Ikonion Journal of Mathematics 6(2) (2024) 30-44

Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

https://doi.org/10.54286/ikjm.1524180

ISSN: 2687-6531

A Qualitative Investigation of a System of Third-Order Difference Equations with Multiplicative Reciprocal Terms

Durhasan Turgut Tollu[1], **İbrahim Yalçınkaya**[2]

Keywords
Boundedness, Equilibrium point, System of difference equations

Abstract — In this paper, we study the system of third-order difference equations

$$x_{n+1} = a + \frac{a_1}{y_n} + \frac{a_2}{y_{n-1}} + \frac{a_3}{y_{n-2}}, \quad y_{n+1} = b + \frac{b_1}{x_n} + \frac{b_2}{x_{n-1}} + \frac{b_3}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where the parameters a, a_i, b, b_i $(i = 1,2,3)$ and the initial values x_{-j}, y_{-j} $(j = 0,1,2)$ are positive real numbers. We first prove a general convergence theorem. By applying this convergence theorem to the system, we show that positive equilibrium is a global attractor. We also study the local asymptotic stability of the equilibrium and show that it is globally asymptotically stable. Finally, we study the invariant set of solutions.

Subject Classification (2020): 39A10, 39A20, 39A30.

1. Introduction

Difference equations have been studied with great interest for the last thirty years. Determining the qualitative behavior of solutions, which is very important in applications, forms the basis of these studies. Difference equations have become a significant topic in mathematics and other disciplines because they can be discrete analogs of differential equations or mathematical models of phenomena. For some examples of discrete analogs of differential equations, see [1]. For some mathematical models, see [8]. In our opinion, this fact is the basis of the intense interest mentioned above. But whatever the reason, some classes of difference equations are being studied for the development of the theory of difference equations, even though they are not any mathematical models. The main idea, of course, is to discover new classes of difference equations and to develop new techniques and methods for determining the qualitative behavior of solutions of difference equations.

Since many mathematical models are nonlinear, nonlinear difference equations are studied quite frequently. Rational difference equations, as a subclass of nonlinear difference equations, are also frequently encountered in the literature. Below, we list some old and new studies that we encounter in the literature on the rational difference equations that we think are related to our research.

[1]dttollu@erbakan.edu.tr (Corresponding Author); [2]iyalcinkaya@erbakan.edu.tr
[1]Department of Mathematics and Computer Sciences, Faculty of Science, Necmettin Erbakan University, Konya, Türkiye
[2]Department of Mathematics and Computer Sciences, Faculty of Science, Necmettin Erbakan University, Konya, Türkiye
Article History: Received: 29.07.2024 - Accepted: 10.09.2024 - Published: 23.09.2024

In [6], DeVault et al. conducted a boundedness study on positive solutions of the second-order difference equation

$$x_{n+1} = \frac{A}{x_n^p} + \frac{B}{x_{n-1}^q}, \quad n \in \mathbb{N}_0,$$

where p, q, A, B, and the initial values are positive real numbers.

In [7], DeVault et al. showed that every positive solution of the third-order equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where $A \in (0, \infty)$, converges to a two-periodic solution of the equation.

In [28], Philos et al. studied the attractivity of the unique positive equilibrium of the higher-order equation

$$x_{n+1} = a + \sum_{k=1}^{m} \frac{b_k}{x_{n-k}}, \quad n \in \mathbb{N}_0,$$

where a and b_k $(k = 1, 2, \ldots, m)$ are nonnegative real parameters with $B = \sum_{k=1}^{m} b_k > 0$.

In [9], El-Metwally et al. established a global convergence result and applied it to the higher-order equation

$$x_{n+1} = \sum_{i=0}^{m} \frac{A_i}{x_{n-2i}}, \quad n \in \mathbb{N}_0,$$

where A_i $(i = 1, 2, \ldots, m)$ are nonnegative and the initial values are positive. They showed that every positive solution of the equation converges to a two-periodic solution.

In [10], El-Metwally et al. established a global convergence result and applied it to the higher-order equation

$$x_{n+1} = \sum_{i=0}^{k-1} \frac{A_i}{x_{n-i}}, \quad n \in \mathbb{N}_0,$$

where A_i $(i = 0, 1, \ldots, k-1)$ are nonnegative with $A = \sum_{i=1}^{k-1} A_i > 0$, and the initial values are positive. They showed that every positive solution of the equation converges to a $p-$periodic solution.

The study of two-dimensional systems, which are generally symmetric, of difference equations is a process initiated by Papaschinopoulos and Schinas in the late nineties. See, e.g. [22–26, 29]. Their work encouraged other authors, especially in the area of mathematics, to work on such systems. In the 2000s, studies on nonlinear rational difference equations and their systems gathered speed, and a rich literature emerged. Although this speed is not at the initial level, new studies are being published, especially on difference equation systems.

Fuzzy difference equations, which are a type of difference equation that is by definition particularly related to symmetric systems, also began to be studied during this process. For example, in [27], Papaschinopoulos and Papadopoulos considered the fuzzy difference equation

$$x_{n+1} = A + \frac{B}{x_n}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where A, B, x_0 are fuzzy numbers. Due to the nature of fuzzy difference equations, to study the solutions of

Eq.(1.1), they were interested in the system of classical difference equations

$$y_{n+1}=\alpha+\frac{\beta}{z_n},\quad z_{n+1}=\gamma+\frac{\delta}{y_n},\quad n\in\mathbb{N}_0,$$

which is a special case of the system stated in the abstract of this paper.

In [13], in line with [27], Hatir et al. investigated the behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1}=A+\frac{B}{x_{n-1}},\quad n\in\mathbb{N}_0, \tag{1.2}$$

where the parameters A, B, and the initial values x_{-1}, x_0 are fuzzy numbers. Naturally, to study the positive solutions of Eq.(1.1), they discussed the positive solutions of the system of classical difference equations

$$y_{n+1}=\alpha+\frac{\beta}{z_{n-1}},\quad z_{n+1}=\gamma+\frac{\delta}{y_{n-1}},\quad n\in\mathbb{N}_0,$$

which is another special case of the system in the abstract. For similar studies on fuzzy difference equations, see references [34, 35]. Apart from these, many systems of difference equations have been studied. For some examples, see [2, 3, 5, 11, 12, 14–18, 21, 30–33, 36, 37].

