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Investigation of Exact Solutions of Perturbed Nonlinear Schrödinger's Equation by $\exp(-\Phi(\xi))$-Expansion Method

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Abstract: This paper presents an analytic study on optical solitons of a perturbed nonlinear Schrödinger's equation (NLSE). An integration tool that is the $\exp(-\Phi(\xi))$-expansion approach is used to find exact solutions. As a consequence, hyperbolic, trigonometric and rational function solutions are extracted by this approach.

$\exp(-\Phi(\xi))$-Açılım Metodu ile Pertürbe Edilmiş Lineer Olmayan Schrödinger Denkleminin Tam Çözümlerinin Araştırılması

Anahtar Kelimeler
Solitonlar,
Pertürbe edilmiş lineer olmayan Schrödinger denklemi,
$\exp(-\Phi(\xi))$-açılım metodu

Özet: Bu makale, pertürbe edilmiş lineer olmayan Schrödinger denkleminin optik solitonları üzerine analitik bir çalışma sunar. Tam çözümler elde etmek için $\exp(-\Phi(\xi))$-açılım metodu kullanılır. Sonuç olarak bu metot ile hiperbolik, trigonometrik ve rasyonel fonksiyon çözümler elde edilir.

1. Introduction

The NLSE (nonlinear Schrödinger equation) has a central importance in many natural sciences as well as engineering with numerous interpretations and applications concerning eg. nonlinear optics, protein chemistry, plasma physics and fluid dynamics. This paper will consider the perturbed NLSE which governs the dynamics of solitons in negative-index material with non-Kerr nonlinearity and third-order dispersion, and the dimensionless form of the equation is given by [1, 2]

$$\begin{aligned} &iu_t + au_{xx} + bu_{xt} + cF(|u|^2)u = \\ &- i\lambda u_x - is\left(|u|^2u\right)_x - i\mu\left(|u|^2\right)_x u - i\theta|u|^2u_x \\ &- i\gamma u_{xxx} - \theta_1\left(|u|^2u\right)_{xx} - \theta_2|u|^2u_{xx} - \theta_3 u^2 u^*_{xx}, \end{aligned} \quad (1)$$

where $u(x,t)$ is the complex field amplitude. a, b, and c are the coefficients of group velocity dispersion, spatial-temporal dispersion and non-Kerr nonlinearity, and λ, s, μ, θ and γ account for the inter-modal dispersion, self-steepening, Raman effect, nonlinear dispersion and third-order dispersion, respectively. The last three terms appear in the context of negative-index material [1–6].

Recently, various analytical and numerical methods have been introduced to obtain solutions of nonlinear evolution equations. Some of these methods are F-function method [7], exp-function method [8], Hirota's bilinear method [9], homotopy perturbation method [10], variational iteration method [11, 12], Adomian Pade approximation [13], Lie group method [14], homogeneous balance method [15], inverse scattering transform method [16], Jacobi elliptic expansion method [17], sine-cosine method [18], (G'/G)-expansion method [19] and improved $\tan(\Phi(\xi)/2)$-expansion method [20].

The main aim of this study is to extract exact solitons to Eq. (1) using the $\exp(-\Phi(\xi))$-expansion approach [21–23]. Four kinds of nonlinearity are considered for Eq. (1). They are Kerr law, power law, parabolic law and dual-power law.

2. Analytical Solutions

In order to solve Eq. (1), we use the wave transformation as

$$u(x,t) = P(\xi)e^{i\Phi(x,t)}, \quad (2)$$

where $P(\xi)$ represents the shape of the pulse and

$$\xi = x - vt, \quad (3)$$

$$\Phi(x,t) = -\kappa x + \omega t + \zeta. \quad (4)$$

In Eq. (2), the function $\Phi(x,t)$ gives the phase component of the soliton. Then, in Eq. (4), κ, ω and ζ respectively represent the frequency, wave number and phase constant. Finally in Eq. (3), v shows the velocity of the soliton. Inserting (2) into (1) and then decomposing into real and

