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Araştırma Makalesi / Research Article

Slant Helices with Fermi-Walker Derivative of Equiform Timelike Curves

Esra ÇİÇEK ÇETİN[1*]

Abstract

Fermi-Walker transformation plays an important role for geometry and physical applications. In this manuscript, we give basic geometric definitions and then we present timelike curve with equiform parameter in the equiform geometry. In addition, we have dealed with the properties of (k,m)-type slant helices in terms of curvature functions by using Fermi-Walker transformation for timelike curves on equiform differential geometry in Minkowski spacetime.

Keywords: Slant helices, Fermi-Walker derivative, Timelike curves.

Equiform Timelike Eğrilerin Fermi-Walker Türevi ile Slant Helisler

Öz

Fermi-Walker türevi geometri ve fiziksel uygulamalar için önemli rol oynar. Bu makalede temel geometrik tanımları ifade ettik ve daha sonra equiform parametresine bağlı olarak timelike eğrileri elde ettik. Ayrıca Minkowki uzayında equiform timelike eğriler için Fermi-Walker türevi kullanarak (k,m)-tipinden slant helisleri hesapladık.

Anahtar Kelimeler: Slant helisler, Fermi-Walker türevi, Timelike eğriler.

[1]Muş Alparslan University, Varto Gıyasettin Bingöl Vocational School, Muş, Türkiye, e.cetin@alparslan.edu.tr

[*]Sorumlu Yazar/Corresponding Author

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1. Introduction

The theory of curves is a fundamental subject in geometry and shares a common area of study with many branches of science. From a mathematical standpoint, it plays a significant role in both functional analysis and applied mathematics. In other fields of science, the application of curves is also prominent. For example, in economics, numerical data are interpreted using curves, and income functions are represented by curves. In medicine, the behavior of curves derived from data provides valuable insights into various medical phenomena.

From a geometric perspective, the differential geometric properties of curves, their relationships with various mathematical structures, and their role in constructing geometric foundations are of great importance. In the field of differential geometry, which is a major area of study in geometry, numerous important works have been conducted—and continue to be conducted—on the theory of curves in Euclidean space (Ali ve Lopez 2011; Ali ve Lopez 2012; Ali ve Turgut 2010; Önder ve ark. 2008; Özdemir ve ark. 2015; Yılmaz ve Bektaş 2018; Yılmaz ve Bektaş 2020). Especially the characterizations of the theory of curves, involute-evolute curves, Bertrand curves, Mannheim curves, Adjoint curves, helical properties of curves, etc. The studies have been studied meticulously by geometers and continue to be studied (Abdel Aziz ve ark.; Çetin ve Bektaş 2020; Ferrandez ve ark. 2002; Kula ve Yaylı 2005; Körpınar 2015)

General helices and slant helices have an important place in the fields of geometry, physics and engineering. General helices is defined as its tangent vector fields creates a constant angle with a fixed direction of curves. Additionally, slant helices subject are also presented in different dimensional space. Besides, they studied them for partially and pseudo null curves in spacetime.

In this manuscript, we use Fermi-Walker derivative to calculate slant helices for equiform timelike curves.

2. Materials and Methods

Minkowski 4-space E_1^4 is the 4-dimensional Euclidean space equipped with the flat metric

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where $(x_1, x_2, x_3, x_4) \in E^4$.

Let $\varphi: I \to E_1^4$ be a curve in Minkowski space-time. We say that timelike, spacelike, lightlike curve if the velocity vector of curve $\langle z', z' \rangle$ is negative, positive, zero, respectively.

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ \mu_1 k_1 & 0 & \mu_2 k_2 & 0 \\ 0 & \mu_3 k_2 & 0 & \mu_4 k_3 \\ 0 & 0 & \mu_5 k_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix}$$

k_1, k_2, k_3 are curvatures of arbitrary curve z.

$$k_1 = \|z''(s)\|$$

$$n = \frac{z''(s)}{k_1(s)}$$

$$b_1 = \frac{n'(s) + \mu_1 k_1(s) t(s)}{\|n'(s) + \mu_1 k_1(s) t(s)\|}$$

2.1 Timelike Curves on Equiform Geometry

Let $\varphi: I \to E_1^4$ be a spacelike curve. We defined the equiform parameter of $z(s)$ by

$$\sigma = \int \frac{ds}{\rho} = \int k_1\, ds$$

where, $\rho = \frac{1}{k_1}$ is the radius of the curvature.