In this work, we define the system of difference equations

$$x_{n+1}=a+\frac{a_1}{y_n}+\frac{a_2}{y_{n-1}}+\frac{a_3}{y_{n-2}},\quad y_{n+1}=b+\frac{b_1}{x_n}+\frac{b_2}{x_{n-1}}+\frac{b_3}{x_{n-2}},\quad n\in\mathbb{N}_0, \tag{1.3}$$

where the parameters a, a_i, b, b_i $(i=1,2,3)$ and the initial values x_{-j}, y_{-j} $(j=0,1,2)$ are positive real numbers. We investigate the qualitative behavior of positive solutions of system (1.3). More specifically, we establish a global convergence result and apply it to (1.3) to study the global stability of the positive equilibrium.

For the methods followed in our study, the references [4, 19, 20] can be consulted.

2. Main Results

In this section, the main results of the paper are given and proven. This section is divided into two subsections.

2.1. A result of convergence

The following theorem states a general convergence result and enables us to prove that the unique positive equilibrium of (1.3) is the global attractor.

Theorem 2.1. Let $[\alpha,\beta]$ and $[\gamma,\delta]$ be intervals of positive real numbers and assume that $h_1:[\gamma,\delta]^{k+1}\to[\alpha,\beta]$ and $h_2:[\alpha,\beta]^{k+1}\to[\gamma,\delta]$ are continuous functions satisfying the following properties:

(a) Both $h_1(y_1,y_2,\dots,y_{k+1})$ and $h_2(x_1,x_2,\dots,x_{k+1})$ are decreasing in all of the arguments.

(b) If $(m_1,M_1,m_2,M_2)\in[\alpha,\beta]^2\times[\gamma,\delta]^2$ is a solution of the system

$$\begin{aligned} m_1 &= h_1(M_2,M_2,\dots,M_2),\quad M_1=h_1(m_2,m_2,\dots,m_2),\\ m_2 &= h_2(M_1,M_1,\dots,M_1),\quad M_2=h_2(m_1,m_1,\dots,m_1), \end{aligned} \tag{2.1}$$

then $m_1 = M_1$ and $m_2 = M_2$. Then the system

$$\left.\begin{array}{l} x_{n+1} = h_1(y_n, y_{n-1}, \dots, y_{n-k}) \\ y_{n+1} = h_2(x_n, x_{n-1}, \dots, x_{n-k}) \end{array}\right\}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

has a unique positive equilibrium $(\overline{x}, \overline{y}) \in [\alpha, \beta] \times [\gamma, \delta]$ and its every positive solution converges to this equilibrium.

Proof.
Let

$$m_1^0 := \alpha, \quad M_1^0 := \beta, \quad m_2^0 := \gamma, \quad M_2^0 := \delta$$

and

$$\begin{aligned} m_1^{i+1} &: = h_1(M_2^i, M_2^i, \dots, M_2^i), \quad M_1^{i+1} := h_1(m_2^i, m_2^i, \dots, m_2^i), \\ m_2^{i+1} &: = h_2(M_1^i, M_1^i, \dots, M_1^i), \quad M_2^{i+1} := h_2(m_1^i, m_1^i, \dots, m_1^i). \end{aligned}$$

For each $i = 0, 1, \dots$, we have

$$\begin{aligned} \alpha &\leq h_1(\delta, \delta, \dots, \delta) \leq h_1(\gamma, \gamma, \dots, \gamma) \leq \beta, \\ \gamma &\leq h_2(\beta, \beta, \dots, \beta) \leq h_2(\alpha, \alpha, \dots, \alpha) \leq \delta \end{aligned}$$

and so,

$$\begin{aligned} m_1^0 &= \alpha \leq h_1(M_2^0, M_2^0, \dots, M_2^0) = m_1^1 \leq h_1(m_2^0, m_2^0, \dots, m_2^0) = M_1^1 \leq \beta = M_1^0, \\ m_2^0 &= \gamma \leq h_2(M_1^0, M_1^0, \dots, M_1^0) = m_2^1 \leq h_2(m_1^0, m_1^0, \dots, m_1^0) = M_2^1 \leq \delta = M_2^0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} m_1^1 &= h_1(M_2^0, M_2^0, \dots, M_2^0) \\ &\leq h_1(M_2^1, M_2^1, \dots, M_2^1) \\ &= m_1^2 \\ &\leq h_1(m_2^1, m_2^1, \dots, m_2^1) \\ &= M_1^2 \\ &\leq h_1(m_2^0, m_2^0, \dots, m_2^0) \\ &= M_1^1, \end{aligned}$$

and

$$\begin{aligned} m_2^1 &= h_2(M_1^0, M_1^0, \ldots, M_1^0) \\ &\leq h_2(M_1^1, M_1^1, \ldots, M_1^1) \\ &= m_2^2 \\ &\leq h_2(m_1^1, m_1^1, \ldots, m_1^1) \\ &= M_2^2 \\ &\leq h_2(m_1^0, m_1^0, \ldots, m_1^0) \\ &= M_2^1. \end{aligned}$$

By induction, one can see for $i = 0, 1, \ldots,$ that

$$\begin{aligned} \alpha &= m_1^0 \leq m_1^1 \leq \cdots \leq m_1^{i-1} \leq m_1^i \leq M_1^i \leq M_1^{i-1} \leq \cdots \leq M_1^1 \leq M_1^0 = \beta, \\ \gamma &= m_2^0 \leq m_2^1 \leq \cdots \leq m_2^{i-1} \leq m_2^i \leq M_2^i \leq M_2^{i-1} \leq \cdots \leq M_2^1 \leq M_2^0 = \delta. \end{aligned}$$

It follows that the sequences $(m_1^i)_i$ and $(m_2^i)_i$ (resp. $(M_1^i)_i$ and $(M_2^i)_i$) are increasing (resp. decreasing) and also bounded, and therefore they are convergent sequences. Then we can assume that

$$m_1 = \lim_{i \to +\infty} m_1^i, \quad M_1 = \lim_{i \to +\infty} M_1^i, \quad m_2 = \lim_{i \to +\infty} m_2^i, \quad M_2 = \lim_{i \to +\infty} M_2^i.$$