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imaginary parts yield a pair of relations. Real part gives

$$\begin{aligned}&(a-bv+3\kappa\gamma)P''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa+\gamma\kappa^3\right)P\\&+cF(P^2)P+\left(s\kappa+\theta\kappa-\theta_1\kappa^2-\theta_2\kappa^2-\theta_3\kappa^2\right)P^3\\&+6\theta_1P(P')^2+(3\theta_1+\theta_2+\theta_3)P^2P''=0,\end{aligned}\tag{5}$$

while imaginary part leads to

$$\begin{aligned}&(-v-2a\kappa+b\omega+b\kappa v+\lambda-3\gamma\kappa^2)P'\\&+(3s+2\mu+\theta-2\kappa(3\theta_1+\theta_2-\theta_3))P^2P'+\gamma P'''=0.\end{aligned}\tag{6}$$

The imaginary part equation implies the relations given by

$$\gamma=0,\tag{7}$$

$$v=-\frac{2a\kappa-b\omega-\lambda}{1-b\kappa},\tag{8}$$

$$3s+2\mu+\theta-2\kappa(3\theta_1+\theta_2-\theta_3)=0.\tag{9}$$

Therefore, Eq. (5) by virtue of Eq. (7) reduces to

$$\begin{aligned}&(a-bv)P''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)P+cF(P^2)P\\&+\left(s\kappa+\theta\kappa-\kappa^2\theta_1-\kappa^2\theta_2-\kappa^2\theta_3\right)P^3\\&+(3\theta_1+\theta_2+\theta_3)P^2P''+6\theta_1P\left(P'\right)^2=0.\end{aligned}\tag{10}$$

To obtain an analytic solution, one applies the transformations $\theta_1=0$, $\theta_2=-\theta_3$ and $s=-\theta$ in Eq. (10) to find

$$(a-bv)P''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)P+cF(P^2)P=0,\tag{11}$$

where

$$\theta-\mu-2\theta_3\kappa=0.\tag{12}$$

2.1. Kerr law

For Kerr law nonlinearity

$$F(q)=q,\tag{13}$$

Eq. (1) reduces to

$$\begin{aligned}&iu_t+au_{xx}+b|u|^2u+bu_{xt}+c|u|^2u=\\&-i\lambda u_x-is\left(|u|^2u\right)_x-i\mu\left(|u|^2\right)_xu-i\theta|u|^2u_x\\&-i\gamma u_{xxx}-\theta_1\left(|u|^2u\right)_{xx}-\theta_2|u|^2u_{xx}-\theta_3u^2u^*_{xx},\end{aligned}\tag{14}$$

and Eq. (11) simplifies to

$$(a-bv)P''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)P+cP^3=0.\tag{15}$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be used, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (14). According to the homogeneous balance method, Eq. (15) has the solution as

$$P(\xi)=A_0+A_1\exp[-\Phi(\xi)],\tag{16}$$

where A_0, A_1 will be determined later, such that A_1 is nonzero constant and $\Phi=\Phi(\xi)$ satisfies the auxiliary ODE is given by

$$\Phi'(\xi)=\exp[-\Phi(\xi)]+\rho\exp[\Phi(\xi)]+\tau.\tag{17}$$

Eq. (17) has solutions in the following forms:

When $\rho\neq0$ and $\tau^2-4\rho>0$,

$$\Phi(\xi)=\ln\left(-\frac{\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}(\xi+C)\right)+\tau}{2\rho}\right).\tag{18}$$

When $\rho\neq0$ and $\tau^2-4\rho<0$,

$$\Phi(\xi)=\ln\left(\frac{\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}(\xi+C)\right)-\tau}{2\rho}\right).\tag{19}$$

