]. Implies, there exist $\widetilde{O}_{\widetilde{y}^e_r}, \widetilde{O}_{\widetilde{F}} \in \widetilde{\tau}$ such that $\widetilde{O}_{\widetilde{y}^e_r} \not\Uparrow \widetilde{O}_{\widetilde{F}}$. It follows that, $\widetilde{O}^Y_{\widetilde{y}^e_r} = \widetilde{O}_{\widetilde{y}^e_r}\widetilde{\cap}\widetilde{Y} \notin \widetilde{O}_{\widetilde{F}}\widetilde{\cap}\widetilde{Y} = \widetilde{O}^Y_{\widetilde{F}}$ and $\widetilde{O}^Y_{\widetilde{y}^e_r}, \widetilde{O}^Y_{\widetilde{F}} \in \widetilde{\tau}_Y$. Hence, $(Y,\widetilde{\tau}_Y,E)$ is a SDR_2.

Theorem 4.12. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a SDT_3-space $(X,\widetilde{\tau},E)$ is a SDT_3.

Proof. It follows from theorem 4.7 and theorem 4.11.

Theorem 4.13. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a SDT_3^*-space $(X,\widetilde{\tau},E)$ is a SDT_3^*.

Proof. It follows from theorem 4.8 and theorem 4.11.

Theorem 4.14. A SD-subspace $(Y,\widetilde{\tau}_Y,E)$ of a $SDT_i^{**}-$space $(X,\widetilde{\tau},E)$ is a $SDT_i^{**}, (i=0,1,2,3)$.

Proof. It is obvious.

5 Some Properties of the SD-continuous Functions

In this section, we study the behavior of the separation axioms under open (homeomorphism) mappings.

Definition 5.1. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping and $\widetilde{F}_E \in SD(X)_E$.

1. $f_{\beta\psi}$ is called SD-open if $f_{\beta\psi}(\widetilde{F}_E) \in \widetilde{\eta}$, $\forall \widetilde{F}_E \in \widetilde{\tau}$.

2. $f_{\beta\psi}$ is called SD-closed if $f_{\beta\psi}(\widetilde{F}_E) \in \widetilde{\eta}^c$, $\forall \widetilde{F}_E \in \widetilde{\tau}^c$.

Theorem 5.2. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping and $\widetilde{F}_E \in SD(X)_E$. Then, $f_{\beta\psi}$ is SD-closed iff $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))$, $\forall \widetilde{F}_E \in SD(X)_E$.

Proof. Suppose $f_{\beta\psi}$ is SD-closed and $\widetilde{F}_E \in SD(X)_E$, then $\widetilde{F}_E \widetilde{\subseteq} cl_{\widetilde{\tau}}(\widetilde{F}_E)$, and so $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} cl_{\widetilde{\eta}}(f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))) = f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E)), cl_{\widetilde{\tau}}(\widetilde{F}_E) \in \widetilde{\tau}^c$.
Therefore, $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))$.

Conversely, suppose $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{F}_E))$, $\forall \widetilde{F}_E \in SD(X)_E$. Let $\widetilde{F}_E$ be an SD-closed in X, then $cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E)) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{F}_E)$. But $f_{\beta\psi}(\widetilde{F}_E) \widetilde{\subseteq} cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E))$, so that $f_{\beta\psi}(\widetilde{F}_E) = cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{F}_E))$. Therefore, $f_{\beta\psi}$ is SD-closed. Hence, the result.

Lemma 5.3. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDTS and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a (one-one) and onto mapping. Then:

1. If $\widetilde{y}_t^k \in SDP(Y)_K$, then $\exists\, x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k, \widetilde{x}_t^e \in SDP(X)_E$ and $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$.

2. If $\widetilde{y}_t^k \in SDP(Y)_K$, then $f^{-1}(\widetilde{y}_t^e) \in SDP(X)_E$.

3. If $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K, \widetilde{y_1}_t^{k_1} \notin \widetilde{y_2}_r^{k_2}$, then $\exists\, x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}, \widetilde{x_1}_t^{e_1} \notin \widetilde{x_2}_r^{e_2}$.

