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ON THE LAGRANGE INTERPOLATIONS OF THE JACOBSTHAL AND JACOBSTHAL-LUCAS SEQUENCES

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Abstract. This study explores the formation of polynomials of at most degree n using the first $n+1$ terms of the Jacobsthal and Jacobsthal-Lucas sequences through Lagrange interpolation. The paper provides a detailed examination of the recurrence relations and various identities associated with the Jacobsthal and Jacobsthal-Lucas Lagrange Interpolation Polynomials.

1. Introduction

As is well known, Fibonacci numbers have been prominently featured in applied sciences. There have been many studies on Fibonacci numbers and their generalizations over the centuries. Lucas numbers, which share the same recurrence relation but have different initial conditions from Fibonacci numbers, have many relationships with the Fibonacci numbers. Both Fibonacci and Lucas numbers are sequences of second-order recurrence relations. There are many sequences of the same order, some of which include Pell, Jacobsthal, Pell-Lucas, and Jacobsthal-Lucas sequences. Among the generalizations of the Fibonacci sequence, the Tribonacci sequence has a third-order recurrence relation. Some sequences with a third-order recurrence relation are the Narayana, Perrin, and Padovan sequences. The purpose of this study is to establish a relationship between the Lagrange interpolation with the Jacobsthal sequences.
Jacobsthal numbers have attracted a lot of interest due to their intriguing characteristics. Jacobsthal and Jacobsthal-Lucas numbers appear respectively as the integer sequences A001045 and A014551 from [21, 22]. The Jacobsthal sequence $\{J_n\}_{n\geq 0}$ is

$$J_{n+2} = J_{n+1} + 2J_n. \tag{1.1}$$

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with the initial elements $J_0 = 1$ and $J_1 = 1$. First few terms of the sequence $\{J_n\}$ are

$$1, 1, 3, 5, 11, 21, 43, 85, 171, 341.$$

The Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ is

$$j_{n+2} = j_{n+1} + 2j_n. \tag{1.2}$$

with the initial elements $j_0 = 2$ and $j_1 = 1$. First few terms of the sequence $\{j_n\}$ are

$$2, 1, 5, 7, 17, 31, 65, 127, 257, 511.$$

Some studies related to the sequences $\{J_n\}$ and $\{j_n\}$ can be found in [1–12, 14–16, 18, 19, 23]. The characteristic equation of the recurrences $\{J_n\}$ and $\{j_n\}$ is

$$x^2 - x - 2 = 0 \tag{1.3}$$

where the roots of the equation (1.3) are

$$x_1 = 2 \quad \text{and} \quad x_2 = -1$$

in order for,

$$x_1 + x_2 = 1, \quad x_1 - x_2 = 3 \quad \text{and} \quad x_1 x_2 = -2.$$

The Binet-like formulas of the sequences $\{J_n\}$ and $\{j_n\}$ are

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1.4}$$

and

$$j_n = 2^n + (-1)^n, \tag{1.5}$$

respectively. Some interrelationships are

$$J_n + j_n = 2J_{n+1} \tag{1.6}$$

$$3J_n + j_n = 2^{n+1} \tag{1.7}$$

The Lagrange interpolating polynomial is essentially a rephrased version of the Newton polynomial that eliminates the need to calculate divided differences. Lagrange interpolation is beneficial because it is effective for data points that are unevenly spaced along the independent variable. In the realm of numerical analysis, interpolation refers to the method of identifying the most suitable function based on certain given points. The most basic form of interpolation uses a polynomial. This implies that for a set of specified points, there is a polynomial that intersects all these points. This polynomial approximates the underlying function closely. One technique for polynomial interpolation is the Lagrange interpolation method [20].

Let P_n be the set of all real-valued polynomials of degree at most n defined over the set $\mathbb{R}$ of real numbers, given that n is a nonnegative integer. The basic interpolation problem is as follows: identify a polynomial $p_0 \in P_0$ such that $p_0(x_0) = y_0$, given x_0 and y_0 in $\mathbb{R}$. This can be solved by using the formula $p_0(x) \equiv y_0$. Examining the subsequent, more general problem is the primary purpose [13].