When $\rho=0$, $\tau\neq0$ and $\tau^2-4\rho>0$,

$$\Phi(\xi)=-\ln\left(\frac{\tau}{\exp(\tau(\xi+C))-1}\right).\tag{20}$$

When $\rho\neq0$, $\tau\neq0$ and $\tau^2-4\rho=0$,

$$\Phi(\xi)=\ln\left(-\frac{2(\tau(\xi+C)+2)}{\tau^2(\xi+C)}\right).\tag{21}$$

When $\rho=0$, $\tau=0$ and $\tau^2-4\rho=0$,

$$\Phi(\xi)=\ln(\xi+C).\tag{22}$$

Here C is the integration constant. Inserting (16) along with (17) into Eq. (15), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one obtains the following results:

$$\begin{aligned}A_0&=\pm\frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}},\\A_1&=\pm\frac{2\sqrt{\tau^2(bv-a)}}{\sqrt{2c}\tau},\\\omega&=\frac{-2\kappa\lambda+bv\left(4\rho-\tau^2\right)+a\left(2\kappa^2-4\rho+\tau^2\right)}{2b\kappa-2},\end{aligned}\tag{23}$$

where ρ and τ are arbitrary constants. Substituting the solution set (23) into Eq. (16), the solution formula of (15) can be written in the form

$$P(\xi)=\pm\frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}}\left\{1+\frac{2}{\tau}\exp[-\Phi(\xi)]\right\}.\tag{24}$$

Substituting the general solutions of (17) into Eq. (50) and inserting the result into the hypothesis (2), one recovers the hyperbolic, trigonometric and plane wave solutions to Eq. (14) as below :

When $\rho\neq0$ and $\tau^2-4\rho>0$, hyperbolic function solutions are:

$$\begin{aligned}u(x,t)&=\pm\frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}}\\&\times\left\{1-\frac{4\rho}{\tau\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}(\xi+C)\right)+\tau^2}\right\}\\&\times e^{i\left\{-\kappa x+\left(\frac{-2\kappa\lambda+bv\left(4\rho-\tau^2\right)+a\left(2\kappa^2-4\rho+\tau^2\right)}{2b\kappa-2}\right)t+\theta\right\}}.\end{aligned}\tag{25}$$

When $\rho \neq 0$ and $\tau^2 - 4\rho < 0$, trigonometric function solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}} \times \left\{ 1 + \frac{4\rho}{\tau\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}(\xi+C)\right) - \tau^2} \right\} \times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho-\tau^2\right) + a\left(2\kappa^2-4\rho+\tau^2\right)}{2b\kappa-2}\right)t + \theta\right\}}. \quad (26)$$

When $\rho = 0$, $\tau \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}} \left\{ 1 + \frac{2}{\exp(\tau(\xi+C)) - 1} \right\} \times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho-\tau^2\right) + a\left(2\kappa^2-4\rho+\tau^2\right)}{2b\kappa-2}\right)t + \theta\right\}}. \quad (27)$$

When $\rho \neq 0$, $\tau \neq 0$ and $\tau^2 - 4\rho = 0$, rational function solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}} \left\{ 1 - \frac{\tau(\xi+C)}{\tau(\xi+C)+2} \right\} \times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho-\tau^2\right) + a\left(2\kappa^2-4\rho+\tau^2\right)}{2b\kappa-2}\right)t + \theta\right\}}. \quad (28)$$

When $\rho = 0$, $\tau = 0$ and $\tau^2 - 4\rho = 0$, plane wave solutions are:

$$u(x,t) = \pm \frac{\sqrt{\tau^2(bv-a)}}{\sqrt{2c}} \left\{ 1 + \frac{2}{\tau(\xi+C)} \right\} \times e^{i\left\{-\kappa x + \left(\frac{-2\kappa\lambda + bv\left(4\rho-\tau^2\right) + a\left(2\kappa^2-4\rho+\tau^2\right)}{2b\kappa-2}\right)t + \theta\right\}}. \quad (29)$$

2.2. Power law

In this case,

$$F(q) = q^n, \quad (30)$$

for power law nonlinear medium. Based on this nonlinearity, (1) reduces to

$$\begin{aligned} &iu_t + au_{xx} + b|u|^2u + bu_{xt} + c|u|^{2n}u = \\ &-i\lambda u_x - is\left(|u|^2u\right)_x - i\mu\left(|u|^2\right)_x u - i\theta|u|^2u_x \\ &-i\gamma u_{xxx} - \theta_1\left(|u|^2u\right)_{xx} - \theta_2|u|^2u_{xx} - \theta_3 u^2 u^*_{xx}, \end{aligned} \quad (31)$$

so that (11) simplifies to

$$(a-bv)P'' - \left((1-b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)P + cP^{2n+1} = 0. \quad (32)$$