Proof. 1. $f_{\beta\psi}(\widetilde{x}_t^e)(k)$
$= \beta(\bigcup_{e \in \psi^{-1}(k)} \widetilde{x}_t^e(e))$
$= \beta(\widetilde{x}_t^e(e))$
$= \beta(\widetilde{x}_t)$
$= (\widetilde{y}_t), \psi(e) = k$
$= \widetilde{y}_t^k(k)$.
Therefore, $f_{\beta\psi}(\widetilde{x}_t^e) = \widetilde{y}_t^k$.

2. $f_{\beta\psi}^{-1}(\widetilde{y_1}_t^k)(e_1)$
$= \beta^{-1}(\widetilde{y_1}_t^k(\psi(e_1)))$
$= \beta^{-1}(\widetilde{y_1}_t(k)), \psi(e_1) = k$
$= \widetilde{x_1}_t(e_1), \psi^{-1}(k) = e_1$

$= \widetilde{x_1}_t^{e_1}(e_1)$.
Thus, $f_{\beta\psi}^{-1}(\widetilde{y_1}_t^k) = \widetilde{x_1}_t^{e_1}$.
Hence, the result.

3. $f_{\beta\psi}(\widetilde{x_1}_t^{e_1})(k)$
$= \beta(\bigcup_{e\in\psi^{-1}(k)} \widetilde{x_1}_t^{e_1}(e))$
$= \beta(\widetilde{x}_{1t}), e = e_1$
$= (\widetilde{y}_{1t}), \psi(e_1) = k$
$= \widetilde{y_1}_t^k(k)$.
Therefore, $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^k$.
Similarly, we can see that $f_{\beta\psi}(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k'}$.
Now, since $\widetilde{y_1}_t^{k_1} \notin \widetilde{y_2}_r^{k_2}$, then $\widetilde{y_1}_t^{k_1} \widetilde{\subseteq} (\widetilde{y_2}_r^{k_2})^c$. So that, $f_{\beta\psi}^{-1}(\widetilde{y_1}_t^{k_1}) \widetilde{\subseteq} f_{\beta\psi}^{-1}((\widetilde{y_2}_r^{k_2})^c) = (f_{\beta\psi}^{-1}(\widetilde{y_2}_r^{k_2}))^c$ [by Proposition 2.15]. Thus, $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (\widetilde{x_2}_r^{e_2})^c$. Therefore, $\widetilde{x_1}_t^{e_1} \notin \widetilde{x_2}_r^{e_2}$.

Definition 5.4. Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be a mapping. $f_{\beta\psi}$ is called SD-homeomorphism if it is SD-continuous, SD-closed, one-one and onto.

Theorem 5.5. The property of being SDT_0^* is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be an SD-homeomorphism mapping.

Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \notin \widetilde{y_2}_r^{k_2}$, $y_1 \neq y_2$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X,\ x_1 \neq x_2,\ e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \notin \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_0^*-space, then $\widetilde{x_1}_t^{e_1} \notin cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})$ or $\widetilde{x_2}_r^{e_2} \notin cl_{\widetilde{\tau}}(\widetilde{x_1}_t^{e_1})$, so that $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c$, implies $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c = (f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})))^c$ [by proposition 3.5]. Thus, $\widetilde{y_1}_t^{k_1} \widetilde{\in} (cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x_2}_r^{e_2})))^c$ (as $f_{\beta\psi}$ is SD-homeomorphism). It follows that, $\widetilde{y_1}_t^{k_1} \notin cl_{\widetilde{\eta}}(\widetilde{y_2}_r^{k_2})$. similarly, we also have $\widetilde{y_2}_t^{k_2} \notin cl_{\widetilde{\eta}}(\widetilde{y_1}_r^{k_1})$. Hence, $(Y, \widetilde{\eta}, K)$ is a SDT_0^*.

