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On the Quaternionic Focal Curves

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Abstract: In this study, a brief summary about quaternions and quaternionic curves are firstly presented. Also, the definition of focal curve is given. The focal curve of a smooth curve consists of the centers of its osculating hypersphere. By using this definition and the quaternionic osculating hyperspheres of these curves, the quaternionic focal curves in the spaces $\mathbb{Q}$ and $\mathbb{Q}_\nu$ with index $\nu = \{1,2\}$ are discussed. Some relations about spatial semi-real quaternionic curves and semi-real quaternionic curves are examined by using focal curvatures and "scalar Frenet equations" between the focal curvatures. Then, the notions: such as vertex, flattenings, a symmetry point are defined for these curves. Moreover, the relation between the Frenet apparatus of a quaternionic curve and the Frenet apparatus of its quaternionic focal curve are presented.

Kuaterniyonik Fokal Eğriler Üzerine

Anahtar Kelimeler
Kuaterniyonlar,
Kuaterniyonik eğriler,
Oskülatör hiperküre,
Fokal eğriler,
Yarı-Öklidyen uzay

Özet: Bu çalışmada, ilk olarak kuaterniyonlar ve kuaterniyonik eğriler hakkında kısa bir özet sunulmuş, ayrıca fokal eğri tanımı verilmiştir. Düzgün bir eğrinin fokal eğrisi oskülatör hiperküresinin merkezlerinden oluşur. Bu tanım ve bu eğrilerin kuaterniyonik oskülatör hiperküreleri kullanılarak, $\nu = \{1,2\}$ olmak üzere $\mathbb{Q}$ ve $\mathbb{Q}_\nu$ uzaylarında kuaterniyonik fokal eğri tanımı ele alınmıştır. Uzaysal yarı-reel kuaterniyonik eğriler ve yarı-reel kuaterniyonik eğriler hakkında bazı ilişkiler, fokal eğrilikler ve fokal eğriler arasındaki "skalar Frenet denklemleri" kullanılarak verilmiştir. Ayrıca bu eğriler için tepe noktası, basıklık ve simetri noktası gibi kavramlar tanımlanmıştır. Bunların yanında bir kuaterniyonik eğrinin Frenet elemanları ve onun kuaterniyonik fokal eğrisinin Frenet elemanları arasındaki ilişki sunulmuştur.

1. Introduction

The quaternions were first introduced by William R. Hamilton in 1843. It is a number system that extend the complex numbers and there is a strictly correspondence between the quaternion set and four-dimensional vector space $\mathbb{E}^4$. Addition, multiplication by scalar and quaternion multiplication are defined on the set of quaternions. The addition and scalar multiplication of quaternions are defined same as in the Euclidean space $\mathbb{E}^4$. An effectual feature of quaternions is that the quaternion multiplication between two quaternions is noncommutative.

Besides, the product of two elements of the set of quaternions necessitates a choice of basis for $\mathbb{E}^4$. Every member of this basis, as usual denoted as e_1, e_2, e_3 and $e_4 = 1$, can be written uniquely as $ae_1 + be_2 + ce_3 + de_4$, where a, b, c and d are real numbers. Therefore, quaternions are usually written $ae_1 + be_2 + ce_3 + de_4$, suppressing the basis element $e_4 = 1$ [1].

In 1987, the properties of smooth quaternionic curves in the Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$ are examined by [2]. Elements of $\mathbb{E}^4$ were identified with quaternions. The Frenet-Serret formulas, which prove geometric properties of the curve itself irrespective of any motion, were given by these researchers for quaternionic curves. After that, Frenet-Serret formulas are given by Tuna in [3] in semi-Euclidean space for quaternionic curves. By the aid of these formulae, remarkable studies of quaternionic curves are worked in literature [4, 5].

In $\mathbb{E}^3$, there exists a unique sphere which contacts a curve α at the third order at $\alpha(0)$. The intersection of this sphere with the curves' osculating plane is a circle which contacts at the second order at $\alpha(0)$, [6–8]. This notion examined with regards to real quaternionic curves in $\mathbb{E}^4$ by [9]. In [10], the osculating sphere and the osculating circle of a curve are obtained in semi-Euclidean spaces $\mathbb{E}^3_1$, $\mathbb{E}^4_1$ and $\mathbb{E}^4_2$. Additionally, the definition of the osculating spheres

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for semi-real quaternionic curves in $\mathbb{E}_2^4$ was presented by [11].

On the other hand, R. Uribe-Vargas investigated the concept of the focal geometry and observed the properties of the focal curves [12]. The notion of the focal curve of a smooth curve of α consists of the centers of its osculating sphere. The focal curve is represented by C_α for the main curve α such that $C_\alpha = \alpha + \sum c_i N_i$, $1 \leq i \leq n$, where N_i is the Frenet-Serret apparatus of α. In these statement, c_i is called *focal curvatures of* α , [12].

In [12], some definitions, results and theorems were given with respect to focal curvatures. Also, by using focal curves' notations, the the results for the curves in the Euclidean n-space were examined. Besides, the focal curvatures of a non-lightlike curve in Minkowski $(m+1)$-space which satisfy the "scalar Frenet equations" were investigated by [13]. By using the focal curvatures, he gave necessary and sufficient conditions for a point of a non-lightlike curve to be a vertex. Moreover, the fundamental results of the focal curves which defined in $\mathbb{E}^n$ were given by favour of Darboux vector. The relationships between the focal curves and the concept of 2-planar [14]. H. Şimşek investigated some characteristic features of focal curves and focal curvatures with regard to Darboux frame on the timelike and spacelike surfaces [15]. Singularities of focal surfaces of null Cartan curves in Minkowski 3-space were studied by Wang and et.al. [16]. Lightlike hypersurfaces and lightlike focal sets of null Cartan curves in Lorentz-Minkowski spacetime were given by Liu and Wang [17]. The focal curves of a null Cartan curve was examined by [18]. Furthermore, different applications of focal curves were given in the studies [19–21].

In this present paper, we will examine the real quaternionic and semi-real quaternionic focal curves in original sections 3-5 by using the quaternionic osculating hyperspheres and the Frenet-Serret formulas in the spaces $\mathbb{Q}$ and $\mathbb{Q}_v$. Then, we will obtain some characterizations of real quaternionic focal curves with the aim of focal curvatures.

2. Preliminaries

In this section, the fundamental information about the quaternions in the spaces $\mathbb{E}^4$, $\mathbb{E}_1^3$ and $\mathbb{E}_2^4$ are briefly presented, [1].

i) Real Quaternions

In the Euclidean space $\mathbb{E}^4$, a real quaternion consists of a set of four ordered real numbers a,b,c,d with four units e_1,e_2,e_3 and e_4, respectively such that: $q = ae_1 + be_2 + ce_3 + de_4$ or $q = \mathbf{V}_q + S_q$, where the symbols $S_q = d$ (scalar part of q) and $\mathbf{V}_q = ae_1 + be_2 + ce_3$ (vector part of q). The three units e_1,e_2,e_3 and e_4 have the following properties:

$$\begin{cases} e_i \times e_i = -e_4, & (e_4 = +1,\ 1 \leq i \leq 3) \\ e_i \times e_j = e_k = -e_j \times e_i, & (\ 1 \leq i,j \leq 3). \end{cases}$$

We symbolize all real quaternions by $\mathbb{Q}$. The multiplication $p = \mathbf{V}_p + S_p$ and $q = \mathbf{V}_q + S_q$ is given below:

$$p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q,\ \forall p,q \in \mathbb{Q}, \tag{1}$$

where the symbols "$\langle , \rangle$" and "$\wedge$" represent the scalar and cross products in $\mathbb{E}^3$. The conjugate of q is denoted by $\overline{q}$ and defined as follows:

$$\overline{q} = S_q - \mathbf{V}_q = de_4 - ae_1 - be_2 - ce_3.$$

This conjugate provides the following bilinear form:

$$\begin{aligned} \langle\, , \rangle : \mathbb{Q} \times \mathbb{Q} \ & \to \mathbb{R} \\ (p,q) \ & \to h(p,q) = \tfrac{1}{2}(p \times \overline{q} + q \times \overline{p}) \ \text{ for } \ \mathbb{E}^4. \end{aligned}$$