Let $n \geq 1$, and assume that x_i for $i = 0, 1, \ldots, n$ are distinct real numbers (i.e., $x_i \neq x_j$ for $i \neq j$), and y_i for $i = 0, 1, \ldots, n$ are real numbers. We want to find $p_n \in \mathcal{P}_n$ such that $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

There exist polynomials $L_k \in \mathcal{P}_n$ for $k = 0, 1, \ldots, n$, such that

$$L_k(x_i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

for all $i, k = 0, 1, \ldots, n$. Furthermore,

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

satisfies the interpolation conditions mentioned above. In other words, $p_n \in \mathcal{P}_n$ and $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$. For each fixed k, $0 \leq k \leq n$, L_k is required to have n zeros at x_i for $i = 0, 1, \ldots, n$ and $i \neq k$. Thus, $L_k(x)$ is of the form

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

Once these basis polynomials are constructed, the Lagrange interpolation polynomial can be expressed as follows:

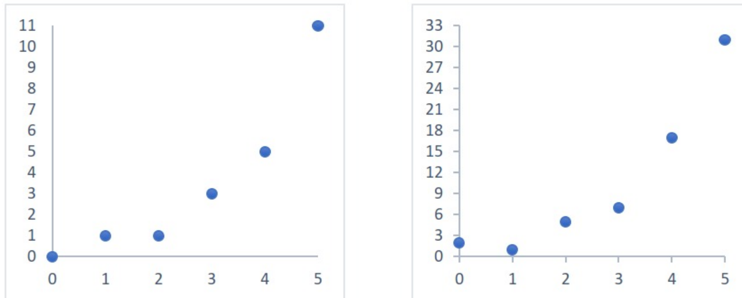
$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k = \sum_{k=0}^{n} \left(\prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} \right) y_k. \tag{1.8}$$

Based on these statements, Mufid and at al. showed that a polynomial of degree n at most can be created from the first $n+1$ terms of the Fibonacci sequence using Lagrange interpolation, and that this Fibonacci Lagrange Interpolation Polynomial (FLIP) can be obtained both recursively and implicitly [17].

In this study, we first investigate the formation of polynomials of degree at most n using the first $n+1$ terms of the sequences $\{J_n\}$ and $\{j_n\}$es via Lagrange interpolation. Then, we present a detailed examination of the recurrence relations and various identities associated with the Jacobsthal and Jacobsthal-Lucas Lagrange Interpolation Polynomials.

2. The Lagrange interpolation of the Jacobsthal sequences

Before we commence the interpolations of the sequences $\{J_n\}$ and $\{j_n\}$, we will plot these sequences on the $xy-$coordinate system. Let's denote the Jacobsthal point as $p_n = (n, J_n)$ and the Jacobsthal-Lucas point as $q_n = (n, j_n)$, representing the points associated with the $n-$th terms of the sequences $\{J_n\}$ and $\{j_n\}$, respectively. For illustration, the points of the sequences $\{J_n\}$ and $\{j_n\}$ from $n = 0$ to $n = 5$ are depicted in Figure 1.



Jacobsthal points

Jacobsthal-Lucas points

FIGURE 1. The points of the sequences $\{J_n\}$ and $\{j_n\}$

We define $\mathbb{J}_n(x)$ as the polynomial constructed using the Lagrange interpolation from the points j_k for $k \in \{0, 1, \ldots, n\}$. Specifically, we interpolate using the points $(x_k, y_k) = (k, J_k)$. Accordingly, with $x_k = k$ and $y_k = J_k$ in equation (1.8), we write

$$\mathbb{J}_n(x) = \sum_{k=0}^{n} \left(\prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x-i}{k-i} \right) J_k. \tag{2.1}$$

The factors $(k - i)$ in equation (2.1) can be simplified as follows.

$$\prod_{\substack{i=0 \\ i \neq k}}^{n} (k-i) = (-1)^{n-k}(n-k)!k! = (-1)^{n-k}\binom{n}{k} n! \tag{2.2}$$

Upon incorporating equation (2.2) into equation (2.1), we obtain:

$$\mathbb{J}_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \left((-1)^{n-k} \binom{n}{k} \prod_{\substack{i=0 \\ i \neq k}}^{n} (x-i) \right) J_k. \tag{2.3}$$

For instance, we can obtain $\mathbb{J}_n(x)$ for $n = 1, 2, 3, 4$ using equation (2.3) as follows:

$$\mathbb{J}_1(x) = 1,$$

$$\mathbb{J}_2(x) = x^2 - x + 1,$$

$$\mathbb{J}_3(x) = -\frac{1}{6}\left(2x^3 - 12x^2 + 10x - 6\right),$$

$$\mathbb{J}_4(x) = \frac{1}{24}\left(6x^4 - 44x^3 + 114x^2 - 76x + 24\right).$$

In Figure 2, the graphs of the above polynomials are displayed. Recall that $\mathbb{J}_n(k) = J_k$ for $k \in \{0, 1, \ldots, n\}$.