Theorem 5.6. The property of being SDT_0 is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDTS and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be an SD-homeomorphism mapping.
Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \notin \widetilde{y_2}_r^{k_2}$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \notin \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_0-space, then $\widetilde{x_1}_t^{e_1} \notin cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})$ or $\widetilde{x_2}_r^{e_2} \notin cl_{\widetilde{\tau}}(\widetilde{x_1}_t^{e_1})$. So that, $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c$, implies $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c = (f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})))^c$ [by proposition 3.5]. Thus, $\widetilde{y_1}_t^{k_1} \widetilde{\in} (cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x_2}_r^{e_2})))^c$ (as $f_{\beta\psi}$ is SD-homeomorphism). It follows that, $\widetilde{y_1}_t^{k_1} \notin cl_{\widetilde{\eta}}(\widetilde{y_2}_r^{k_2})$. similarly, we also have $\widetilde{y_2}_t^{k_2} \notin cl_{\widetilde{\eta}}(\widetilde{y_1}_r^{k_1})$. Hence, $(Y, \widetilde{\eta}, K)$ is a SDT_0.

Theorem 5.7. The property of being a $SDT_{\frac{1}{2}}-$space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-open, SD-closed, one-one, onto.

Let $\widetilde{y}_r^k \in SDP(Y)$. Then, by lemma 5.3 $\exists\ x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k$ and $f_{\beta\psi}(\widetilde{x}_t^e) = \widetilde{y}_r^k$. Since $(X, \widetilde{\tau}, E)$ is $SDT_{\frac{1}{2}}-$space, then $\widetilde{x}_t^e$ is an open or a closed SD-point in X. Since $f_{\beta\psi}$ is SD-open and SD-closed, then $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$ is open SD-set and closed SD-set in Y. Hence, $(Y, \widetilde{\eta}, K)$ is $SDT_{\frac{1}{2}}$.

Theorem 5.8. The property of being a SDT_1-space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping.

Let $\widetilde{y}_r^k \in SDP(Y)_K$. Then, by lemma 5.3 $\exists x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k, \widetilde{x}_t^e \in SDP(X)_E$ and $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$. Since $(X, \widetilde{\tau}, E)$ is SDT_1-space, then $\widetilde{x}_t^e = cl_{\widetilde{\tau}}(\widetilde{x}_t^e)$. Thus, $f_{\beta\psi}(\widetilde{x}_t^e) = f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x}_t^e)) = cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x}_t^e)) = cl_{\widetilde{\tau}}(\widetilde{y}_r^k)$ (as $f_{\beta\psi}$ is SD-homeomorphism). Therefore, $\widetilde{y}_r^k = cl_{\widetilde{\eta}}(\widetilde{y}_r^k)$. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_1.

Theorem 5.9. The property of being SDT_1^*-space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping.

Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \not\widetilde{q}\, \widetilde{y_2}_r^{k_2}$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \not{q}\ \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_1^*-space, then $\widetilde{x_1}_t^{e_1} \not\widetilde{q}\ cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})$ and $\widetilde{x_2}_r^{e_2} \not\widetilde{q}\ cl_{\widetilde{\tau}}(\widetilde{x_1}_t^{e_1})$. So that $\widetilde{x_1}_t^{e_1} \widetilde{\subseteq} (cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c$, implies $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\subseteq} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2}))^c = (f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{x_2}_r^{e_2})))^c$ [by proposition 3.5]. Thus, $\widetilde{y_1}_t^{k_1} \widetilde{\in} (cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{x_2}_r^{e_2})))^c$ (as $f_{\beta\psi}$ is SD-homeomorphism). It follows that, $\widetilde{y_1}_t^{k_1} \not\widetilde{q}\ cl_{\widetilde{\eta}}(\widetilde{y_2}_r^{k_2})$. similarly, we also have $\widetilde{y_2}_t^{k_2} \not\widetilde{q}\ cl_{\widetilde{\eta}}(\widetilde{y_1}_r^{k_1})$. Hence, $(Y, \widetilde{\eta}, K)$ is a SDT_1^*.