This bilinear form is called *quaternion inner product*. The norm of q is denoted by

$$\|q\|^2 = q \times \overline{q} = \overline{q} \times q = a^2 + b^2 + c^2 + d^2,\ \forall q \in \mathbb{Q}.$$

ii) Semi-Real Quaternions

In the semi-Euclidean space $\mathbb{E}_2^4$, a semi-real quaternion is written by $q = ae_1 + be_2 + ce_3 + de_4$ or $q = \mathbf{V}_q + S_q$, where the symbols $S_q = d$ (scalar part of q) and $\mathbf{V}_q = ae_1 + be_2 + ce_3$ (vector part of q. The units e_1,e_2,e_3 and e_4 have the following properties:

$$\begin{cases} e_i \times e_i & = & -\varepsilon_{e_i}, & 1 \leq i \leq 3, & e_4 = 1 \\ e_i \times e_j & = & \varepsilon_{e_i}\varepsilon_{e_j} e_k, & 1 \leq i,j,k \leq 3, & \text{in } \mathbb{E}_1^3 \\ e_i \times e_j & = & -\varepsilon_{e_i}\varepsilon_{e_j} e_k, & 1 \leq i,j,k \leq 3, & \text{in } \mathbb{E}_2^4, \end{cases}$$

where (ijk) is an even permutation of (123) [4]. The symbol ε_{e_i} means that:

$$\varepsilon_{e_i} = \begin{cases} -1, & e_i \quad \text{timelike} \\ +1, & e_i \quad \text{spacelike}. \end{cases}$$

We denote all semi-real quaternions by $\mathbb{Q}_v$ with an index $v = \{1,2\}$ such that

$$\mathbb{Q}_v = \left\{ \begin{array}{l} q|\ q = ae_1 + be_2 + ce_3 + de_4;\ a,b,c,d \in \mathbb{R},\ e_i \in \ \mathbb{E}_1^3, \\ h_v = (e_i,e_i) = \varepsilon_{e_i}, 1 \leq i \leq 3 \end{array} \right\}$$

where the h_v is the h-inner product. For every $p,q \in \mathbb{Q}_v$, the h-inner product $h_v : \mathbb{Q}_v \times \mathbb{Q}_v \to \mathbb{R}$ is defined by:

$$h_1(p,q) = \tfrac{1}{2}\left[\varepsilon_p \varepsilon_{\overline{q}} (p \times \overline{q}) + \varepsilon_q \varepsilon_{\overline{p}} (q \times \overline{p})\right] \ \text{ for } \ \mathbb{E}_1^3$$

and

$$h_2(p,q) = -\tfrac{1}{2}\left[\varepsilon_p \varepsilon_{\overline{q}} (p \times \overline{q}) + \varepsilon_q \varepsilon_{\overline{p}} (q \times \overline{p})\right] \ \text{ for } \ \mathbb{E}_2^4.$$

The quaternion multiplication of p and q is given below:

$$p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q,\ \forall p,q \in \mathbb{Q}_v, \tag{2}$$

where the symbols "$\langle , \rangle$" and "$\wedge$" represent the scalar and cross products in $\mathbb{E}_1^3$. The conjugate of q is denoted by $\overline{q}$ and defined by $\overline{q} = S_q - \mathbf{V}_q = de_4 - ae_1 - be_2 - ce_3$. The real number $[h_v(q,q)]^{1/2}$ is called the norm of q and symbolized by $\|q\|$. Thus

$$\|q\|^2 = |h_v(q,q)| = \left|a^2 \varepsilon_{e_1} + b^2 \varepsilon_{e_2} + c^2 \varepsilon_{e_3} + d^2\right|$$

for $v = \{1,2\}$, [22]. Furthermore, if $h(p,q) = 0$, then p and q are *orthogonal*. If $q + \overline{q} = 0, q \in \mathbb{Q}_v$, then q is called a *spatial semi-real quaternion* [3]. If $\|q\| = 1$, then q is called *unit semi-real quaternion.*

iii) The Frenet-Serret formulas

The Frenet-Serret formulas for real quaternionic curves in the Euclidean spaces $\mathbb{E}^3$, $\mathbb{E}^4$ and for semi-real quaternionic curves in the semi-Euclidean spaces $\mathbb{E}^3_1$, $\mathbb{E}^4_2$ are given namely:

Let

$$\begin{aligned} \alpha:[0,1]\subset\mathbb{R} &\to \mathbb{Q} \\ s &\to \alpha(s)=\sum_{i=1}^{3}\alpha_i(s)e_i, \end{aligned}$$

be an unit speed curve with nonzero curvatures $\{k,r\}$ and the Frenet frame $\{\mathbf{t},\mathbf{n_1},\mathbf{n_2}\}$. Then, Frenet-Serret formulas for α in $\mathbb{E}^3$ are defined by

$$\begin{aligned} &\mathbf{t}'(s)=k\mathbf{n_1}(s)\\ &\mathbf{n_1'}(s)=-k\mathbf{t}(s)+r\mathbf{n_2}(s)\\ &\mathbf{n_2'}(s)=-r\mathbf{n_1}(s), \end{aligned} \tag{3}$$

where k is the principal curvature and r is torsion of α, [2].

Let

$$\begin{aligned} \beta:[0,1]\subset\mathbb{R} &\to \mathbb{Q} \\ s &\to \beta(s)=\sum_{i=1}^{4}\alpha_i(s)e_i, \quad (e_4=1), \end{aligned}$$

be an unit speed curve with nonzero curvatures $\{K,k,(r-K)\}$ and the Frenet frame $\{\mathbf{T},\mathbf{N_1},\mathbf{N_2},\mathbf{N_3}\}$. Then, Frenet-Serret formulas for β in $\mathbb{E}^4$ are defined by

$$\begin{aligned} &\mathbf{T}'(s)=K\mathbf{N_1}(s)\\ &\mathbf{N_1'}(s)=-K\mathbf{T}(s)+k\mathbf{N_2}(s)\\ &\mathbf{N_2'}(s)=-k\mathbf{N_1}(s)+(r-K)\mathbf{N_3}(s)\\ &\mathbf{N_3'}(s)=-(r-K)\mathbf{N_2}(s), \end{aligned} \tag{4}$$

where K is the principal curvature, k is torsion and $(r-K)$ is bitorsion of β, [2].

Let

$$\begin{aligned} \gamma:[0,1]\subset\mathbb{R} &\to \mathbb{Q}_v \\ s &\to \gamma(s)=\sum_{i=1}^{3}\gamma_i(s)e_i, \end{aligned}$$

be an arc-length curve with nonzero curvatures $\{k,r\}$ and the Frenet frame $\{\mathtt{t},\mathtt{n_1},\mathtt{n_2}\}$. Then, Frenet-Serret formulas for γ in $\mathbb{E}^3_1$ are given by

$$\begin{aligned} &\mathtt{t}'(s)=\varepsilon_{\mathtt{n_1}}k\mathtt{n_1}(s)\\ &\mathtt{n_1}'(s)=-\varepsilon_{\mathtt{t}}k\mathtt{t}(s)+\varepsilon_{\mathtt{n_1}}r\mathtt{n_2}(s)\\ &\mathtt{n_2}'(s)=-\varepsilon_{\mathtt{n_2}}r\mathtt{n_1}(s) \end{aligned} \tag{5}$$

where k is the principal curvature, r is torsion of γ and $\langle\mathtt{t},\mathtt{t}\rangle_p=\varepsilon_{\mathtt{t}}$, $\langle\mathtt{n_1},\mathtt{n_1}\rangle_p=\varepsilon_{\mathtt{n_1}}$, $\langle\mathtt{n_2},\mathtt{n_2}\rangle_p=\varepsilon_{\mathtt{n_2}}$, [3].