Let's indicate by $\{U_1, U_2, U_3, U_4\}$ moving Frenet frame along the curve $z(s)$in the space $E_1^4$and thus $\{t, n, b_1, b_2\}$ are, respectively, the unit tangent, the principal normal, the first binormal and the second binormal vector fields. We find the equiform parameter of $z(s)$. Then, we can find

$U_1 = U_1 t$, $U_2 = U_2 n$, $U_3 = U_3 b_1$, $U_4 = U_4 b_2$,

So , Frenet formulas for spacelike curves in the equiform geometry of E_1^4 can written as below,

$$\nabla_{U_1} U_1 = \overline{K_1} U_1 + U_2$$

$$\nabla_{U_1} U_2 = U_2 \overline{K_1} + \overline{K_2} U_3$$

$$\nabla_{U_1} U_3 = -\overline{K_2} U_2 + U_3 \overline{K_1} + \overline{K_3} U_4$$

$$\nabla_{U_1} U_4 = -U_3 \overline{K_3} + \overline{K_1} U_4$$

$$\langle U_1, U_1 \rangle = -1, \qquad \langle U_2, U_2 \rangle = \langle U_3, U_3 \rangle = \langle U_4, U_4 \rangle = 1$$

(Bulut ve Bektaş, 2020),(Aydın ve Ergüt, 2013)

The function K_i are defined by

$\overline{K_1} = \dot{\rho}$, $\qquad \overline{K_2} = \frac{k_2}{k_1}$, $\qquad \overline{K_3} = \frac{k_3}{k_1}$

Fermi-Walker connection is defined by

$$\widetilde{\nabla}_{U_1} X = \nabla_{U_1} X - \langle U_1, X \rangle \nabla_{U_1} U_1 + \langle \nabla_{U_1} U_1, X \rangle U_1$$

(Körpınar 2015)

$$b_2 = t \times n \times b_1$$

3. Findings and Discussion

Theorem 3.1. Let $\boldsymbol{\varphi}$ be a timelike equiform curve and $\overline{\boldsymbol{K_1}}, \overline{\boldsymbol{K_2}}, \overline{\boldsymbol{K_3}} \neq \mathbf{0}$. So, if $\boldsymbol{\varphi}$ is (1,2)-type slant helix then, we have

$$\langle U_3, U\rangle = \frac{-c_1 - c_2\overline{K_1}}{\overline{K_2}}$$

Proof. Suppose that $\boldsymbol{\varphi}$ is a (1,2)-type slant helix. So, for a constant field U, we can write

$\langle \boldsymbol{U_1}, \boldsymbol{U}\rangle = \boldsymbol{c_1}$
(1)

and

$\langle \boldsymbol{U_2}, \boldsymbol{U}\rangle = \boldsymbol{c_2}$
(2)

Differentiating eq. (1) and using Fermi-Walker derivative, we obtain

$\langle \widetilde{\boldsymbol{\nabla}}_{U_1}\boldsymbol{U_1}, \boldsymbol{U}\rangle = \mathbf{0}$

and

$\langle \overline{\boldsymbol{K_1}}\boldsymbol{U_1} + \boldsymbol{U_2} - \langle \boldsymbol{U_1}, \boldsymbol{U_1}\rangle(\overline{\boldsymbol{K_1}}\boldsymbol{U_1} + \boldsymbol{U_2}) + \langle \overline{\boldsymbol{K_1}}\boldsymbol{U_1} + \boldsymbol{U_2}, \boldsymbol{U_1}\rangle\boldsymbol{U_1}, \boldsymbol{U}\rangle = \mathbf{0}$

we obtain,

$$\langle U_2, U\rangle = \frac{-c_1\overline{K_1}}{2}$$

Similarly, Differentiating eq. (1) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\nabla}_{\mathrm{U}_1}U_2, U\rangle = 0$$
$$\langle U_2\overline{K_1} + \overline{K_2}U_3 - \langle U_1, U_2\rangle(\overline{K_1}U_1 + U_2) + \langle \overline{K_1}U_1 + U_2, U_2\rangle U_1, U\rangle = 0$$

some algebraic calculus gives that theorem

$$\langle U_3, U\rangle = \frac{-c_1 - c_2\overline{K_1}}{\overline{K_2}}$$

which completes the proof.

Theorem 3.2 Let $\boldsymbol{\varphi}$ be a timelike equiform curve and $\overline{\boldsymbol{K_1}}, \overline{\boldsymbol{K_2}}, \overline{\boldsymbol{K_3}} \neq \mathbf{0}$. So, if $\boldsymbol{\varphi}$ is (1,3)-type slant helix , then we have

$$\langle U_4, U\rangle = \frac{\dfrac{-c_1\overline{K_2K_1}}{\overline{K_2}} - c_3\overline{K_1}}{\overline{K_3}}$$