Theorem 5.10. The property of being a SDT_2-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping.
Let $\widetilde{y}_r^k \in SDP(Y)_K$. Then, by lemma 5.3 $\exists x \in X$ and $e \in E$ such that $\beta(x) = y, \psi(e) = k, \widetilde{x}_t^e \in SDP(X)_E$ and $f(\widetilde{x}_t^e) = \widetilde{y}_t^k$. Since $(X, \widetilde{\tau}, E)$ is SDT_2-space, then $\widetilde{x}_t^e = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})$.
Thus, $f_{\beta\psi}(\widetilde{x}_t^e) = f_{\beta\psi}(\widetilde{\bigcap}_{\widetilde{O}_{\widetilde{x}_t^e} \in N^q_{(\widetilde{x}_t^e)_E}} cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})) = \widetilde{\bigcap}_{\widetilde{O}_{f_{\beta\psi}(\widetilde{x}_t^e)} \in N^q_{(f_{\beta\psi}(\widetilde{x}_t^e))_K}} f_{\beta\psi}(cl_{\widetilde{\tau}}(\widetilde{O}_{\widetilde{x}_t^e})) =$
$\widetilde{\bigcap}_{\widetilde{O}_{f_{\beta\psi}(\widetilde{x}_t^e)} \in N^q_{(f_{\beta\psi}(\widetilde{x}_t^e))_K}} cl_{\widetilde{\eta}}(f_{\beta\psi}(\widetilde{O}_{f_{\beta\psi}(\widetilde{x}_t^e)})) = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^k} \in N^q_{(\widetilde{y}_r^k)_K}} cl_{\widetilde{\eta}}(\widetilde{O}_{\widetilde{y}_r^k})$.
Therefore, $\widetilde{y}_r^k = \widetilde{\bigcap}_{\widetilde{O}_{\widetilde{y}_r^k} \in N^q_{(\widetilde{y}_r^k)_K}} cl_{\widetilde{\eta}}(\widetilde{O}_{\widetilde{y}_r^k})$. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_2.

Theorem 5.11. The property of being SDT_2^*-space is a topological property.

Proof. Suppose that $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-open, one-one and onto.

Let $\widetilde{y_1}_t^{k_1}, \widetilde{y_2}_r^{k_2} \in SDP(Y)_K$ such that $\widetilde{y_1}_t^{k_1} \not\widetilde{q}\ \widetilde{y_2}_r^{k_2}$. Then, by lemma 5.3 $\exists\ x_1, x_2 \in X, e_1, e_2 \in E$ such that $\beta(x_i) = y_i, \psi(e_i) = k_i, (i = 1, 2)$. Also, $\widetilde{x_1}_t^{e_1} \not{q}\ \widetilde{x_2}_r^{e_2}$ and $f(\widetilde{x_1}_t^{e_1}) = \widetilde{y_1}_t^{k_1}, f(\widetilde{x_2}_r^{e_2}) = \widetilde{y_2}_r^{k_2}$. Since $(X, \widetilde{\tau}, E)$ is SDT_2^*-space, then there exist $\widetilde{F}_E, \widetilde{G}_E \in \widetilde{\tau}$ such that $\widetilde{x_1}_t^{e_1} \widetilde{\in} \widetilde{F}_E, \widetilde{x_2}_r^{e_2} \widetilde{\in} \widetilde{G}_E$ and $\widetilde{F}_E \not\widetilde{q}\ \widetilde{G}_E$. Thus, $f_{\beta\psi}(\widetilde{x_1}_t^{e_1}) \widetilde{\in} f_{\beta\psi}(\widetilde{F}_E)$, $f_{\beta\psi}(\widetilde{x_2}_t^{e_2}) \widetilde{\in} f_{\beta\psi}(\widetilde{G}_E)$ and $f_{\beta\psi}(\widetilde{F}_E) \not\widetilde{q}\ f_{\beta\psi}(\widetilde{G}_E)$ [by proposition 3.5]. Therefore, $\widetilde{y_1}_t^{k_1} \widetilde{\in} f_{\beta\psi}(\widetilde{F}_E), \widetilde{y_2}_r^{k_2} \widetilde{\in} f_{\beta\psi}(\widetilde{G}_E)$ and $f_{\beta\psi}(\widetilde{F}_E) \not\widetilde{q}\ f_{\beta\psi}(\widetilde{G}_E), (f_{\beta\psi}(\widetilde{F}_E), f_{\beta\psi}(\widetilde{G}_E) \in \widetilde{\eta})$.
Hence, $(Y, \widetilde{\eta}, K)$ is SDT_2^*.