Similarly, let

$$\begin{aligned} \theta:[0,1]\subset\mathbb{R} &\to \mathbb{Q}_v \\ s &\to \theta(s)=\sum_{i=1}^{4}\gamma_i(s)e_i, \quad (e_4=1), \end{aligned}$$

be an unit speed curve in $\mathbb{E}^4_2$ with nonzero curvatures $\{K,k,(r-K)\}$ and the Frenet frame $\{\mathtt{T},\mathtt{N_1},\mathtt{N_2},\mathtt{N_3}\}$. Then, Frenet-Serret formulas for θ in $\mathbb{E}^4_2$ are defined by

$$\begin{aligned} &\mathtt{T}'(s)=\varepsilon_{\mathtt{N_1}}K\mathtt{N_1}(s)\\ &\mathtt{N_1}'(s)=-\varepsilon_{\mathtt{N_1}}\varepsilon_{\mathtt{t}}K\mathtt{T}(s)+\varepsilon_{\mathtt{n_1}}k\mathtt{N_2}(s)\\ &\mathtt{N_2}'(s)=-\varepsilon_{\mathtt{t}}k\mathtt{N_1}(s)+\varepsilon_{\mathtt{n_1}}\left(r-K\varepsilon_{\mathtt{T}}\varepsilon_{\mathtt{t}}\varepsilon_{\mathtt{N_1}}\right)\mathtt{N_3}(s)\\ &\mathtt{N_3}'(s)=-\varepsilon_{\mathtt{N_2}}\left(r-K\varepsilon_{\mathtt{T}}\varepsilon_{\mathtt{t}}\varepsilon_{\mathtt{N_1}}\right)\mathtt{N_2}(s), \end{aligned} \tag{6}$$

where $\langle\mathtt{T},\mathtt{T}\rangle=\varepsilon_{\mathtt{T}}$, $\langle\mathtt{N_1},\mathtt{N_1}\rangle=\varepsilon_{\mathtt{N_1}}$, $\langle\mathtt{N_2},\mathtt{N_2}\rangle=\varepsilon_{\mathtt{N_2}}$ and $K=\varepsilon_{\mathtt{N_1}}\left\|\mathtt{T}'(s)\right\|$, [3].

iv) Focal curves

Definition 2.1. The curve which originated of the centres of the osculating hyperspheres of the fundamental curve is introduced the focal curve. The focal curve is represented by C_α for the main curve α.

For the focal curve, the following property can be given:

$$C_\alpha=\alpha+\sum c_iN_i,\ 1\le i\le n$$

where c_i and N_i are the focal curvatures and the Frenet-Serret apparatus of α, respectively. Besides, the focal curvatures of α satisfy the Frenet-Serret equations [12].

Theorem 2.2. *Let C_α be the focal curve of α. Then the focal curvatures are satisfied the following statement*

$$\kappa_i=\frac{\sum c_ic_i'}{c_{i-1}c_i},\ \textit{for } i\ge 2$$

where $\kappa_i's$ are the curvatures of α [12].

3. Quaternionic Focal Curves

In this section, we will analyze quaternionic focal curves of a unit speed real quaternionic curve β in $\mathbb{Q}$.

Definition 3.1. Let us take the quaternionic curve

$$\begin{aligned} \beta:[0,1]\subset\mathbb{R} &\to \mathbb{Q} \\ s &\to \beta(s)=\sum_{i=1}^{4}\alpha_i(s)e_i, \quad (e_4=1). \end{aligned}$$

We consider a quaternionic sphere

$$\langle x-M,x-M\rangle=(R_M)^2,$$

where M is origin, R is radius and $x=(x_1,x_2,x_3,x_4)$. Let us write

$$f(s)=\langle\beta(s)-M,\beta(s)-M\rangle-(R_M)^2.$$

If the followings hold

$$f(0)=0, f'(0)=0, f''(0)=0, f'''(0)=, f^{(4)}(0)=0, f^{(5)}\neq 0, \tag{7}$$

then the sphere contacts at fourth order to β at $\beta(0)$. This sphere is called *quaternionic osculating sphere* in $\mathbb{E}^4$, [9].

Theorem 3.2. *The quaternionic focal curve of a smooth real quaternionic curve $\beta : [0,1] \subset \mathbb{R} \to \mathbb{Q}$ consists of the centers its quaternionic osculating hyperspheres. At the points that β have nonzero curvatures, the centers of the quaternionic osculating hypersphere of β are well-defined. In this situation, the quaternionic focal curve C_β can be written as*

$$C_\beta(s) = (\beta + c_1\mathbf{N_1} + c_2\mathbf{N_2} + c_3\mathbf{N_3})(s),$$

where

$$\begin{aligned} c_1 &= \tfrac{1}{K} \\ c_2 &= -\tfrac{K'}{K^2k} \\ c_3 &= -\tfrac{K''Kk+2(K')^2k+(K')^2K+K^2k^3}{K^3k^2(r-K)} \end{aligned}$$

are smooth functions and $\{\mathbf{T},\mathbf{N_1},\mathbf{N_2},\mathbf{N_3}\}$ is the Frenet frame of β.

Proof. Let us take a quaternionic sphere from the above definition, $f(s) = \langle \beta(s) - M, \beta(s) - M\rangle - (R_M)^2$ which satisfies (7) and contacts at fourth order to β at $\beta(0)$. From the Frenet-Serret equations (4), we have

$$\begin{aligned} f'(0) = 0 \quad &\Rightarrow f' = 2\langle \beta', \beta - M\rangle = 0 \\ &\Rightarrow \langle \mathbf{T}, \beta - M\rangle = 0 \\ f''(0) = 0 \quad &\Rightarrow f'' = 2[\langle \mathbf{T}', \beta - M\rangle + \langle \mathbf{T}, \beta'\rangle] = 0 \\ &\Rightarrow \langle \mathbf{N_1}, \beta - M\rangle = -\tfrac{1}{K} \\ f'''(0) = 0 \quad &\Rightarrow f''' = K'\langle \mathbf{N_1}, \beta - M\rangle - K^2\langle \mathbf{T}, \beta - M\rangle \\ &+Kk\langle \mathbf{N_2}, \beta - M\rangle = 0 \\ &\Rightarrow \langle \mathbf{N_2}, \beta - M\rangle = \tfrac{K'}{K^2k} \\ f^4(0) = 0 \quad &\Rightarrow \langle \mathbf{N_3}, \beta - M\rangle = \tfrac{K''Kk+2(K')^2k+(K')^2K+K^2k^3}{K^3k^2(r-K)}. \end{aligned} \tag{8}$$

Then, let us take the function as follows:

$$\beta(0) - M = \lambda_1\mathbf{T} + \lambda_2\mathbf{N_1} + \lambda_3\mathbf{N_2} + \lambda_4\mathbf{N_3}. \tag{9}$$

By using the equations (8) and (9), we obtained that

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= \langle \mathbf{N_1}, \beta - M\rangle = -\tfrac{1}{K} \\ \lambda_3 &= \langle \mathbf{N_2}, \beta - M\rangle = \tfrac{K'}{K^2k} \\ \lambda_4 &= \langle \mathbf{N_3}, \beta - M\rangle = \tfrac{K''Kk+2(K')^2k+K'k'K+K^2k^3}{K^3k^2(r-K)}. \end{aligned}$$

From the definition (3.1), $\beta(s)$ can be written as follows:

$$C_\beta(s) = (\beta + c_1\mathbf{N_1} + c_2\mathbf{N_2} + c_3\mathbf{N_3})(s),$$

where $c_i = -\lambda_{i+1}$ for $1 \le i \le 3$. ☐

Definition 3.3. Let $\beta : [0,1] \to \mathbb{Q}$ be a real quaternionic curve and $C_\beta(s) = (\beta + c_1\mathbf{N_1} + c_2\mathbf{N_2} + c_3\mathbf{N_3})(s)$ be its quaternionic focal curve. Then, c_i are called *ith quaternionic focal curvatures* of β $(1 \le i \le 3)$.