Balancing P'' with P^{2n+1} in Eq. (32) gives $N = \frac{1}{n}$. To obtain an analytic solution, one employs the transformation

$$P = U^{\frac{1}{2n}}, \quad (33)$$

in Eq. (32) to find

$$\begin{aligned} &(a-bv)\left((1-2n)(U')^2 + 2nUU''\right) \\ &-4n^2\left((1-b\kappa)\omega + a\kappa^2 - \lambda\kappa\right)U^2 + 4cn^2U^3 = 0. \end{aligned} \quad (34)$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be utilized, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (31). According to the homogeneous balance method, Eq. (34) has the solution as

$$U(\xi) = A_0 + A_1\exp[-\Phi(\xi)] + A_2(\exp[-\Phi(\xi)])^2, \quad (35)$$

where A_i $(i = 0,1,2)$ will be determined later, such that A_2 is non-zero constant and $\Phi = \Phi(\xi)$ satisfies Eq. (17). Inserting (35) along with (17) into Eq. (34), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one obtains the following results:

$$\begin{aligned} A_0 &= -\frac{\rho(1+n)(a-bv)}{cn^2}, \\ A_1 &= -\frac{\tau(1+n)(a-bv)}{cn^2}, \\ A_2 &= -\frac{(1+n)(a-bv)}{cn^2}, \\ \omega &= \frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2 + 4\rho - \tau^2\right) + bv\left(\tau^2 - 4\rho\right)}{4n^2(b\kappa-1)}, \end{aligned} \quad (36)$$

where ρ and τ are arbitrary constants. Substituting the solution set (36) into Eq. (35), the solution formula of Eq. (34) can be written as

$$U(\xi) = -\frac{(1+n)(a-bv)}{cn^2} \times \left\{\rho + \tau\exp[-\Phi(\xi)] + (\exp[-\Phi(\xi)])^2\right\}. \quad (37)$$

Consequently, one recovers the hyperbolic, trigonometric and plane wave solutions to Eq. (31) in the forms:

When $\rho \neq 0$ and $\tau^2 - 4\rho > 0$, hyperbolic function solutions are:

$$u(x,t) = \left\{ -\frac{(1+n)(a-bv)}{cn^2} \times \left(\rho - \frac{2\rho\tau}{\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}(\xi+C)\right) + \tau} + \frac{4\rho^2}{\left[\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}(\xi+C)\right) + \tau\right]^2} \right) \right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x + \left(\frac{-4n^2\kappa\lambda + a\left(4n^2\kappa^2+4\rho-\tau^2\right) + bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t + \theta\right\}}. \quad (38)$$

When $\rho \neq 0$ and $\tau^2 - 4\rho < 0$, trigonometric function solutions are:

$$u(x,t) = \left\{ -\frac{(1+n)(a-bv)}{cn^2} \times \left(\rho + \frac{2\rho\tau}{\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}(\xi+C)\right) - \tau} \right. \right.$$

$$+\frac{4\rho^2}{\left[\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}(\xi+C)\right)-\tau\right]^2}\Bigg)\Bigg\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (39)$$

When $\rho=0$, $\tau\neq 0$ and $\tau^2-4\rho>0$, hyperbolic function solutions are:

$$u(x,t)=\Bigg\{-\frac{(1+n)(a-bv)}{cn^2} \times\left(\rho+\frac{\tau^2}{\exp(\tau(\xi+C))-1}+\frac{\tau^2}{[\exp(\tau(\xi+C))-1]^2}\right)\Bigg\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (40)$$

When $\rho\neq 0$, $\tau\neq 0$ and $\tau^2-4\rho=0$, rational function solutions are:

$$u(x,t)=\Bigg\{-\frac{(1+n)(a-bv)}{cn^2} \times\left(\rho-\frac{\tau^3(\xi+C)}{2(\tau(\xi+C)+2)}+\frac{\tau^4(\xi+C)^2}{[2(\tau(\xi+C)+2)]^2}\right)\Bigg\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (41)$$