Then,

$$\alpha \leq m_1 \leq M_1 \leq \beta, \quad \gamma \leq m_2 \leq M_2 \leq \delta.$$

By taking limits in the equalities

$$\begin{aligned} m_1^{i+1} &= h_1(M_2^i, M_2^i, \ldots, M_2^i), \quad M_1^{i+1} = h_1(m_2^i, m_2^i, \ldots, m_2^i), \\ m_2^{i+1} &= h_2(M_1^i, M_1^i, \ldots, M_1^i), \quad M_2^{i+1} = h_2(m_1^i, m_1^i, \ldots, m_1^i), \end{aligned}$$

and using that h_1 and h_2 are continuous, we obtain system (2.1). So, from (b), it follows that $m_1 = M_1$ and $m_2 = M_2$. It can be concluded from the hypothesis that

$$m_1^0 = \alpha \leq x_n \leq \beta = M_1^0, \quad m_2^0 = \gamma \leq y_n \leq \delta = M_2^0, \quad n = 1, 2, \ldots.$$

Therefore, we obtain

$$\begin{aligned} m_1^1 &= h_1(M_2^0, M_2^0, \ldots, M_2^0) \leq h_1(y_n, y_{n-1}, \ldots, y_{n-2}) = x_{n+1} \leq h_1(m_2^0, m_2^0, \ldots, m_2^0) = M_1^1, \\ m_2^1 &= h_2(M_1^0, M_1^0, \ldots, M_1^0) \leq h_2(x_n, x_{n-1}, \ldots, x_{n-2}) = y_{n+1} \leq h_2(m_1^0, m_1^0, \ldots, m_1^0) = M_2^1, \end{aligned}$$

for $n = 2, 3, \ldots,$ and

$$\begin{aligned} m_1^2 &= h_1(M_2^1, M_2^1, \ldots, M_2^1) \leq h_1(y_n, y_{n-1}, \ldots, y_{n-2}) = x_{n+1} \leq h_1(m_2^1, m_2^1, \ldots, m_2^1) = M_1^2, \\ m_2^2 &= h_2(M_1^1, M_1^1, \ldots, M_1^1) \leq h_2(x_n, x_{n-1}, \ldots, x_{n-2}) = y_{n+1} \leq h_2(m_1^1, m_1^1, \ldots, m_1^1) = M_2^2. \end{aligned}$$

for $n = 4, 5, \ldots$, and

$$\begin{aligned} m_1^3 &= h_1(M_2^2, M_2^2, \ldots, M_2^2) \le h_1(y_n, y_{n-1}, \ldots, y_{n-2}) = x_{n+1} \le h_1(m_2^2, m_2^2, \ldots, m_2^2) = M_1^3, \\ m_2^3 &= h_2(M_1^2, M_1^2, \ldots, M_1^2) \le h_2(x_n, x_{n-1}, \ldots, x_{n-2}) = y_{n+1} \le h_2(m_1^2, m_1^2, \ldots, m_1^2) = M_2^3 \end{aligned}$$

for $n = 6, 7, \ldots$. Moreover, by induction, it follows for $i = 0, 1, \ldots$, that

$$m_1^i \le x_n \le M_1^i, \quad m_2^i \le y_n \le M_2^i, \quad n \ge 2i + 1.$$

It is obvious that $i \to +\infty$ implies $n \to +\infty$. Also, since $m_1 = M_1$ and $m_2 = M_2$, we obtain

$$\lim_{n\to+\infty} x_n = M_1, \quad \lim_{n\to+\infty} y_n = M_2.$$

Moreover, in this case, since system (2.1) reduces to

$$M_1 = h_1(M_2, M_2, \ldots, M_2), \quad M_2 = h_2(M_1, M_1, \ldots, M_1),$$

we obtain

$$M_1 = \overline{x}, \quad M_2 = \overline{y}.$$

Therefore, the proof is completed.

2.2. Dynamics of system (1.3)

We here begin our study on system (1.3). For the sake of simplicity, let $a_1 + a_2 + a_3 = \alpha$ and $b_1 + b_2 + b_3 = \beta$. The equilibrium points of system (1.3) correspond to the solutions of the system

$$\overline{x} = a + \frac{\alpha}{\overline{y}}, \quad \overline{y} = b + \frac{\beta}{\overline{x}}, \tag{2.3}$$

from which it follows that

$$\begin{aligned} \overline{x} &= \frac{\beta - \alpha - ab \pm \sqrt{\Delta}}{2b}, \\ \overline{y} &= \frac{\alpha - \beta - ab \pm \sqrt{\Delta}}{2a}, \end{aligned}$$

where

$$\begin{aligned} \Delta &= (\alpha - \beta - ab)^2 + 4ab\alpha \\ &= (\beta - \alpha - ab)^2 + 4ab\beta \\ &> 0. \end{aligned}$$

Hence, system (1.3) possesses the positive equilibrium point

$$(\overline{x}, \overline{y}) = \left(\frac{\beta - \alpha - ab + \sqrt{\Delta}}{2b}, \frac{\alpha - \beta - ab + \sqrt{\Delta}}{2a} \right).$$

Theorem 2.2. The equilibrium $(\overline{x}, \overline{y})$ of system (1.3) is locally asymptotically stable.