Lemma 3.4. *Let us take β in $\mathbb{Q}$ with nonzero curvatures $\{K, k, (r-K)\}$ at any point and with Frenet frame $\{\mathbf{T},\mathbf{N_1},\mathbf{N_2},\mathbf{N_3}\}$. The velocity vector $C'_\beta(s)$ of a quaternionic focal curve of β is proportional to the $\mathbf{N_3}$ vector of β at the point s.*

Proof. Let us consider

$$\begin{aligned} F : \mathbb{E}^4 \times \mathbb{R} &\to \mathbb{R} \\ (C, s) &\to F_C(s) = \tfrac{1}{2}||C - \beta(s)||^2, \end{aligned}$$

where the caustic family of F is given by the set:

$$\left\{C \in \mathbb{E}^4 : \exists s \in \mathbb{R} : F'_C(s) = 0 \text{ and } F''_C(s) = 0\right\}.$$

Then we can write

$$-F = -\frac{\langle C_\beta, C_\beta\rangle}{2} + \langle C, \beta\rangle - \sigma,$$

where $\sigma = \frac{\langle \beta,\beta\rangle}{2}$. Therefore, the following equation system defines the focal curve $C_\beta(s)$ as below:

$$\begin{aligned} \langle \beta', C_\beta\rangle - \sigma' &= 0 \\ \langle \beta'', C_\beta\rangle - \sigma'' &= 0 \\ \langle \beta''', C_\beta\rangle - \sigma''' &= 0 \\ \langle \beta^4, C_\beta\rangle - \sigma^4 &= 0. \end{aligned} \tag{10}$$

By differentiating each equation with respect to s, we have

$$\begin{aligned} \langle \beta', C'_\beta\rangle + \langle \beta'', C_\beta\rangle - \sigma'' &= 0 \\ \langle \beta'', C'_\beta\rangle + \langle \beta''', C_\beta\rangle - \sigma''' &= 0 \\ \langle \beta''', C'_\beta\rangle + \langle \beta^{(4)}, C_\beta\rangle - \sigma^{(4)} &= 0 \\ \langle \beta^{(4)}, C'_\beta\rangle + \langle \beta^{(5)}, C_\beta\rangle - \sigma^{(5)} &= 0. \end{aligned} \tag{11}$$

By combining the equations (11) and (10) together, we calculate

$$\begin{aligned} \langle \beta', C'_\beta\rangle &= 0 \\ \langle \beta'', C'_\beta\rangle &= 0 \\ \langle \beta''', C'_\beta\rangle &= 0. \end{aligned} \tag{12}$$

Hence, $C'_\beta(s)$ is perpendicular to the osculating hyperplane of β, i.e., $C'_\beta(s)$ and $\mathbf{N_3}$ are linearly dependent. ☐

Theorem 3.5. *The focal curvatures of a unit speed real quaternionic curve $\beta : [0,1] \to \mathbb{Q}$, fulfill the below relation:*

$$\begin{bmatrix} 1 \\ c'_1 \\ c'_2 \\ c'_3 - \frac{((R_M)^2)'}{2c_3} \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r-K) \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

for $c_3 \neq 0$. The above relation is called "scalar Frenet equations".

Proof. Let $C_\beta(s) = (\beta + c_1\mathbf{N_1} + c_2\mathbf{N_2} + c_3\mathbf{N_3})(s)$ be a quaternionic focal curve of β. By differentiating C_β with respect to arc-length of β and considering the equations (4), we have

$$\begin{aligned} C'_\beta = \; &\beta' + c'_1\mathbf{N_1} + c'_2\mathbf{N_2} + c'_3\mathbf{N_3} + c_1\mathbf{N'_1} + c_2\mathbf{N'_2} \\ &+ c_3\mathbf{N'_3} \end{aligned}$$

and we get

$$\begin{aligned} C'_\beta = \; &(1 - c_1K)\mathbf{T} + (c'_1 - c_2k)\mathbf{N_1} \\ &+ (c'_2 + c_1k - c_3(r-K))\mathbf{N_2} + (c'_3 + (r-K)c_2)\mathbf{N_3}. \end{aligned}$$

From the statement of Lemma (3.4), $C'_\beta(s)$ is proportional to the vector $\mathbf{N_3}$ and the components of the vectors $\mathbf{T}, \mathbf{N_1}$ and $\mathbf{N_2}$ are zero. Hence, we get

$$\begin{aligned} 1 &= c_1 K \\ c'_1 &= c_2 k \\ c'_2 &= -c_1 k + c_3(r-K). \end{aligned} \tag{13}$$

So, we have $C'_\beta = (c'_3 + (r-K)c_2)\mathbf{N_3}$. Due to the fact that, R_M satisfies the equation $(R_M)^2 = \|C_\beta - \beta\|^2$.

Accordingly, we get the equation as follows:

$$((R_M)^2)' = 2c_3(c'_3 + (r-K)c_2).$$

Since $c_3 \neq 0$, we obtain that

$$c'_3 - \frac{((R_M)^2)'}{2c_3} = -(r-K)c_2.$$

The above equation together with the equation (13) proves the theorem. □

Definition 3.6. A vertex of a real quaternionic curve of β in $\mathbb{Q}$ is a point at least 6-point contact with its quaternionic osculating hypersphere.

Definition 3.7. If a point contact osculating hyperplane of β in $\mathbb{Q}$ at least 5, the point is called flattening of β. For these points $K \neq 0, k \neq 0, (r-K) = 0$ and $(r-K)' \neq 0$. These exists 4-point contact between flattenings and ordinary points.

Definition 3.8. The point where the center of the osculating hypersphere lies in the osculating hyperplane is called symmetry point of a real quaternionic curve in $\mathbb{Q}$, namely $c_3 = 0$.

Lemma 3.9. *A non-flattening point $\beta(s)$ of β in $\mathbb{Q}$ is a vertex of $\Leftrightarrow C'_\beta = 0$ at that point.*

Proof. Let $\beta(s)$ be a vertex of β, then moreover equation system (10), we have the equation as follows:

$$\langle \beta^{(5)}, C_\beta \rangle - g^{(5)} = 0. \tag{14}$$

By substituting (14) with the system (11), we obtain the equation as below:

$$\langle \beta^{(4)}, C'_\beta \rangle = 0.$$

The above equation together with (14) implicates that for a non flat vertex $\beta(s)$ of β, $C'_\beta(s)$ is zero.

To the contrary, if $\beta(s_0)$ is non-vertex, then the corresponding point C_β satisfies:

$$\langle \beta^{(5)}(s_0), C_\beta(s_0) \rangle - g^{(5)}(s_0) \neq 0. \tag{15}$$

The equation (15) and the equation (12) imply that $C'_\beta(s) \neq 0$ for $s = s_0$. □

Theorem 3.10. *A non-flattening of β in $\mathbb{Q}$, is a vertex $\Leftrightarrow$ $c'_3 + (r-K)c_2 = 0$.*

Proof. With reference to Lemma (3.9), we get $C'_\beta = (c'_3 + (r-K)c_2)\mathbf{N_3} = 0$. Hence, $c'_3 + (r-K)c_2 = 0$. □

Corollary 3.11. *β is spherical if and only if*

$$c'_3 + (r-K)c_2 = 0.$$

Theorem 3.12. *The curvatures of unit speed curve β in $\mathbb{Q}$, can be obtained in the sense of the quaternionic focal curvatures as follows:*

$$\begin{aligned} K &= \tfrac{1}{c_1} \\ k &= \tfrac{c_1 c'_1}{c_1 c_2} \\ r-K &= \tfrac{c_2 c'_2 + c_1 c'_1}{c_2 c_3}. \end{aligned}$$

Proof. It is apparent from the Theorem 3.2. □

Theorem 3.13. *Let $\beta(s) \in \mathbb{Q}$ be a real quaternionic curve without flattenings. Let $\{\mathbf{T}, \mathbf{N_1}, \mathbf{N_2}, \mathbf{N_3}\}$ be its Frenet frame and $\{K, k, (r-K)\}$ be its quaternionic curvatures. Besides, let $\{\mathbf{T}^c, \mathbf{N_1}^c, \mathbf{N_2}^c, \mathbf{N_3}^c\}$ and $\{K^c, k^c, (r-K)^c\}$ be Frenet frame and curvatures of C_β, respectively. For all non-vertex $\beta(s)$ of β, let $\varepsilon(s)$ be the sign of $(c'_3 + (r-K)c_2)(s)$ and $\delta_n(s)$ be the sign of $((-1)^n \varepsilon(r-K))(s), 1 \leq n \leq 3$. For any non-vertex of β, the above statements hold:*

i) The Frenet frame $\{\mathbf{T^c}, \mathbf{N_1^c}, \mathbf{N_2^c}, \mathbf{N_3^c}\}$ of C_β at $C_\beta(s)$ is given by

$$\begin{aligned} \mathbf{T^c} &= \varepsilon \mathbf{N_3} \\ \mathbf{N_1^c} &= \delta_1 \mathbf{N_2} \\ \mathbf{N_2^c} &= \delta_2 \mathbf{N_1} \\ \mathbf{N_3^c} &= \pm \mathbf{T}, \end{aligned}$$

where we write $\pm\mathbf{T}$ so as to have a positive basis.