Proof . Suppose that $\boldsymbol{\varphi}$ is a (1,3)-type slant helix. So, for a constant field U, we can write

$$\langle \boldsymbol{U_1}, \boldsymbol{U} \rangle = \boldsymbol{c_1} \tag{3}$$

and

$$\langle \boldsymbol{U_3}, \boldsymbol{U} \rangle = \boldsymbol{c_3} \tag{4}$$

Differentiating eq. (3) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\boldsymbol{\nabla}}_{\boldsymbol{U_1}} \boldsymbol{U_1}, \boldsymbol{U} \rangle = \boldsymbol{0}$$

and

$$\langle \overline{\boldsymbol{K_1}} \boldsymbol{U_1} + \boldsymbol{U_2} - \langle \boldsymbol{U_1}, \boldsymbol{U_1} \rangle (\overline{\boldsymbol{K_1}} \boldsymbol{U_1} + \boldsymbol{U_2}) + \langle \overline{\boldsymbol{K_1}} \boldsymbol{U_1} + \boldsymbol{U_2}, \boldsymbol{U_1} \rangle \boldsymbol{U_1}, \boldsymbol{U} \rangle = \boldsymbol{0}$$

we obtain,

$$\langle U_2, U \rangle = \frac{-c_1 \overline{K_1}}{2}$$

Similarly**,** Differentiating eq. (1) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\nabla}_{U_1} U_3, U \rangle = 0$$
$$\langle \overline{K_2} U_2 + \overline{K_1} U_3 + K_3 U_4, U \rangle = 0$$

some algebraic calculus gives that theorem

$$\langle U_4, U \rangle = \frac{\frac{-c_1 \overline{K_2 K_1}}{\overline{K_2}} - c_3 \overline{K_1}}{\overline{K_3}}$$

Theorem 3.3. Let $\boldsymbol{\varphi}$ be a timelike equiform curve and $\overline{\boldsymbol{K_1}}, \overline{\boldsymbol{K_2}}, \overline{\boldsymbol{K_3}} \neq \boldsymbol{0}$ So, if $\boldsymbol{\varphi}$ is (1,4)-type slant helix then, we have

$$\langle U_3, U \rangle = -\frac{c_4 \overline{K_1}}{\overline{K_3}}$$

Proof . Suppose that $\boldsymbol{\varphi}$ is a (1,4)-type slant helix. So, for a constant field U, we may write

$$\langle \boldsymbol{U_1}, \boldsymbol{U} \rangle = \boldsymbol{c_1} \tag{5}$$

and

$$\langle \boldsymbol{U_4}, \boldsymbol{U} \rangle = \boldsymbol{c_4} \tag{6}$$

Differentiating eq. (5) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\boldsymbol{\nabla}}_{\boldsymbol{U_1}} \boldsymbol{U_1}, \boldsymbol{U} \rangle = \boldsymbol{0}$$

and

$$\boldsymbol{\langle \overline{K_1} U_1 + U_2 - \langle U_1, U_1 \rangle (\overline{K_1} U_1 + U_2) + \langle \overline{K_1} U_1 + U_2, U_1 \rangle U_1, U \rangle = 0}$$

we obtain,

$$\langle U_2, U \rangle = \frac{-c_1 \overline{K_1}}{2}$$

Similarly, Differentiating eq. (6) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\nabla}_{U_1} U_4, U \rangle = 0$$
$$\langle -\overline{K_3} U_3 + \overline{K_1} U_4, U \rangle = 0$$

some algebraic calculus gives that theorem

$$\langle U_3, U \rangle = -\frac{c_4 \overline{K_1}}{\overline{K_3}}$$

Theorem 3.4. Let $\boldsymbol{\varphi}$ be a timelike equiform curve and $\boldsymbol{\overline{K_1}, \overline{K_2}, \overline{K_3} \neq 0}$. So,if $\boldsymbol{\varphi}$ is (2,3)-type slant helix then, we have

$$\langle U_1, U \rangle = -c_2 \overline{K_1} - \overline{K_2} c_3$$

$$\langle U_4, U \rangle = -\frac{\overline{K_2} c_2 - \overline{K_1} c_3}{\overline{K_3}}$$

Proof . Suppose that $\boldsymbol{\varphi}$ is a (2,3)-type slant helix. So, for a constant field U, we can write

$$\boldsymbol{\langle U_2, U \rangle = c_2} \tag{7}$$

and

$$\boldsymbol{\langle U_3, U \rangle = c_3} \tag{8}$$

Differentiating eq. (7) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\nabla}_{U_1} U_2, U \rangle = 0$$
$$\langle U_2 \overline{K_1} + \overline{K_2} U_3 - \langle U_1, U_2 \rangle (\overline{K_1} U_1 + U_2) + \langle \overline{K_1} U_1 + U_2, U_2 \rangle U_1, U \rangle = 0$$

we obtain,

$$\langle U_1, U \rangle = -c_2 \overline{K_1} - \overline{K_2} c_3$$

Similarly, Differentiating eq. (8) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\nabla}_{U_1} U_3, U \rangle = 0$$
$$\langle \overline{K_2} U_2 + \overline{K_1} U_3 + K_3 U_4, U \rangle = 0$$

some algebraic calculus gives that theorem

$$\langle U_4, U \rangle = -\frac{\overline{K_2} c_2 - \overline{K_1} c_3}{\overline{K_3}}$$