Proof.
Let

$$\begin{aligned} f &: = a + \frac{a_1}{y_n} + \frac{a_2}{y_{n-1}} + \frac{a_3}{y_{n-2}}, \\ f_1 &: = x_n, \\ f_2 &: = x_{n-1}, \\ g &: = b + \frac{b_1}{x_n} + \frac{b_2}{x_{n-1}} + \frac{b_3}{x_{n-2}}, \\ g_1 &: = y_n, \\ g_2 &: = y_{n-1}. \end{aligned}$$

Then, we can define a map $T : (0,\infty)^6 \longrightarrow (0,\infty)^6$ and the system corresponding to T as follows:

$$W_{n+1} = T(W_n), \tag{2.4}$$

where $W_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^t$, ($t$ states the transpose operation)

$$T\begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix} = \begin{pmatrix} a + \frac{a_1}{y_n} + \frac{a_2}{y_{n-1}} + \frac{a_3}{y_{n-2}} \\ x_n \\ x_{n-1} \\ b + \frac{b_1}{x_n} + \frac{b_2}{x_{n-1}} + \frac{b_3}{x_{n-2}} \\ y_n \\ y_{n-1} \end{pmatrix}.$$

In this case, the equilibrium of (2.4) is $E = (\overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y})^t$. For $i = 0, 1, 2$, we obtain

$$\begin{aligned} \frac{\partial f}{\partial x_{n-i}}|_E &= 0, \quad \frac{\partial f}{\partial y_{n-i}}|_E = -\frac{a_{i+1}}{\overline{y}^2}, \\ \frac{\partial f_1}{\partial x_n}|_E &= 1, \quad \frac{\partial f_1}{\partial x_{n-1}}|_E = \frac{\partial f_1}{\partial x_{n-2}}|_E = 0, \quad \frac{\partial f_1}{\partial y_{n-i}}|_E = 0, \\ \frac{\partial f_2}{\partial x_n}|_E &= 0, \quad \frac{\partial f_2}{\partial x_{n-1}}|_E = 1, \quad \frac{\partial f_2}{\partial x_{n-2}}|_E = 0, \quad \frac{\partial f_2}{\partial y_{n-i}}|_E = 0, \\ \frac{\partial g}{\partial x_{n-i}}|_E &= -\frac{b_{i+1}}{\overline{x}^2}, \quad \frac{\partial g}{\partial y_{n-i}}|_E = 0, \\ \frac{\partial g_1}{\partial x_{n-i}}|_E &= 0, \quad \frac{\partial g_1}{\partial y_n}|_E = 1, \quad \frac{\partial g_1}{\partial y_{n-1}}|_E = \frac{\partial g_1}{\partial y_{n-2}}|_E = 0, \\ \frac{\partial g_1}{\partial x_{n-i}}|_E &= 0, \quad \frac{\partial g_1}{\partial y_n}|_E = 0, \quad \frac{\partial g_1}{\partial y_{n-1}}|_E = 1, \quad \frac{\partial g_1}{\partial y_{n-2}}|_E = 0. \end{aligned}$$

By these partial derivatives, one can obtain the Jacobian of the map T evaluated at E as follows:

$$J_T(E) = \begin{pmatrix} 0 & 0 & 0 & -\frac{a_1}{\overline{y}^2} & -\frac{a_2}{\overline{y}^2} & -\frac{a_3}{\overline{y}^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{b_1}{\overline{x}^2} & -\frac{b_2}{\overline{x}^2} & -\frac{b_3}{\overline{x}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix $J_F(E)$ has the characteristic polynomial

$$\begin{aligned} P(\lambda) &= \lambda^6 - \frac{a_1 b_1 \lambda^4 + (a_1 b_2 + a_2 b_1)\lambda^3 + (a_1 b_3 + a_2 b_2 + a_3 b_1)\lambda^2 + (a_2 b_3 + a_3 b_2)\lambda + a_3 b_3}{\overline{x}^2 \overline{y}^2} \\ &= \lambda^6 - \frac{\left(a_1 \lambda^2 + a_2 \lambda + a_3\right)\left(b_1 \lambda^2 + b_2 \lambda + b_3\right)}{\overline{x}^2 \overline{y}^2}. \end{aligned}$$

We need to ensure that all roots of P are less than 1 in absolute value. For this, let

$$\Phi(\lambda) = \lambda^6$$

and

$$\Psi(\lambda) = -\frac{\left(a_1 \lambda^2 + a_2 \lambda + a_3\right)\left(b_1 \lambda^2 + b_2 \lambda + b_3\right)}{\overline{x}^2 \overline{y}^2}.$$

It is easily seen that every root of Φ satisfies the condition $|\lambda| < 1$. That is, those are all less than 1 in absolute value. So, if we assume

$$|\Psi(\lambda)| \le \frac{(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)}{\overline{x}^2 \overline{y}^2} < 1 = |\Phi(\lambda)|, \quad \forall \lambda \in \mathbb{C}, \quad |\lambda| = 1,$$

then every root of P will satisfy the condition $|\lambda| < 1$ according to Rouché's theorem. After some arrangements, we get the inequality

$$\alpha\beta < \overline{x}^2 \overline{y}^2. \tag{2.5}$$

From (2.3), we obtain

$$\overline{xy} = ab + \frac{b\alpha}{\overline{y}} + \frac{a\beta}{\overline{x}} + \frac{\alpha\beta}{\overline{xy}} \quad \Leftrightarrow \quad \overline{x}^2 \overline{y}^2 = ab\overline{xy} + b\alpha\overline{x} + a\beta\overline{y} + \alpha\beta,$$

and therefore

$$\overline{x}^2 \overline{y}^2 - \alpha\beta = ab\overline{xy} + b\alpha\overline{x} + a\beta\overline{y} > 0,$$

which shows that the inequality in (2.5) is always satisfied. This completes the proof.

Theorem 2.3. Every positive solution of (1.3) is bounded.

Proof.
Let $\left\{(x_n, y_n)\right\}_{n=-2}^{\infty}$ be a positive solution of (1.3). Then, we obtain from (1.3) that

$$x_n \ge a > 0, \quad y_n \ge b > 0 \tag{2.6}$$

for all $n \in \mathbb{N}$. That is, x_n and y_n are bounded from below. Also, it follows from system (1.3) and (2.6) that

$$\begin{aligned} x_{n+1} &= a+\frac{a_1}{y_n}+\frac{a_2}{y_{n-1}}+\frac{a_3}{y_{n-2}} \leq a+\frac{\alpha}{b}<\infty, \\ y_{n+1} &= b+\frac{b_1}{x_n}+\frac{b_2}{x_{n-1}}+\frac{b_3}{x_{n-2}} \leq b+\frac{\beta}{a}<\infty \end{aligned}$$

for all $n \in \mathbb{N}$. That is, x_n and y_n are bounded from above. This completes the proof.