ii) There is a relation between the curvatures $\{K^c, k^c, (r-K)^c\}$ of $C_\beta(s)$ and the curvatures $\{K, k, (r-K)\}$ of β such as

$$\frac{K^c}{(r-K)} = \frac{k^c}{k} = \frac{(r-K)^c}{K} = \frac{1}{c'_3 + (r-K)c_2},$$

where the sign of $(r-K)^c$ is equal to δ_3 times the sign chosen in $\pm\mathbf{T}$. Thus, the Frenet matrix of C_β at $C_\beta(s)$ is

$$\frac{1}{c'_3 + (r-K)c_2} \begin{bmatrix} 0 & (r-K) & 0 & 0 \\ -(r-K) & 0 & k & 0 \\ 0 & -k & 0 & \pm\delta_3 K \\ 0 & 0 & \pm\delta_3 K & 0 \end{bmatrix}.$$

Proof. Let t be arc-length parameter of C_β at $C_\beta(s)$. Suppose that the orientations of the unit speed $C_\beta(s)$ and $C_\beta(t)$ coincide. It is obvious that $C'_\beta \neq 0$ for any vertex of β. Hence, we can write the unit tangent vector of C_β such that

$$\mathbf{T^c} = \frac{C'_\beta}{\|C'_\beta\|} = \varepsilon \mathbf{N_3}, \tag{16}$$

where $\varepsilon = \frac{c_3'+(r-K)c_2}{|c_3'+(r-K)c_2|}$.

Besides, for any non-vertex

$$\frac{ds}{dt} = \frac{1}{c_3'+(r-K)c_2}. \tag{17}$$

If we derive the equation (16) with respect to t and perform the equations (4) considering that the curvatures $\{K, k, (r-K)\}$ are always positive, we have

$$\mathbf{N_1^c} = \delta_1 \mathbf{N_2}$$

and

$$K^c = \frac{(r-K)}{c_3'+(r-K)c_2}.$$

Moreover, we use equation (17) to obtain $\mathbf{N_2^c} = \delta_2 \mathbf{N_1}$ and $k^c = \frac{k}{c_3'+(r-K)c_2}$. □

4. Spatial Semi-Real Quaternionic Focal Curves

In this section, we will examine spatial semi-real quaternionic focal curves of a unit speed spatial semi-real quaternionic curve in $\mathbb{Q}_v$.

Definition 4.1. Let us take the spatial semi-real quaternionic curve

$$\begin{aligned} \gamma : [0,1] \subset \mathbb{R} &\to \mathbb{Q}_v \\ s &\to \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i, \end{aligned}$$

where $\{q \in \mathbb{Q}_v \mid q + \overline{q} = 0\}$. Let us take a spatial semi-real quaternionic sphere $\langle y-m, y-m\rangle_p = r_m^2$, where m is origin, r is radius and $y = (y_1, y_2, y_3)$. Let us define $h(s) = \langle \gamma(s)-m, \gamma(s)-m\rangle_p - r_m^2$. If the undermentioned relations hold

$$h(0) = h'(0) = h''(0) = h'''(0) = 0, \; h^{(4)} \neq 0, \tag{18}$$

then the sphere contacts at third order to γ at $\gamma(0)$. This sphere is called *spatial semi-real quaternionic osculating sphere* for spatial real quaternionic curves in $\mathbb{E}_1^3$ [11].

Theorem 4.2. *The spatial semi-real quaternionic focal curve of $\gamma : [0,1] \subset \mathbb{R} \to \mathbb{Q}_v$ consists of the centers its semi-real quaternionic osculating hyperspheres. If all the curvatures of γ are nonzero, then the centers of this hypersphere are well-defined and in this manner, the spatial semi-real quaternionic focal curve C_γ can be written as follows:*

$$C_\gamma(s) = (\gamma + c_1 \mathtt{n_1} + c_2 \mathtt{n_2})(s),$$

where c_1, c_2 are smooth functions and $\{\mathtt{t}, \mathtt{n_1}, \mathtt{n_2}\}$ is Frenet frame of γ. And c_1, c_2 are defined by

$$\begin{aligned} c_1 &= \frac{\varepsilon_{\mathtt{t}}}{\varepsilon_{\mathtt{n_1}} k} \\ c_2 &= -\frac{\varepsilon_{\mathtt{t}} k'}{k^2 r}. \end{aligned} \tag{19}$$

Proof. From the above mentioned definition (4.1), we can take the function $h(s) = \langle \gamma(s)-m, \gamma(s)-m\rangle - r_m^2$ which satisfies (18). By using the equations (18), we obtain that

$$\begin{aligned} h'(0) = 0 \quad & \Rightarrow h' = 2\langle \gamma', \gamma-m\rangle = 0 \Rightarrow \langle \mathtt{t}, \gamma-m\rangle = 0 \\ h''(0) = 0 \quad & \Rightarrow h'' = 2[\langle \mathtt{t}', \gamma-m\rangle + \langle \mathtt{t}, \gamma'\rangle] = 0 \\ & \Rightarrow \langle \mathtt{n_1}, \gamma-m\rangle = -\frac{\varepsilon_{\mathtt{t}}}{\varepsilon_{\mathtt{n_1}} k} \\ h'''(0) = 0 \quad & \Rightarrow h''' = \varepsilon_{\mathtt{n_1}} k' \langle \mathtt{n_1}, \gamma-m\rangle \\ & + \varepsilon_{\mathtt{n_1}} k \langle -\varepsilon_{\mathtt{t}} k \mathtt{t} + \varepsilon_{\mathtt{n_1}} r \mathtt{n_2}, \gamma-m\rangle = 0 \\ & \Rightarrow \langle \mathtt{n_2}, \gamma-m\rangle = \frac{\varepsilon_{\mathtt{t}} k'}{k^2 r}. \end{aligned} \tag{20}$$

Then, let us take the function as follows:

$$\gamma(0) - m = \lambda_1 \mathtt{t} + \lambda_2 \mathtt{n_1} + \lambda_3 \mathtt{n_2}. \tag{21}$$

By taking into account the equations (20) and (21), we have

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -\frac{\varepsilon_{\mathtt{t}}}{\varepsilon_{\mathtt{n_1}} k} \\ \lambda_3 &= \frac{\varepsilon_{\mathtt{t}} k'}{k^2 r}. \end{aligned}$$

From the definition (4.1), $C_\gamma(s)$ can be written namely:

$$C_\gamma(s) = (\gamma + c_1 \mathtt{n_1} + c_2 \mathtt{n_2})(s),$$

where $c_i = -\lambda_{i+1}$ for $1 \leq i \leq 2$. □

Definition 4.3. Let $C_\gamma(s) = (\gamma + c_1 \mathtt{n_1} + c_2 \mathtt{n_2})(s)$ be spatial semi-real quaternionic focal curve of γ. Then, c_1 and c_2 are called *spatial semi-real quaternionic focal curvatures* of γ.

Lemma 4.4. *Let*

$$\begin{aligned} \gamma : [0,1] &\to \mathbb{Q}_v \\ s &\to \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i \end{aligned}$$

be a spatial semi-real quaternionic curve with nonzero curvatures $\{k, r\}$ and Frenet frame $\{\mathtt{t}, \mathtt{n_1}, \mathtt{n_2}\}$. At the point s, the velocity vector $C_\gamma'(s)$ and $\mathtt{n_2}$ vector of a spatial semi-real quaternionic focal curve of γ are linearly dependent.