When $\rho=0$, $\tau=0$ and $\tau^2-4\rho=0$, plane wave solutions are:

$$u(x,t)=\left\{-\frac{(1+n)(a-bv)}{cn^2}\left(\rho+\frac{\tau}{\xi+C}+\frac{1}{(\xi+C)^2}\right)\right\}^{\frac{1}{2n}} \times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (42)$$

2.3. Parabolic law

In this case,

$$F(q)=q+\eta q^2, \quad (43)$$

where η is a real-valued constant. Based on this nonlinearity, (1) reduces to

$$iu_t+au_{xx}+b|u|^2u+bu_{xt}+c(|u|^2+\eta|u|^4)u = -i\lambda u_x-is\left(|u|^2u\right)_x-i\mu\left(|u|^2\right)_x u-i\theta|u|^2u_x -i\gamma u_{xxx}-\theta_1\left(|u|^2u\right)_{xx}-\theta_2|u|^2u_{xx}-\theta_3u^2u^*_{xx}, \quad (44)$$

so that equation (11) simplifies to

$$(a-bv)P''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)P+cP^3+c\eta P^5=0. \quad (45)$$

Balancing P'' with P^5 gives $N=\frac{1}{2}$. To obtain an analytic solution, one employs the transformation

$$P=U^{\frac{1}{2}}, \quad (46)$$

in Eq. (45) to find

$$(a-bv)\left(2UU''-(U')^2\right)-4\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)U^2 +4cU^3+4c\eta U^4=0. \quad (47)$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be implemented, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (44). According to the homogeneous balance method, Eq. (47) has the solution as

$$U(\xi)=A_0+A_1\exp[-\Phi(\xi)], \quad (48)$$

where A_0, A_1 will be determined later, such that A_1 is nonzero constant and $\Phi=\Phi(\xi)$ satisfies Eq. (17). Inserting (48) along with (17) into Eq. (47), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one recovers the following results:

$$\begin{aligned}
A_0&=-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2(\tau^2-4\rho)}\right)}{8\eta^2(4\rho-\tau^2)},\\
A_1&=\frac{3\tau}{4\sqrt{\eta^2\tau^2(\tau^2-4\rho)}},\\
\omega&=\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4},\\
c&=\frac{4}{3}\eta(a-bv)\left(4\rho-\tau^2\right),
\end{aligned} \quad (49)$$

where ρ and τ are arbitrary constants. Substituting the solution set (49) into Eq. (48), the solution formula of Eq. (47) can be written as

$$U(\xi)=-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2(\tau^2-4\rho)}\right)}{8\eta^2(4\rho-\tau^2)} +\frac{3\tau}{4\sqrt{\eta^2\tau^2(\tau^2-4\rho)}}\exp[-\Phi(\xi)]. \quad (50)$$

Consequently, one obtains the hyperbolic, trigonometric and plane wave solutions to Eq. (44) in the following forms:

When $\rho\neq 0$ and $\tau^2-4\rho>0$, hyperbolic function solutions are:

$$u(x,t)=\Bigg\{-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2(\tau^2-4\rho)}\right)}{8\eta^2(4\rho-\tau^2)} -\frac{3\tau}{4\sqrt{\eta^2\tau^2(\tau^2-4\rho)}} \times\left(\frac{2\rho}{\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}(\xi+C)\right)+\tau}\right)\Bigg\}^{\frac{1}{2}} \times e^{i\left\{-\kappa x+\left(\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4}\right)t+\theta\right\}}. \quad (51)$$

When $\rho\neq 0$ and $\tau^2-4\rho<0$, trigonometric function solutions are:

$$u(x,t)=\left\{-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}\right)}{8\eta^2\left(4\rho-\tau^2\right)}+\frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}}\times\left(\frac{2\rho}{\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}\left(\xi+C\right)\right)-\tau}\right)\right\}^{\frac{1}{2}}\times e^{i\left\{-\kappa x+\left(\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4}\right)t+\theta\right\}}. \tag{52}$$