Theorem 2.4. The positive equilibrium $(\overline{x}, \overline{y})$ of system (1.3) is globally asymptotically stable.

Proof.
Theoretically, for the equilibrium $(\overline{x}, \overline{y})$ to be globally asymptotically stable, it must be locally asymptotically stable. See [19]. But we have already proven this in Theorem 2.2. Then, we only need to show that $(\overline{x}, \overline{y})$ is the global attractor of the positive solutions. That is, we will show that

$$\lim_{n\to\infty} x_n = \bar{x} \text{ and } \lim_{n\to\infty} y_n = \overline{y}.$$

To do this, we apply Theorem 2.1 to (1.3). We know from Theorem 2.3 that x_n and y_n are bounded for all $n \geq 1$. Then, it follows that $a \leq m_1 := \lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n := M_1 \leq a+\frac{\alpha}{b}$ and $b \leq m_2 := \lim_{n\to\infty} \inf y_n \leq \lim_{n\to\infty} \sup y_n := M_2 \leq b+\frac{\beta}{a}$. It suffices to show that $m_1 = M_1$ and $m_2 = M_2$.
Consider the system

$$M_1 = a+\frac{a_1}{m_2}+\frac{a_2}{m_2}+\frac{a_3}{m_2}, \tag{2.7}$$
$$m_1 = a+\frac{a_1}{M_2}+\frac{a_2}{M_2}+\frac{a_3}{M_2}, \tag{2.8}$$
$$M_2 = b+\frac{b_1}{m_1}+\frac{b_2}{m_1}+\frac{b_3}{m_1}, \tag{2.9}$$
$$m_2 = b+\frac{b_1}{M_1}+\frac{b_2}{M_1}+\frac{b_3}{M_1}. \tag{2.10}$$

Then, from (2.7) and (2.10), it follows that

$$bM_1^2+\left(\beta-\alpha-ab\right)M_1-a\beta = 0, \tag{2.11}$$
$$bm_1^2+\left(\beta-\alpha-ab\right)m_1-a\beta = 0, \tag{2.12}$$

from (2.8) and (2.9), it follows that

$$aM_2^2+\left(\alpha-\beta-ab\right)M_2-b\alpha = 0, \tag{2.13}$$
$$am_2^2+\left(\alpha-\beta-ab\right)m_2-b\alpha = 0. \tag{2.14}$$

Note that (2.11) and (2.12) are equations that have the same solutions. Also, since

$$\left(\beta-\alpha-ab\right)^2+4ab\beta>0, \quad -\frac{a}{b}\beta<0,$$

(2.11) and (2.12) have simple real roots such that one is positive and another is negative. Therefore, the positive solutions of them are the same, and so we have $M_1 = m_1$. Similarly, (2.13) and (2.14) are equations that have the same solutions, and since

$$\left(\alpha-\beta-ab\right)^2+4ab\alpha>0,\quad -\frac{b}{a}\alpha<0,$$

(2.13) and (2.14) have simple real roots such that one is positive and another is negative. Therefore, the positive solutions of them are the same, and so we have $M_2 = m_2$. Consequently, by Theorem 2.1, $(\overline{x},\overline{y})$ is a global attractor and thus globally asymptotically stable.The proof is complete.

According to Theorem 2.3, for all $n\in\mathbb{N}$, the inequalities $a\leq x_n\leq a+\frac{\alpha}{b}$ and $b\leq y_n\leq b+\frac{\beta}{a}$ exist. That is, the positive solutions of system (1.3) are bounded. However, depending on the subset that initial conditions are found, the solutions in question can be always found in this subset. Such subsets are called invariant sets. In the next theorem, the invariant sets of system (1.3) are examined.

Theorem 2.5. The following statements are true:

(a) $\left[a,\overline{x}\right]\times\left[\overline{y},b+\frac{\beta}{a}\right]$ is an invariant set of system (1.3).

(b) $\left[\overline{x},a+\frac{\alpha}{b}\right]\times\left[b,\overline{y}\right]$ is an invariant set of system (1.3).

Proof.

Let the functions

$$\widehat{h}_1\left(\overline{x}\right)=a+\frac{\alpha}{b+\frac{\beta}{\overline{x}}}-\overline{x},\quad \widehat{h}_2\left(\overline{y}\right)=b+\frac{\beta}{a+\frac{\alpha}{\overline{y}}}-\overline{y}$$

be defined, taking into account the system in (2.3). In this case, we can see that

$$\begin{aligned}
\widehat{h}_1\left(a\right) &= a+\frac{\alpha}{b+\frac{\beta}{a}}-a=\frac{\alpha}{b+\frac{\beta}{a}}>0,\\
\widehat{h}_1\left(a+\frac{\alpha}{b}\right) &= a+\frac{\alpha}{b+\frac{\beta}{a+\frac{\alpha}{b}}}-a-\frac{\alpha}{b}\\
&= \frac{\alpha}{b+\frac{b\beta}{ab+\alpha}}-\frac{\alpha}{b}\\
&= \frac{\alpha}{b}\left(\frac{1}{1+\frac{\beta}{ab+\alpha}}-1\right)\\
&< 0,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{h}_2\left(b\right) &= b+\frac{\beta}{a+\frac{\alpha}{b}}-b=\frac{\beta}{a+\frac{\alpha}{b}}>0,\\
\widehat{h}_2\left(b+\frac{\beta}{a}\right) &= b+\frac{\beta}{a+\frac{\alpha}{b+\frac{\beta}{a}}}-b-\frac{\beta}{a}\\
&= \frac{\beta}{a+\frac{a\alpha}{ab+\beta}}-\frac{\beta}{a}\\
&= \frac{\beta}{a}\left(\frac{1}{1+\frac{\alpha}{ab+\beta}}-1\right)\\
&< 0.
\end{aligned}$$

Hence, we obtain

$$\left(\overline{x},\overline{y}\right)\in\left[a,a+\frac{\alpha}{b}\right]\times\left[b,b+\frac{\beta}{a}\right].$$