Proof. Let us take into consideration

$$\begin{aligned} F : \mathbb{E}_1^3 \times \mathbb{R} &\to \mathbb{R} \\ (C, s) &\to F_C(s) = \tfrac{1}{2} ||C - \gamma(s)||^2, \end{aligned}$$

where the caustic family of F is given by the set

$$\left\{ C \in \mathbb{E}_1^3 : \exists s \in \mathbb{R} : F_C'(s) = 0 \text{ and } F_C''(s) = 0 \right\}.$$

Then we can write

$$-F = -\frac{\langle C_\gamma, C_\gamma\rangle}{2} + \langle C_\gamma, \gamma\rangle - \rho,$$

where $\rho = \frac{\langle \gamma, \gamma\rangle}{2}$. Then, the following equation system defines the focal curve $C_\gamma(s)$ as below:

$$\begin{aligned} \langle \gamma', C_\gamma\rangle - \rho' &= 0 \\ \langle \gamma'', C_\gamma\rangle - \rho'' &= 0 \\ \langle \gamma''', C_\gamma\rangle - \rho''' &= 0. \end{aligned} \tag{22}$$

By differentiating each equation with respect to s, we obtain that

$$\begin{aligned}\langle\gamma',C_\gamma'\rangle+\langle\gamma'',C_\gamma\rangle-\rho''&=0\\ \langle\gamma'',C_\gamma'\rangle+\langle\gamma''',C_\gamma\rangle-\rho'''&=0\\ \langle\gamma''',C_\gamma'\rangle+\langle\gamma^{(4)},C_\gamma\rangle-\rho^{(4)}&=0.\end{aligned}\tag{23}$$

By combining the equation (23) and (22) together, we get

$$\begin{aligned}\langle\gamma',C_\gamma'\rangle&=0\\ \langle\gamma'',C_\gamma'\rangle&=0.\end{aligned}\tag{24}$$

So, the velocity vector $C_\gamma'(s)$ is perpendicular to the osculating hyperplane of γ, i.e., $C_\gamma'(s)$ and $\mathtt{n_2}$ are linearly dependent. □

Theorem 4.5. *The focal curvatures of $\gamma:[0,1]\to\mathbb{Q}_v$, satisfy the below equations called "scalar Frenet equations":*

$$\begin{bmatrix}\varepsilon_{\mathtt{t}}\\ \varepsilon_{\mathtt{n_2}}c_1'\\ \frac{c_2'}{\varepsilon_{\mathtt{n_1}}}-\frac{((r_m)^2)'}{2\varepsilon_{\mathtt{n_1}}\varepsilon_{\mathtt{n_2}}c_2}\end{bmatrix}=\begin{bmatrix}0&k&0\\-k&0&r\\0&-r&0\end{bmatrix}\begin{bmatrix}0\\c_1\\c_2\end{bmatrix}\tag{25}$$

for $c_2\neq 0$.

Proof. Let $C_\gamma(s)=(\gamma+c_1\mathtt{n_1}+c_2\mathtt{n_2})(s)$ be a spatial semi-real quaternionic focal curve. By differentiating C_γ with respect to s and considering the equations (5), we have

$$C_\gamma'=(1-\varepsilon_{\mathtt{t}}kc_1)\mathtt{t}+(c_1'-\varepsilon_{\mathtt{n_2}}rc_2)\mathtt{n_1}+(c_2'+\varepsilon_{\mathtt{n_1}}rc_1)\mathtt{n_2}.$$

The statement of the Lemma (4.4) $C_\gamma'(s)$ is proportional to the vector $\mathtt{n_2}$ and the components of the vectors $\mathtt{t}$ and $\mathtt{n_1}$ are zero. Therefore, the following equations are calculated:

$$\begin{aligned}1&=\varepsilon_{\mathtt{t}}kc_1\\ c_1'&=\varepsilon_{\mathtt{n_2}}rc_2\\ C_\gamma'&=(c_2'+\varepsilon_{\mathtt{n_1}}rc_1)\mathtt{n_2}.\end{aligned}\tag{26}$$

Owing to the fact that, $(r_m)^2=\|C_\gamma-\gamma\|^2$, we get

$$((r_m)^2)'=2\varepsilon_{\mathtt{n_2}}c_2(c_2'+\varepsilon_{\mathtt{n_1}}rc_1).$$

Finally, we have

$$c_2'=\frac{((r_m)^2)'}{2\varepsilon_{\mathtt{n_2}}c_2}-\varepsilon_{\mathtt{n_1}}rc_1$$

for $c_2\neq 0$. The last equation together with (26) proves the theorem. □

Definition 4.6. Let $\gamma\in\mathbb{Q}_v$. Then,

i) A vertex of a γ is a point at least 5-point contact with its spatial semi-real quaternionic osculating hypersphere.

ii) If a point of γ has at least 4-point contact with osculating hyperplane of γ is called flattening of γ. For these points $k\neq 0, r=0$ and $r'\neq 0$. These exists 3-point contact between flattenings and ordinary points.

iii) The point where the center of the semi-real quaternionic osculating hypersphere lies in the osculating hyperplane is called symmetry point of γ. For these points $c_2=0$.

Lemma 4.7. *A non-flattening point of γ in $\mathbb{Q}_v$ is a vertex $\Leftrightarrow C_\gamma'(s)=0$.*

Proof. Let $\gamma(s)$ be a vertex of γ, so we have

$$\langle\gamma^{(4)},C_\gamma(s)\rangle-h^{(4)}=0.\tag{27}$$

By using (27) in the system (23), we obtain the equation as below:

$$\langle\gamma^{(3)},C_\gamma'(s)\rangle=0.$$

Furthermore, the last equation and the equation (24) state that for a non flat vertex $\gamma(s)$ of γ, the vector $C_\gamma'(s)$ is zero.

The sufficiency condition, if $\gamma(s_0)$ is non-vertex then the corresponding point of C_γ has the following relation:

$$\langle\gamma^{(4)}(s_0),C_\gamma(s_0)\rangle-h^{(4)}(s_0)\neq 0.\tag{28}$$

The equation (28) and (26) means that $C_\gamma'(s)\neq 0$ for $s=s_0$. □

Theorem 4.8. *A non-flattening of γ in $\mathbb{Q}_v$, is a vertex $\Leftrightarrow$ $c_2'+\varepsilon_{\mathtt{n_1}}rc_1=0$.*

Proof. According to the Lemma (4.7), we get $C_\gamma'=(c_2'+\varepsilon_{\mathtt{n_1}}rc_1)\mathtt{n_2}=0$. In this way, $c_2'+\varepsilon_{\mathtt{n_1}}rc_1=0$. □

Corollary 4.9. *γ in $\mathbb{Q}_v$ is spherical if and only if $c_2'+\varepsilon_{\mathtt{n_1}}rc_1=0$.*

Theorem 4.10. *The curvatures of γ in $\mathbb{Q}_v$ can be calculated with regards to semi-real quaternionic focal curvatures of γ, by the formulae:*

$$\begin{aligned}k&=\frac{\varepsilon_t}{\varepsilon_{\mathtt{n_1}}c_1}\\ r&=-\frac{\varepsilon_t c_1'}{c_1^2c_2}.\end{aligned}$$

Proof. It is clear from the Theorem 4.2. □

Theorem 4.11. *Let $\gamma(s)\in\mathbb{Q}_v$ be a spatial semi-real quaternionic curve without flattenings. Let $\{\mathtt{t},\mathtt{n_1},\mathtt{n_2}\}$ be its Frenet frame and $\{k,r\}$ be its semi-real quaternionic curvatures. Additionally, let $\{\mathtt{t}^c,\mathtt{n}_1^c,\mathtt{n}_2^c\}$ and $\{K^c,k^c,(r-K)^c\}$ be Frenet frame and curvatures of C_γ, respectively. For all non-vertex $\gamma(s)$ of γ, let $\varepsilon(s)$ be the sign of $((c_2'+\varepsilon_{\mathtt{n_1}}rc_1)\mathtt{n_2})(s)$ and $\delta_n(s)$ be the sign of $((-1)^n\varepsilon r)(s), n=1,2$. For any non-vertex $\gamma(s)$, the followings hold:*

i) The Frenet frame $\{\mathtt{t}^c,\mathtt{n}_1^c,\mathtt{n}_2^c\}$ C_γ at $C_\gamma(s)$ is given as below:

$$\begin{aligned}\mathtt{t}^c&=\varepsilon\mathtt{n_2}\\ \mathtt{n}_1^c&=\delta_1\mathtt{n_1}\\ \mathtt{n}_2^c&=\pm\mathtt{t},\end{aligned}$$

where we write $\pm\mathtt{t}$ so as to have a positive basis.

ii) There exists the following relation between k^c and r^c such that:

$$\frac{k^c}{r}=\frac{r^c}{k}=\frac{1}{c_2'+\varepsilon_{\mathtt{n_1}}rc_1}.$$

The sign of r is equal to δ_2 times the sign chosen in $\pm\mathtt{t}$. So, the Frenet matrix of C_γ at $C_\gamma(s)$ given as below:

$$\frac{1}{c_2' + \varepsilon_{\mathtt{n}_1} r c_1} \begin{bmatrix} 0 & r & 0 \\ -r & 0 & \pm\delta_2 k \\ 0 & \pm\delta_2 k & 0 \end{bmatrix}.$$