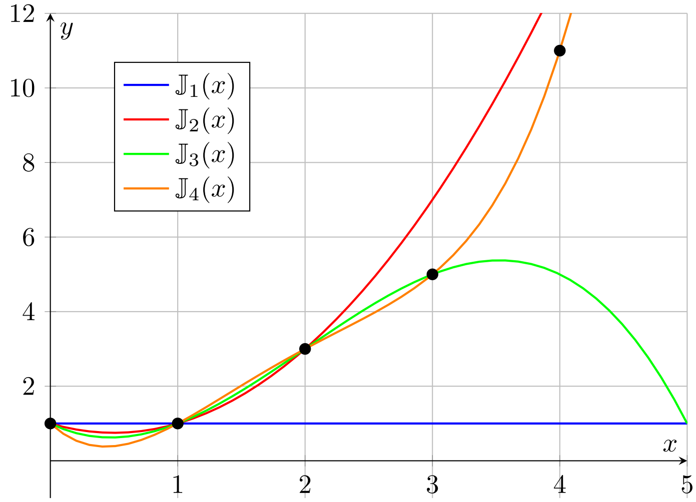


FIGURE 2. Graphs of $\mathbb{J}_1(x)$, $\mathbb{J}_2(x)$, $\mathbb{J}_3(x)$, and $\mathbb{J}_4(x)$

Similarly, the interpolation of the Jacobsthal-Lucas sequence is expressed as:

$$\gimel_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\left((-1)^{n-k}\binom{n}{k}\prod_{\substack{i=0\\ i\neq k}}^{n}(x-i)\right) j_k. \tag{2.4}$$

For instance, we can obtain $\gimel_n(x)$ for $n = 1, 2, 3, 4$ using equation (2.4) as follows:

$$\gimel_1(x) = -x + 2,$$

$$\gimel_2(x) = \frac{1}{2}\left(5x^2 - 7x + 4\right),$$

$$\gimel_3(x) = -\frac{1}{6}\left(7x^3 - 36x^2 + 35x - 12\right),$$

$$\gimel_4(x) = \frac{1}{24}\left(17x^4 - 130x^3 + 331x^2 - 242x + 48\right).$$

In Figure 3, the graphs of the above polynomials are displayed. Recall that $\gimel_n(k) = j_k$ for $k \in \{0, 1, \ldots, n\}$.

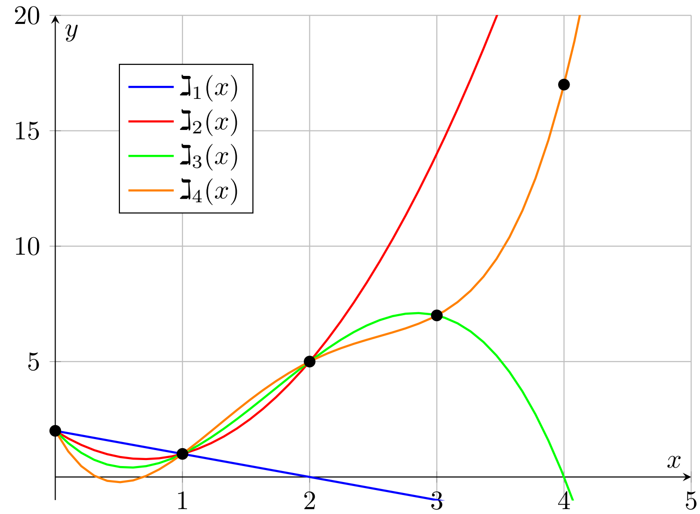


FIGURE 3. Graphs of $\daleth_1(x)$, $\daleth_2(x)$, $\daleth_3(x)$, and $\daleth_4(x)$

We shall establish a leading coefficient theorem for $\mathbb{J}_n(x)$, before obtaining more formulas for it.

Theorem 2.1. *The leading coefficients of* $\mathbb{J}_n(x)$ *and* $\daleth_n(x)$ *are*

$$\frac{(-1)^{n+1}J_n}{n!}$$

and

$$\frac{(-1)^{n+1}j_n}{n!},$$

respectively.