(a) Assume that $\left(x_{-j},y_{-j}\right)\in\left[a,\overline{x}\right]\times\left[\overline{y},b+\frac{\beta}{a}\right]$ for $j=0,1,2$. Then, from system (1.3), we have

$$\begin{aligned}
a &\leq x_1=a+\frac{a_1}{y_0}+\frac{a_2}{y_{-1}}+\frac{a_3}{y_{-2}}\leq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b+\frac{\beta}{a} &\geq y_1=b+\frac{b_1}{x_0}+\frac{b_2}{x_{-1}}+\frac{b_3}{x_{-2}}\geq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
a &\leq x_2=a+\frac{a_1}{y_1}+\frac{a_2}{y_0}+\frac{a_3}{y_{-1}}\leq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b+\frac{\beta}{a} &\geq y_2=b+\frac{b_1}{x_1}+\frac{b_2}{x_0}+\frac{b_3}{x_{-1}}\geq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
&\vdots
\end{aligned}$$

In this case, by induction, one can see that $\left(x_n,y_n\right)\in\left[a,\overline{x}\right]\times\left[\overline{y},b+\frac{\beta}{a}\right]$ for $n\geq-2$.

(b) Assume that $\left(x_{-j},y_{-j}\right)\in\left[\overline{x},a+\frac{\alpha}{b}\right]\times\left[b,\overline{y}\right]$ for $j=0,1,2$. Then, from system (1.3), we have

$$\begin{aligned}
a+\frac{\alpha}{b} &\geq x_1=a+\frac{a_1}{y_0}+\frac{a_2}{y_{-1}}+\frac{a_3}{y_{-2}}\geq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b &\leq y_1=b+\frac{b_1}{x_0}+\frac{b_2}{x_{-1}}+\frac{b_3}{x_{-2}}\leq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
a+\frac{\alpha}{b} &\geq x_2=a+\frac{a_1}{y_1}+\frac{a_2}{y_0}+\frac{a_3}{y_{-1}}\geq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b &\leq y_2=b+\frac{b_1}{x_1}+\frac{b_2}{x_0}+\frac{b_3}{x_{-1}}\leq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
&\vdots
\end{aligned}$$

In this case, by induction, one can see that $\left(x_n,y_n\right)\in\left[\overline{x},a+\frac{\alpha}{b}\right]\times\left[b,\overline{y}\right]$ for $n\geq-2$.

3. Numerical Simulation

This section aims to verify the theoretical results obtained in Section 2 using some specific values of the parameters and the initial values $x_{-2}:=5.21$, $x_{-1}:=2.55$, $x_0:=3.75$, $y_{-2}:=2.13$, $y_{-1}:=4.86$, $y_0:=5.50$. The solutions will be represented by drawings of numerical values.

Example 3.1. Let $a:=2.9$, $a_1:=1.2$, $a_2:=1.55$, $a_3:=4.1$, $b:=3.1$, $b_1:=1.1$, $b_2:=1.40$, $b_3:=3.9$. Then the solution of system (1.3) becomes as in Figure 1.

Example 3.2. Let $a:=2.99$, $a_1:=5.2$, $a_2:=2.55$, $a_3:=0.5$, $b:=0.01$, $b_1:=6.1$, $b_2:=15.4$, $b_3:=0.3$. Then the solution of system (1.3) becomes as in Figure 2.

Example 3.3. Let $a:=0.50$, $a_1:=1.21$, $a_2:=6.05$, $a_3:=14.51$, $b:=0.80$, $b_1:=0.17$, $b_2:=12.42$, $b_3:=2.35$. Then the solution of system (1.3) becomes as in Figure 3.

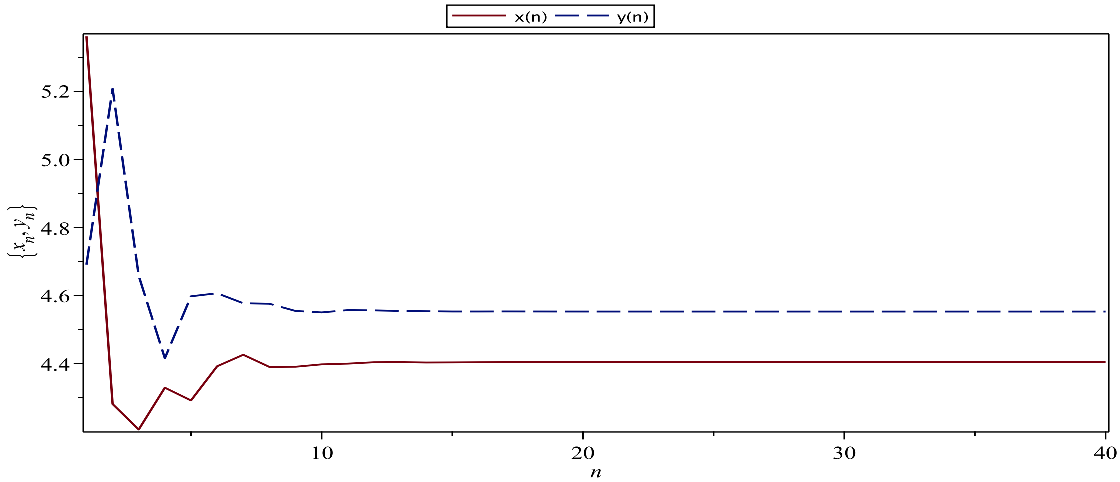

Figure 1. For $a := 2.9$, $a_1 := 1.2$, $a_2 := 1.55$, $a_3 := 4.1$, $b := 3.1$, $b_1 := 1.1$, $b_2 := 1.40$, $b_3 := 3.9$, the solution of system (1.3).