Proof. Let t be arc-length parameter of C_γ at $C_\gamma(s)$. We suppose that $C_\gamma(s)$ and $C_\gamma(t)$ coincide. For any non-vertex $\gamma(s)$, we know that $C_\gamma' \neq 0$. So we can write the unit tangent vector of $C_\gamma(s)$ such that

$$\mathtt{t}^c = \frac{C_\gamma'}{\|C_\gamma'\|} = \varepsilon \mathtt{n}_2, \tag{29}$$

where $\varepsilon = \frac{c_2' + \varepsilon_{\mathtt{n}_1} r c_1}{|c_2' + \varepsilon_{\mathtt{n}_1} r c_1|}$. Additionally, for any non-vertex

$$\frac{ds}{dt} = \frac{1}{c_2' + \varepsilon_{\mathtt{n}_1} r c_1}. \tag{30}$$

By deriving the equation (29) with respect to t and perform the equations (5) with taking into account that the curvatures $\{k, r\}$ are always positive, we get

$$\mathtt{n}_1^c = \delta_1 \mathtt{n}_1$$

and

$$k^c = \frac{r}{c_2' + \varepsilon_{\mathtt{n}_1} r c_1}.$$

In this way, we use (30) equation to obtain $\mathtt{n}_2^c = \pm t$ and $r^c = \frac{r}{c_2' + \varepsilon_{\mathtt{n}_1} r c_1}$. □

5. Semi-Real Quaternionic Focal Curves

In this section, we will investigate semi-real quaternionic focal curves of a unit speed semi-real quaternionic curve in $\mathbb{Q}_v$. All of the proofs belongs to lemmas and theorems can be easily calculated similarly by the same way of the section 4.

Definition 5.1. Let us take the semi-real quaternionic curve

$$\begin{aligned} \theta : [0,1] \subset \mathbb{R} &\to \mathbb{Q}_v \\ s &\to \theta(s) = \sum_{i=0}^{4} \gamma_i(s) e_i. \end{aligned}$$

Let us take a semi-real quaternionic sphere $\langle X - \mathscr{M}, X - \mathscr{M} \rangle = \mathscr{R}^2_{\mathscr{M}}$, where $\mathscr{M}$ is origin, $\mathscr{R}$ is radius, and $X = (X_1, X_2, X_3, X_4)$. Let take $g(s) = \langle \theta(s) - \mathscr{M}, \theta(s) - \mathscr{M} \rangle - \mathscr{R}^2_{\mathscr{M}}$. If the below relations hold:

$$g(0) = g'(0) = g''(0) = g'''(0) = g^4(0) = 0, g^{(5)} \neq 0, \tag{31}$$

then the sphere contacts at fourth order to θ at $\theta(0)$. The sphere is called *semi-real quaternionic osculating sphere* for semi-real quaternionic curves in $\mathbb{E}^4_2$ [11].

Theorem 5.2. *Let θ be a smooth curve in $\mathbb{Q}_v$. Then, its semi-real quaternionic focal curve is defined by its semi-real quaternionic osculating hyperspheres where all its curvatures different from zero. Based upon these statements, the semi-real quaternionic focal curve C_θ can be written as*

$$C_\theta(s) = (\theta + c_1 \mathtt{N}_1 + c_2 \mathtt{N}_2 + c_3 \mathtt{N}_3)(s),$$

where $c_i, 1 \leq i \leq 3$ are smooth functions and $\{\mathtt{T}, \mathtt{N}_1, \mathtt{N}_2, \mathtt{N}_3\}$ is Frenet frame of θ. c_1, c_2, c_3 are defined as follows:

$$\begin{aligned} c_1 &= \frac{\varepsilon_{\mathtt{T}}}{\varepsilon_{\mathtt{N}_1} K} \\ c_2 &= -\frac{\varepsilon_{\mathtt{T}} K'}{\varepsilon_{\mathtt{N}_1} \varepsilon_{\mathtt{n}_1} K^2 k} \\ c_3 &= \frac{\varepsilon_{\mathtt{T}}[K'' + K'k' + \varepsilon_{\mathtt{n}_1} \varepsilon_{\mathtt{t}} K^3]}{\varepsilon_{\mathtt{N}_1} K^2 k [r - K \varepsilon_{\mathtt{T}} \varepsilon_{\mathtt{t}} \varepsilon_{\mathtt{N}_1}]}. \end{aligned}$$

Definition 5.3. Let $\theta \in \mathbb{Q}_v$ and $C_\theta(s) = \theta + c_1 \mathtt{N}_1 + c_2 \mathtt{N}_2 + c_3 \mathtt{N}_3$, be its semi-real quaternionic focal curve. Then, $c_i, 1 \leq i \leq 3$ are called *ith semi-real quaternionic focal curvatures* of θ.

Lemma 5.4. *Let $\theta \in \mathbb{Q}_v$ with nonzero curvatures $\{K, k, (r - K)\}$ at any point and Frenet frame $\{\mathtt{T}, \mathtt{N}_1, \mathtt{N}_2, \mathtt{N}_3\}$. $C_\theta'(s)$ and $\mathtt{N}_3$ are linearly dependent at the point s.*

Theorem 5.5. *The focal curvatures of $\theta : [0,1] \to \mathbb{Q}_v$, satisfy the above "scalar Frenet equations":*

$$\begin{bmatrix} 1 \\ c_1' \\ c_2' \\ c_3' - \frac{((\mathscr{R}_{\mathscr{M}})^2)'}{2\varepsilon_{\mathtt{N}_3} c_3} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & \varepsilon_{\mathtt{N}_1} \varepsilon_{\mathtt{t}} K & 0 & 0 \\ 0 & 0 & \varepsilon_{\mathtt{t}} k & 0 \\ 0 & \varepsilon_{\mathtt{n}_1} k & 0 & -\varepsilon_{\mathtt{n}_2}(r - K \varepsilon_{\mathtt{T}} \varepsilon_{\mathtt{t}} \varepsilon_{\mathtt{N}_1}) \\ 0 & 0 & -\varepsilon_{\mathtt{n}_1}(r - K \varepsilon_{\mathtt{T}} \varepsilon_{\mathtt{t}} \varepsilon_{\mathtt{N}_1}) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

for $c_3 \neq 0$.

Definition 5.6. Let θ be a semi-real quaternionic curve in $\mathbb{Q}_v$. Then,

a) A vertex of θ is a point at least 6-point contact with its semi-real quaternionic osculating hypersphere.

b) If a point of θ has at least 5-point contact with osculating hyperplane of θ is called flattening of θ. For these points $K \neq 0, k \neq 0, (r - K) = 0$ and $(r - K)' \neq 0$. These exists 4-point contact between flattenings and ordinary points.

c) A symmetry point of θ is a point at which the center of the osculating hypersphere lies in the osculating hyperplane, in other words $c_3 = 0$.