When $\rho=0$, $\tau\neq 0$ and $\tau^2-4\rho>0$, hyperbolic function solutions are:

$$u(x,t)=\left\{-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}\right)}{8\eta^2\left(4\rho-\tau^2\right)}+\frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}}\left(\frac{\tau}{\exp\left(\tau\left(\xi+C\right)\right)-1}\right)\right\}^{\frac{1}{2}}\times e^{i\left\{-\kappa x+\left(\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4}\right)t+\theta\right\}}. \tag{53}$$

When $\rho\neq 0$, $\tau\neq 0$ and $\tau^2-4\rho=0$, rational function solutions are:

$$u(x,t)=\left\{-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}\right)}{8\eta^2\left(4\rho-\tau^2\right)}-\frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}}\left(\frac{\tau^2\left(\xi+C\right)}{2\left(\tau\left(\xi+C\right)+2\right)}\right)\right\}^{\frac{1}{2}}\times e^{i\left\{-\kappa x+\left(\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4}\right)t+\theta\right\}}. \tag{54}$$

When $\rho=0$, $\tau=0$ and $\tau^2-4\rho=0$, plane wave solutions are:

$$u(x,t)=\left\{-\frac{3\left(4\eta\rho-\eta\tau^2+\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}\right)}{8\eta^2\left(4\rho-\tau^2\right)}+\frac{3\tau}{4\sqrt{\eta^2\tau^2\left(\tau^2-4\rho\right)}}\left(\frac{1}{\xi+C}\right)\right\}^{\frac{1}{2}}\times e^{i\left\{-\kappa x+\left(\frac{-4\kappa\lambda-a\tau^2+4a\left(\kappa^2+\rho\right)+bv\left(\tau^2-4\rho\right)}{4b\kappa-4}\right)t+\theta\right\}}. \tag{55}$$

2.4. Dual-Power law

In this case,

$$F(q)=q^n+\eta q^{2n}, \tag{56}$$

where η is a real-valued constant. Based on this nonlinearity, (1) reduces to

$$\begin{aligned}&iu_t+au_{xx}+b|u|^2u+bu_{xt}+c(|u|^{2n}+\eta|u|^{4n})u=\\&-i\lambda u_x-is\left(|u|^2u\right)_x-i\mu\left(|u|^2\right)_x u-i\theta|u|^2u_x\\&-i\gamma u_{xxx}-\theta_1\left(|u|^2u\right)_{xx}-\theta_2|u|^2u_{xx}-\theta_3u^2u^*_{xx},\end{aligned} \tag{57}$$

so that equation (11) simplifies to

$$\begin{aligned}&(a-bv)P''-\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)P\\&+cP^{2n+1}+c\eta P^{4n+1}=0.\end{aligned} \tag{58}$$

Balancing P'' with P^{4n+1} gives $N=\frac{1}{2n}$. To obtain an analytic solution, one employs the transformation

$$P=U^{\frac{1}{2n}}, \tag{59}$$

in Eq. (58) to find

$$\begin{aligned}&(a-bv)\left((1-2n)(U')^2+2nUU''\right)\\&-4n^2\left((1-b\kappa)\omega+a\kappa^2-\lambda\kappa\right)U^2\\&+4cn^2U^3+4cn^2\eta U^4=0.\end{aligned} \tag{60}$$

In this subsection, the $\exp(-\Phi(\xi))$-expansion method will be applied, to obtain hyperbolic, trigonometric and rational function solutions to Eq. (57). According to the homogeneous balance method, Eq. (60) has the solution as

$$U(\xi)=A_0+A_1\exp\left[-\Phi(\xi)\right], \tag{61}$$

where A_0, A_1 will be determined later, such that A_1 is nonzero constant and $\Phi=\Phi(\xi)$ satisfies Eq. (17). Inserting (61) along with (17) into Eq. (60), and equating the coefficients of $\exp(-\Phi(\xi))$ to zero, one obtains a system of algebraic equations. Solving it by Mathematica, one recovers the following results:

$$\begin{aligned}&A_0=\frac{\ell-cn^2(1+n)(a-bv)\left(4\rho-\tau^2\right)}{4c^2n^4},\quad A_1=\frac{\ell}{2c^2n^4\tau},\\&\omega=\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)},\\&\eta=\frac{cn^2(1+2n)}{(1+n)^2(a-bv)\left(4\rho-\tau^2\right)},\end{aligned} \tag{62}$$