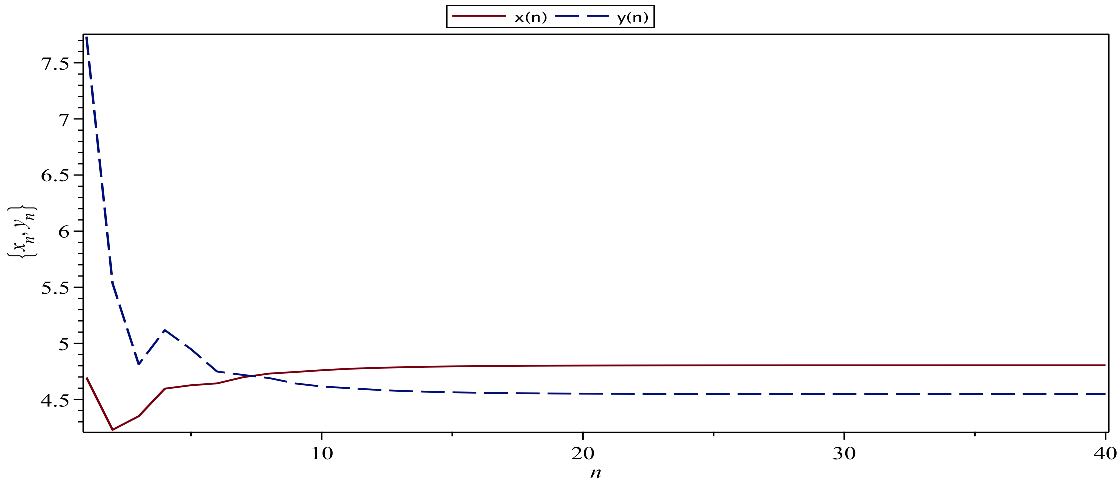

Figure 2. For $a := 2.99$, $a_1 := 5.2$, $a_2 := 2.55$, $a_3 := 0.5$, $b := 0.01$, $b_1 := 6.1$, $b_2 := 15.4$, $b_3 := 0.3$, the solution of system (1.3).

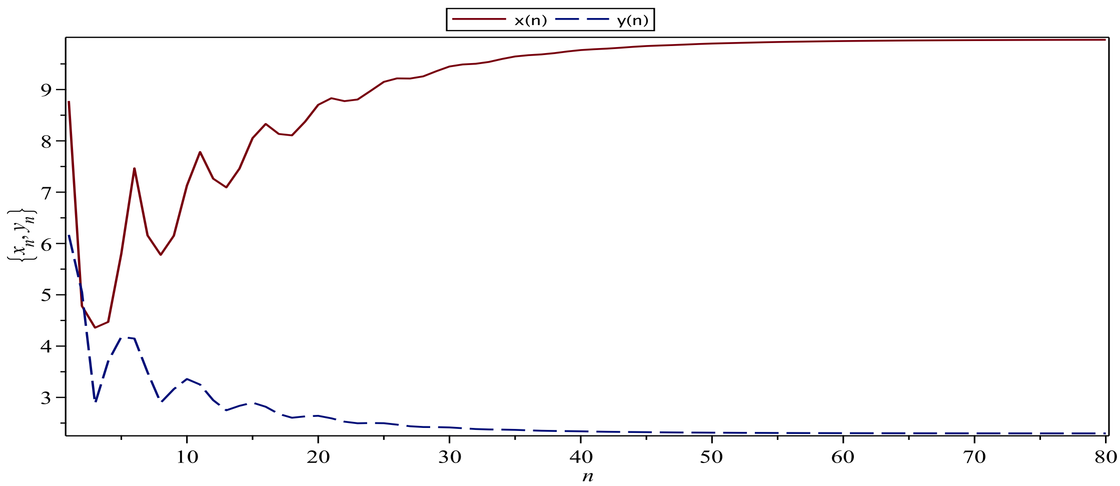

Figure 3. For $a := 2.99$, $a_1 := 5.2$, $a_2 := 2.55$, $a_3 := 0.5$, $b := 0.01$, $b_1 := 6.1$, $b_2 := 15.4$, $b_3 := 0.3$, the solution of system (1.3).

4. Conclusion

In this study, the local and global stability of the positive equilibrium of the system

$$x_{n+1} = a + \frac{a_1}{y_n} + \frac{a_2}{y_{n-1}} + \frac{a_3}{y_{n-2}}, \quad y_{n+1} = b + \frac{b_1}{x_n} + \frac{b_2}{x_{n-1}} + \frac{b_3}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where a, a_i, b, b_i $(i = 1,2,3)$ and x_{-j}, y_{-j} $(j = 0,1,2)$ are positive and real, was investigated. It was concluded that for all positive values of all parameters seen in the system, positive solutions converge to the unique positive equilibrium. Also, it was handled invariant sets to better understand the behavior of the solutions. Finally, the theoretical results were confirmed numerically and illustrated with visuals.

Although the system is a third-order system, it can be expanded to a higher order and similar research can be conducted. One option would be to increase the rational terms. In such a case, the system may be

$$x_{n+1} = a + \sum_{s=1}^{k} \frac{a_s}{y_{n-s+1}}, \quad y_{n+1} = b + \sum_{s=1}^{k} \frac{b_s}{x_{n-s+1}}, \quad n \in \mathbb{N}_0,$$

with positive parameters and positive initial values. Note that this system is a generalization of the above and reduces to it for $k = 3$.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).

[2] N. Akgunes and A. S. Kurbanli, *On the system of rational difference equations* $x_n = f\left(x_{n-a_1}, y_{n-b_1}\right)$, $y_n = g\left(y_{n-b_2}, z_{n-c_1}\right)$, $z_n = g\left(z_{n-c_2}, x_{n-a_2}\right)$, Selcuk Journal of Applied Mathematics, 15(1), (2014), 1-8.

[3] Y. Akrour, M. Kara, N. Touafek and Y. Yazlik, *Solutions formulas for some general systems of nonlinear difference equations*, Miskolc Mathematical Notes, 22(2) (2021), 529–555.

[4] E. Camouzis and G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures*, Chapman and Hall/CRC, (2007).

[5] I. Dekkar, N. Touafek and Y. Yazlik, *Global stability of a third-order nonlinear system of difference equations with period-two coefficients*, Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas, 111(2), (2017) 325-347. Doi:10.1007/s13398-016-0297-z

[6] R. DeVault, G. Ladas and S. W. Schultz, *Necessary and sufficient conditions for the boundedness of* $x_{n+1} = A/x_n^p + B/x_{n-1}^q$, Journal of Difference Equations and Applications, 3(3-4)(1997), 259-266.