Lemma 5.7. *A non-flattening point of $\theta(s) \in \mathbb{Q}_v$ is a vertex if and only $C_\theta' = 0$.*

Theorem 5.8. *A non-flattening point of θ in $\mathbb{Q}_v$ is a vertex $\Leftrightarrow c_3' + \varepsilon_{\mathtt{n}_1}(r - K\varepsilon_{\mathtt{T}}\varepsilon_{\mathtt{t}}\varepsilon_{\mathtt{N}_1})c_2 = 0$*

Corollary 5.9. *θ in $\mathbb{Q}_v$ is spherical if and only if $c_3' + \varepsilon_{\mathtt{n}_1}(r - K\varepsilon_{\mathtt{T}}\varepsilon_{\mathtt{t}}\varepsilon_{\mathtt{N}_1})c_2 = 0$.*

Theorem 5.10. *The curvatures of θ in $\mathbb{Q}_v$, can be obtained with respect to the semi-real quaternionic focal curvatures of θ given as follows:*

$$K = \frac{-\varepsilon_{\mathtt{T}}}{\varepsilon_{\mathtt{N_1}} c_1}$$

$$k = \frac{c_1'}{\varepsilon_{\mathtt{N_1}}}$$

$$r-K = \frac{c_2 c_2' + c_1 c_1'}{c_2 c_3}.$$

Theorem 5.11. *Let $\theta(s) \in \mathbb{Q}_v$ be a semi-real quaternionic curve without flattenings. Let $\{\mathtt{T}, \mathtt{N_1}, \mathtt{N_2}, \mathtt{N_3}\}$ be its a Frenet frame and $\{K, k, (r-K)\}$ be semi-real quaternionic curvatures of θ. Similarly, let $\{\mathtt{T}^c, \mathtt{N}_1^c, \mathtt{N}_2^c, \mathtt{N}_3^c\}$ and $K^c, k^c, (r-K)^c$ be Frenet frame and curvatures of C_θ, respectively. For each non-vertex $\theta(s)$ of θ, let $\varepsilon(s)$ be the sign of $(c_3' + \varepsilon_{\mathtt{n_1}}(r - K\varepsilon_{\mathtt{T}}\varepsilon_{\mathtt{t}}\varepsilon_{\mathtt{N_1}})c_2)(s)$ and $\delta_n(s)$ be the sign of $((-1)^n\varepsilon(r-K))(s), n = 1,2,3$. For any non-vertex $\theta(s)$ the followings hold:*

i) The Frenet frame of C_θ at $C_\theta(s)$ written such that

$$\begin{aligned} \mathtt{T}^c &= \varepsilon \mathtt{N_3} \\ \mathtt{N}_1^c &= \delta_1 \mathtt{N_2} \\ \mathtt{N}_2^c &= \delta_2 \mathtt{N_1} \\ \mathtt{N}_3^c &= \pm \mathtt{T}, \end{aligned}$$

where we write $\pm\mathtt{T}$ so as to have a positive basis.

ii) There is a relation between the curvatures $\{K^c, k^c, (r-K)^c\}$ of $C_\theta(s)$ and the curvatures $\{K, k, (r-K)\}$ of θ such as

$$\frac{K^c}{(r-K)} = \frac{k^c}{k} = \frac{(r-K)^c}{K} \frac{1}{c_3' + \varepsilon_{n_1}(r - K\varepsilon_T \varepsilon_t \varepsilon_{N_1})c_2},$$

where the sign of $(r-K)^c$ is equal to δ_3 times the sign chosen in $\pm\mathtt{T}$. That's why, the Frenet matrix of C_θ at $C_\theta(s)$ is

$$\frac{1}{c_3' + \varepsilon_{n_1}(r - K\varepsilon_T \varepsilon_t \varepsilon_{N_1})c_2} \begin{bmatrix} 0 & (-r-K) & 0 & 0 \\ -(r-K) & 0 & k & 0 \\ 0 & -k & 0 & \pm\delta_3 K \\ 0 & 0 & \pm\delta_3 K & 0 \end{bmatrix}.$$

Example 1. Let us take the following spatial unit speed semi-real quaternionic curve

$$\begin{aligned} \gamma : [0,1] \subset \mathbb{R} &\to \mathbb{Q}_v \\ s &\to \gamma(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3, \end{aligned}$$

with nonzero curvatures $\{k, r\}$ and the Frenet frame $\{\mathtt{t}, \mathtt{n_1}, \mathtt{n_2}\}$ where $\{q \in \mathbb{Q}_v \,|\, q + \overline{q} = 0\}$. Let assume that $\varepsilon_{e_1} = -1, \varepsilon_{e_2} = \varepsilon_{e_3} = 1$. Then γ is a timelike curve. Let us show that the scalar Frenet equations (25) are satisfied by γ.

We can compute the Frenet invariants of γ such that ([23])

$$\mathtt{t} = (\gamma_1', \gamma_2', \gamma_3'),$$

$$\mathtt{n_1} = \frac{1}{\sqrt{-\gamma_1''^2 + \gamma_2''^2 + \gamma_3''^2}} (\gamma_1'', \gamma_2'', \gamma_3''),$$

$$\mathtt{n_2} = \frac{(-\gamma_3'\gamma_2'' + \gamma_2'\gamma_3'', -\gamma_3'\gamma_1'' + \gamma_1'\gamma_3'', \gamma_2'\gamma_1'' - \gamma_1'\gamma_2'')}{\sqrt{-\gamma_1'^2 + \gamma_2'^2 + \gamma_3'^2}},$$

$$k = \sqrt{-\gamma_1''^2 + \gamma_2''^2 + \gamma_3''^2},$$

$$r = \frac{\gamma_1'(-\gamma_3''\gamma_2^{(3)} + \gamma_2''\gamma_3^{(3)}) + \gamma_2'(\gamma_3''\gamma_1^{(3)} - \gamma_1''\gamma_3^{(3)}) + \gamma_3'(-\gamma_2''\gamma_1^{(3)} + \gamma_1''\gamma_2^{(3)})}{\gamma_1''^2 - \gamma_2''^2 - \gamma_3''^2}.$$

By using the equations (19), we can compute the focal curvatures of γ as follows:

$$c_1 = -\frac{1}{\sqrt{-\gamma_1''^2 + \gamma_2''^2 + \gamma_3''^2}},$$

$$c_2 = \frac{\gamma_1''\gamma_1^{(3)} - \gamma_2''\gamma_2^{(3)} - \gamma_3''\gamma_3^{(3)}}{\sqrt{-\gamma_1''^2 + \gamma_2''^2 + \gamma_3''^2}\left(\begin{array}{c} \gamma_1'(-\gamma_3''\gamma_2^{(3)} + \gamma_2''\gamma_3^{(3)}) + \gamma_2'(\gamma_3''\gamma_1^{(3)} - \gamma_1''\gamma_3^{(3)}) \\ +\gamma_3'(-\gamma_2''\gamma_1^{(3)} + \gamma_1''\gamma_2^{(3)}). \end{array}\right)}$$

Then, we have

$$\varepsilon_{\mathtt{n_2}} c_1' = \frac{-\gamma_1''\gamma_1^{(3)} + \gamma_2''\gamma_2^{(3)} + \gamma_3''\gamma_3^{(3)}}{(-\gamma_1''^2 + \gamma_2''^2 + \gamma_3''^2)^{3/2}}$$

$$\frac{c_2'}{\varepsilon_{\mathtt{n_1}}} - \frac{((r_m)^2)'}{2\varepsilon_{\mathtt{n_1}}\varepsilon_{\mathtt{n_2}} c_2} = -\frac{\begin{array}{c} \gamma_1'(-\gamma_3''\gamma_2^{(3)} + \gamma_2''\gamma_3^{(3)}) + \gamma_2'(\gamma_3''\gamma_1^{(3)} - \gamma_1''\gamma_3^{(3)}) \\ +\gamma_3'(-\gamma_2''\gamma_1^{(3)} + \gamma_1''\gamma_2^{(3)}) \end{array}}{(-\gamma_1''^2 + \gamma_2''^2 + \gamma_3''^2)^{3/2}}.$$

It is obvious that $\varepsilon_{\mathtt{t}} = kc_1 = -1$. Also, if we calculate rc_2 and $-rc_1$, we can see that $\varepsilon_{\mathtt{n_2}} c_1' = rc_2$ and

$$\frac{c_2'}{\varepsilon_{\mathtt{n_1}}} - \frac{((r_m)^2)'}{2\varepsilon_{\mathtt{n_1}}\varepsilon_{\mathtt{n_2}} c_2} = -rc_1.$$

It means that the scalar Frenet equations (25) are satisfied by γ.

6. Discussion and Conclusion

In this study, the real quaternionic and semi-real quaternionic focal curves, consists of the centers of its osculating hypersphere, in the spaces $\mathbb{Q}$ and $\mathbb{Q}_v$ with index $v = \{1,2\}$ have been considered. As a set, the space $\mathbb{Q}$ coincides with the space $\mathbb{E}^4$. Similarly, $\mathbb{Q}_v$ coincides with $\mathbb{E}_1^3$ and $\mathbb{E}_2^4$, where $v = 1$ and 2, respectively. By taking this fact account, we have investigated some characterizations of defined focal curves by using focal curvatures and "scalar Frenet equations" between the focal curvatures. The notions: such as vertex, flattenings, a symmetry point, have been defined. Moreover, the relation between the Frenet apparatus of a quaternionic curve and the Frenet apparatus of its quaternionic focal curve has been presented.

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