Proof. The leading coefficient of $\mathbb{J}_n(x)$ is equal to $\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}J_k\binom{n}{k}$. Thus, we just have to demonstrate that

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{n-k}J_k\binom{n}{k} &= \frac{1}{3}\sum_{k=0}^{n}(-1)^{n-k}\left(2^k-(-1)^k\right)\binom{n}{k}\\
&= \frac{1}{3}\left[\sum_{k=0}^{n}(-1)^{n-k}2^k\binom{n}{k}-\sum_{k=0}^{n}(-1)^{n-k}(-1)^k\binom{n}{k}\right]\\
&= \frac{1}{3}\left[(2-1)^n-(-1-1)^n\right] \quad \text{(by identity (1))}\\
&= \frac{(-1)^{n+1}}{3}\left[2^n-(-1)^n\right]\\
&= (-1)^{n+1}\left[\frac{2^n-(-1)^n}{3}\right]\\
&= (-1)^{n+1}J_n
\end{aligned}$$

Similarly, the leading coefficient of $\daleth n(x)$ is found to be $\dfrac{(-1)^{n+1}j_n}{n!}$. □

3. Recurrence relations of the $\mathbb{J}_n(x)$ and $\gimel_n(x)$

In this part, we will derive the additional formulas for $\mathbb{J}_n(x)$ and $\gimel_n(x)$. It's remarkable that $\mathbb{J}_n(x)$ and $\gimel_n(x)$ can be constructed recurrence relations, much like the sequences $\{J_n\}$ and $\{j_n\}$ respectively.

Consider the polynomials $\mathbb{J}_{n+1}(x)$ and $\mathbb{J}_n(x)$. For each $i \in \{0, 1, 2, \ldots, n\}$, we have $\mathbb{J}_{n+1}(i) = \mathbb{J}_n(i)$. In other words, the polynomials $\mathbb{J}_{n+1}(x)$ and $\mathbb{J}_n(x)$ intersect at $n+1$ points. Consequently, we can express the relationship as follows:

$$\mathbb{J}_{n+1}(x) - \mathbb{J}_n(x) = a \cdot x(x-1)\cdots(x-n),$$

where a denotes the leading coefficient of $\mathbb{J}_{n+1}(x)$. As a results, when $P_n(x) = x(x-1)\cdots(x-n)$,

$$\mathbb{J}_{n+1}(x) = \mathbb{J}_n(x) + \frac{(-1)^n J_{n+1}}{(n+1)!} P_n(x)$$

is a recursive formula.

By successively applying the recursive formula for $\mathbb{J}_n(x)$, $\mathbb{J}_{n-1}(x)$, …, $\mathbb{J}_2(x)$, we derive the following implicit formula:

$$\mathbb{J}_{n+1}(x) = \mathbb{J}_1(x) + \sum_{i=1}^{n} \frac{(-1)^i J_{i+1}}{(i+1)!} P_i(x)$$

which simplifies to

$$\mathbb{J}_n(x) = \sum_{i=1}^{n} \frac{(-1)^{i-1} J_i}{i!} P_{i-1}(x). \tag{3.1}$$

Similarly, the recurrence relation of the $\gimel_n(x)$ is derived as:

$$\gimel_n(x) = -2x + 2 + \sum_{i=1}^{n} \frac{(-1)^{i-1} j_i}{i!} P_{i-1}(x). \tag{3.2}$$

Some relationships between recurrence relations (3.1) and (3.2) are as follows:

1.
$$\mathbb{J}_n(x) + \gimel_n(x) = -2x + 2 + 2\sum_{i=1}^{n} \frac{(-1)^{i-1} J_{i+1}}{i!} P_{i-1}(x)$$

2.
$$3\mathbb{J}_n(x) + \gimel_n(x) = -2x + 2 - \sum_{i=1}^{n} \frac{(-2)^i}{i!} P_{i-1}(x)$$

Theorem 3.1. *The Binet-like formulas of the recurrence relations of the $\mathbb{J}_n(x)$ and $\gimel_n(x)$ are, respectively,*

$$\mathbb{J}_n(x) = \sum_{i=1}^{n} \frac{1 - (-2)^i}{3i!} P_{i-1}(x)$$

and

$$\gimel_n(x) = -2x + 2 - \sum_{i=1}^{n} \frac{(-2)^i + 1}{i!} P_{i-1}(x).$$

Proof. It is easily proven using the equalities of (1.4) and (1.5). □

4. Conclusion

This study investigated the formation of polynomials of degree at most n using the first $n+1$ terms of the sequences $\{J_n\}$ and $\{j_n\}$ through Lagrange interpolation. The article provided a detailed examination of recurrence relations and various identities associated with Jacobsthal and Jacobsthal-Lucas Lagrange Interpolation Polynomials.

The interpolation polynomials of the sequences $\{J_n\}$ and $\{j_n\}$ offer valuable tools that reflect the properties and structure of these sequences. These polynomials can be used to determine the value of the independent variable x corresponding to a given function value, even when the parameters are not evenly spaced.

These results lay a foundation for future research, opening new avenues to explore the applicability of these important number sequences and their interpolation polynomials in broader fields. Particularly, there is potential for further use and development of these polynomials in areas such as image processing, numerical analysis, and other engineering applications.

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The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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