Theorem 5.12. The property of being a SDR_2-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-Spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism.

Let $\widetilde{y}_r^k \in SDP(Y)^K$ and $\widetilde{F}_K \in \widetilde{\eta}^c$ such that $\widetilde{y}_r^k \not\widetilde{q}\, \widetilde{F}_K$. Then, by lemma 5.3 $\exists x \in X$ and $e \in E$ such that $\psi(e) = k, \beta(x) = y, \widetilde{x}_t^e \in SDP(X)_E, f(\widetilde{x}_t^e) = \widetilde{y}_t^k$ and $f_{\beta\psi}^{-1}(\widetilde{F}_K) = \widetilde{G}_E, \widetilde{G}_E \in \widetilde{\tau}^c)$ (as $f_{\beta\psi}$ is D-continuous). Also, $\widetilde{x}_t^e \not\widetilde{q}\, \widetilde{G}_E$, $(X, \widetilde{\tau}, E)$ is SDR_2-space, then there exist $\widetilde{H}_E, \widetilde{M}_E \in \widetilde{\tau}$ such that $\widetilde{x}_t^e \widetilde{\in} \widetilde{H}_E, \widetilde{G}_E \widetilde{\subseteq} \widetilde{M}_E$ and $\widetilde{H}_E \not\widetilde{q}\, \widetilde{M}_E$. Thus, $f_{\beta\psi}(\widetilde{x}_t^e) \widetilde{\in} f_{\beta\psi}(\widetilde{H}_E), f_{\beta\psi}(\widetilde{G}_E) \widetilde{\subseteq} f_{\beta\psi}(\widetilde{M}_E)$ and $f_{\beta\psi}(\widetilde{H}_E) \not\widetilde{q}\, f_{\beta\psi}(\widetilde{M}_E)$ [by proposition 3.5]. Therefore, $\widetilde{y}_t^k \widetilde{\in} f_{\beta\psi}(\widetilde{H}_E), \widetilde{F}_K \widetilde{\subseteq} f_{\beta\psi}(\widetilde{M}_E)$ and $f_{\beta\psi}(\widetilde{H}_E) \not\widetilde{q}\, f_{\beta\psi}(\widetilde{M}_E), (f_{\beta\psi}(\widetilde{H}_E), f_{\beta\psi}(\widetilde{M}_E) \in \widetilde{\eta})$. Hence, $(Y, \widetilde{\eta}, K)$ is SDR_2.

Theorem 5.13. The property of being a SDT_3-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping and $(X, \widetilde{\tau}, E)$ is SDT_3-space, then $(Y, \widetilde{\eta}, K)$ is SDT_1 and SDR_2-spaces [by theorems 5.8,5.12]. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_3.

Theorem 5.14. The property of being a SDT_3^*-space is a topological property.

Proof. Suppose $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\eta}, K)$ be two SDT-spaces and let $f_{\beta\psi} : SD(X)_E \to SD(Y)_K$ be SD-homeomorphism mapping and $(X, \widetilde{\tau}, E)$ is SDT_3^*-space, then $(Y, \widetilde{\eta}, K)$ is SDT_1^* and SDR_2-spaces [by theorems 5.9,5.12]. Hence, $(Y, \widetilde{\eta}, K)$ is SDT_3^*.

Theorem 5.15. The property of being a $DT_i^{**}-$space, (i=0, 1, 2, 3) is a topological property.

Proof. Straightforward.

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