[7] R. DeVault, G. Ladas and S. W. Schultz, *On the recursive sequence* $x_{n+1} = A/x_n + 1/x_{n-2}$, Proceedings of the American Mathematical Society, 126(11)(1998), 3257-3261.

[8] S. Elaydi, *An Introduction to Difference Equations, third edition, Undergraduate Texts in Mathematics*, Springer, New York, (1999).

[9] H. El-Metwally, E. A. Grove and G. Ladas, *A global convergence result with applications to periodic solutions*, Journal of Mathematical Analysis and Applications, 245(2000), 161-170.

[10] H. El-Metwally, E. A. Grove, G. Ladas and H. D. Voulov, *On the global attractivity and the periodic character of some difference equations*, Journal of Difference Equations and Applications, 7(6)(2001), 837-850.

[11] N. Haddad, N. Touafek and J. F. T. Rabago, *Solution form of a higher-order system of difference equations and dynamical behavior of its special case*, Mathematical Methods in the Applied Sciences, 40(10), (2017), 3599-3607.

[12] N. Haddad, N. Touafek and J. F. T. Rabago, *Well-defined solutions of a system of difference equations*, Journal of Applied Mathematics and Computing, 56, (2018), 439-458, https://doi.org/10.1007/s12190-017-1081-8

[13] E. Hatir, T. Mansour and I. Yalcinkaya, *On a fuzzy difference equation*, *Utilitas Mathematica*, 93(2014), 135-151.

[14] M. Kara, D. T. Tollu and Y. Yazlik, *Global behavior of two-dimensional difference equations system with two periodic coefficients*, Tbilisi Mathematical Journal, 13(4), (2020), 49-64.

[15] M. Kara, Y. Yazlik and D. T. Tollu, *Solvability of a system of higher order nonlinear difference equations*, Hacettepe Journal of Mathematics and Statistics, 49(5), (2020), 1566-1593.

[16] M. Kara and Y. Yazlik, *On the solutions of three-dimensional system of difference equations via recursive relations of order two and applications*, Journal of Applied Analysis and Computation, 12(2), (2022) 736-753.

[17] M. Kara and Y. Yazlik, *Solvable three-dimensional system of higher-order nonlinear difference equations*, Filomat, 36(10), (2022), 3449-3469.

[18] M. Kara and Y. Yazlik, *On a solvable system of rational difference equations of higher order*, Turkish Journal of Mathematics, 46(2), (2022) 587-611.

[19] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, (1993).

[20] M.R.S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, New York, NY, USA, CRC Press, 2002.

[21] A. S. Kurbanli, C. Çinar and D. Şimşek, *On the periodicity of solutions of the system of rational difference equations*, Applied Mathematics, 2, (2011), 410-413.

[22] G. Papaschinopoulos and C. J. Schinas, *On a system of two nonlinear difference equations*, Journal of Mathematical Analysis and Applications, 219 (2) (1998), 415-426.

[23] G. Papaschinopoulos and C. J. Schinas, *Stability of a class of nonlinear difference equations*, Journal of Mathematical Analysis and Applications, 230 (1999), 211-222.

[24] G. Papaschinopoulos and C. J. Schinas, *Invariants and oscillation for systems of two nonlinear difference equations*, Nonlinear Analysis: Theory, Methods and Applications, 46 (2001), 967–978.

[25] G. Papaschinopoulos and C. J. Schinas, *Oscillation and asymptotic stability of two systems of difference equations of rational form*, Journal of Difference Equations and Applications, 7 (2001), 601-617.

[26] G. Papaschinopoulos and C. J. Schinas, *On the system of two difference equations* $x_{n+1} = \sum_{i=0}^{k} A_i / y_{n-i}^{p_i}$, $y_{n+1} = \sum_{i=0}^{k} B_i / x_{n-i}^{q_i}$, Journal of Mathematical Analysis and Applications, 273 (2) (2002), 294-309.

[27] G. Papaschinopoulos and B. K. Papadopoulos, *On the fuzzy difference equation* $x_{n+1} = A + B/x_n$, Soft Computing, 6(2002), 456-461.

[28] C. G. Philos, I. K. Purnaras and Y. G. Sficas, *Global attractivity in a nonlinear difference equation*, Applied Mathematics and Computation, 62(2-3)(1994), 249-258.

[29] C. J. Schinas, *Invariants for difference equations and systems of difference equations of rational form*, Journal of Mathematical Analysis and Applications, 216(1)(1997), 164-179.

[30] S. Stević and D. T. Tollu, *Solvability and semi-cycle analysis of a class of nonlinear systems of difference equations*, Mathematical Methods in the Applied Sciences, 42, (2019), 3579-3615. https://doi.org/10.1002/mma.5600

[31] S. Stević and D. T. Tollu, *Solvability of eight classes of nonlinear systems of difference equations*, Mathematical Methods in the Applied Sciences, 42, (2019), 4065-4112. https://doi.org/10.1002/mma.5625

[32] N. Taskara, D. T. Tollu, N. Touafek and Y. Yazlik, *A solvable system of difference equations*, Communications of the Korean Mathematical Society, 35 (1) (2020), 301-319.

[33] D. T. Tollu, Y. Yazlik and N. Taskara, *On fourteen solvable systems of difference equations*, Applied Mathematics and Computation, 233, (2014), 310-319.

[34] I. Yalçınkaya, H. El-Metwally and D. T. Tollu, *On the fuzzy difference equation* $z_{n+1} = A + B/z_{n-m}$, Mathematical Notes, 113(2023), 292–302.

[35] I. Yalçınkaya, H. El-Metwally, M. Bayram, et al., *On the dynamics of a higher-order fuzzy difference equation with rational terms*, Soft Computing, 27(2023), 10469–10479. https://doi.org/10.1007/s00500-023-08586-y

[36] Y. Yazlik, E. M. Elsayed and N. Taskara, *On the behaviour of the solutions of difference equation systems*, Journal of Computational Analysis and Applications, 16(5), (2014), 932-941.

[37] Y. Yazlik, D. T. Tollu and N. Taskara, *On the solutions of a three-dimensional system of difference equations*, Kuwait Journal of Science, 43(1) (2016), 95-111.