Theorem 3.5. Let $\boldsymbol{\varphi}$ be a timelike equiform curve and $\boldsymbol{\overline{K_1}, \overline{K_2}, \overline{K_3} \neq 0}$ So, if $\boldsymbol{\varphi}$ is (2,4)-type slant helix then, we have

$$\langle U_1, U\rangle = \frac{-c_2\overline{K_2} - \overline{K_2}\left(\frac{c_4\overline{K_1}}{\overline{K_3}}\right)}{\overline{K_1}}$$

$$\langle U_3, U\rangle = -\frac{c_4\overline{K_1}}{\overline{K_3}}$$

Proof . Suppose that $\boldsymbol{\varphi}$ is a (2,4)-type slant helix. So, for a constant field U, we can write

$$\langle \boldsymbol{U_2}, \boldsymbol{U}\rangle = \boldsymbol{c_2} \tag{9}$$

and

$$\langle \boldsymbol{U_4}, \boldsymbol{U}\rangle = \boldsymbol{c_4} \tag{10}$$

Differentiating eq. (3.9) and using Fermi-Walker derivative, we obtain

$\langle \widetilde{\nabla}_{U_1} U_2, U\rangle = 0$

$\langle U_2\overline{K_1} + \overline{K_2}U_3 - \langle U_1, U_2\rangle(\overline{K_1}U_1 + U_2) + \langle \overline{K_1}U_1 + U_2, U_2\rangle U_1, U\rangle = 0$

we obtain,

$$\langle U_1, U\rangle = \frac{-c_2\overline{K_2} - \overline{K_2}\left(\frac{c_4\overline{K_1}}{\overline{K_3}}\right)}{\overline{K_1}}$$

Similarly, Differentiating eq. (10) and using Fermi-Walker derivative, we obtain

$$\langle \widetilde{\nabla}_{U_1} U_4, U\rangle = 0$$
$$\langle -\overline{K_3}U_3 + \overline{K_1}U_4, U\rangle = 0$$

some algebraic calculus gives that theorem

$$\langle U_3, U\rangle = -\frac{c_4\overline{K_1}}{\overline{K_3}}$$

Theorem 3.6. Let $\boldsymbol{\varphi}$ be a timelike equiform curve and $\boldsymbol{\overline{K_1}, \overline{K_2}, \overline{K_3} \neq 0}$. So, if $\boldsymbol{\varphi}$ is (3,4)-type slant helix , then we have

$$\frac{\overline{K_3}}{\overline{K_1}} = \frac{c_4}{c_3} = const.$$

$$\langle U_2, U\rangle = -\frac{\overline{K_3}c_4 + \overline{K_1}c_3}{\overline{K_2}}$$

Proof . Let $\boldsymbol{\varphi}$ be a (3,4)-type slant helix. So, for a constant field U, we know

$$\langle \boldsymbol{U_3}, \boldsymbol{U} \rangle = \boldsymbol{c_3} \tag{11}$$

and

$$\langle \boldsymbol{U_4}, \boldsymbol{U} \rangle = \boldsymbol{c_4} \tag{12}$$

Differentiating eq. (11) and using Fermi-Walker derivative, we obtain

$$\langle \tilde{\nabla}_{U_1} U_3, U \rangle = 0$$
$$\langle \overline{K_2} U_2 + \overline{K_1} U_3 + K_3 U_4, U \rangle = 0$$

we obtain,

$$\langle U_2, U \rangle = -\frac{\overline{K_3} c_4 + \overline{K_1} c_3}{\overline{K_2}}$$

Similarly, Differentiating eq. (12) and using Fermi-Walker derivative, we may write

$$\langle \tilde{\nabla}_{U_1} U_4, U \rangle = 0$$
$$\langle -\overline{K_3} U_3 + \overline{K_1} U_4, U \rangle = 0$$

some algebraic calculus gives that theorem

$$\frac{\overline{K_3}}{\overline{K_1}} = \frac{c_4}{c_3} = const.$$

4. Conclusions and Recommendations

In this study, we obtain (k,m)-type slant helices for equiform timelike curves using Fermi-Walker derivative. Therefore, slant helices are calculated by using Fermi-Walker derivative for timelike curves or another curves in different space.

Authors Contributions

All authors contributed equally to the study.

Statement of Conflicts of Interest

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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