where ρ, τ are arbitrary constants, and ℓ is given by

$$\ell=\sqrt{c^2n^4(1+n)^2(a-bv)^2\tau^2\left(\tau^2-4\rho\right)}. \tag{63}$$

Substituting the solution set (62) into Eq. (61), the solution formula of Eq. (60) can be written as

$$U(\xi)=\frac{\ell-cn^2(1+n)(a-bv)\left(4\rho-\tau^2\right)}{4c^2n^4}+\frac{\ell}{2c^2n^4\tau}\exp\left[-\Phi(\xi)\right]. \tag{64}$$

Consequently, one obtains the hyperbolic, trigonometric and plane wave solutions to Eq. (57) as follows:

When $\rho\neq 0$ and $\tau^2-4\rho>0$, hyperbolic function solution is:

$$u(x,t)=\left\{\frac{\ell-cn^2(1+n)(a-bv)\left(4\rho-\tau^2\right)}{4c^2n^4}\right.$$
$$\left.-\frac{\ell}{2c^2n^4\tau}\left(\frac{2\rho}{\sqrt{\tau^2-4\rho}\tanh\left(\frac{\sqrt{\tau^2-4\rho}}{2}\left(\xi+C\right)\right)+\tau}\right)\right\}^{\frac{1}{2n}}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (65)$$

When $\rho\neq 0$ and $\tau^2-4\rho<0$, trigonometric function solutions are:

$$u(x,t)=\left\{\frac{\ell-cn^2(1+n)(a-bv)\left(4\rho-\tau^2\right)}{4c^2n^4}\right.$$
$$\left.+\frac{\ell}{2c^2n^4\tau}\left(\frac{2\rho}{\sqrt{4\rho-\tau^2}\tan\left(\frac{\sqrt{4\rho-\tau^2}}{2}\left(\xi+C\right)\right)-\tau}\right)\right\}^{\frac{1}{2n}}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (66)$$

When $\rho=0$, $\tau\neq 0$ and $\tau^2-4\rho>0$, hyperbolic function solutions are:

$$u(x,t)=\left\{\frac{\ell-cn^2(1+n)(a-bv)\left(4\rho-\tau^2\right)}{4c^2n^4}\right.$$
$$\left.+\frac{\ell}{2c^2n^4}\left(\frac{1}{\exp\left(\tau\left(\xi+C\right)\right)-1}\right)\right\}^{\frac{1}{2n}}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (67)$$

When $\rho\neq 0$, $\tau\neq 0$ and $\tau^2-4\rho=0$, rational function solutions are:

$$u(x,t)=\left\{\frac{\ell-cn^2(1+n)(a-bv)\left(4\rho-\tau^2\right)}{4c^2n^4}\right.$$
$$\left.-\frac{\ell}{2c^2n^4}\left(\frac{\tau\left(\xi+C\right)}{2\left(\tau\left(\xi+C\right)+2\right)}\right)\right\}^{\frac{1}{2n}}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (68)$$

When $\rho=0$, $\tau=0$ and $\tau^2-4\rho=0$, plane wave solutions are:

$$u(x,t)=\left\{\frac{\ell-cn^2(1+n)(a-bv)\left(4\rho-\tau^2\right)}{4c^2n^4}\right.$$
$$\left.+\frac{\ell}{2c^2n^4\tau}\left(\frac{1}{\xi+C}\right)\right\}^{\frac{1}{2n}}$$
$$\times e^{i\left\{-\kappa x+\left(\frac{-4n^2\kappa\lambda+a\left(4n^2\kappa^2+4\rho-\tau^2\right)+bv\left(\tau^2-4\rho\right)}{4n^2(b\kappa-1)}\right)t+\theta\right\}}. \quad (69)$$

3. Conclusion

This paper investigated analytically the nonlinear mathematical physical model (1) by using an integration tool called the $\exp\left(-\Phi\left(\xi\right)\right)$-expansion approach. Four kinds of nonlinearities including Kerr law, power law, parabolic law and dual-power law are taken into account As a consequence, hyperbolic, trigonometric and rational function solutions are derived.

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