



Fundamentals of Stars: A Critical Look at Mass-Luminosity Relations and Beyond

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ABSTRACT

Developments on various relations among stellar variables such as the main sequence empirical mass-luminosity (MLR), mass-radius (MRR) and mass-effective temperature (MTR) relations were reviewed. Conceptual changes in their understanding and usages were discussed. After its discovery, MLR was treated as one of the fundamental secrets of the cosmos. Differences between fundamental laws and statistical relations were used to understand long-term developments of MLR, MRR and MTR. Developments show a break point, initiated by Andersen (1991), in the line of progress. Before the break when reliable data were limited, MLR and MRR were calibrated using M , L , and R of binary components of all kinds visual, spectroscopic, and eclipsing for two purposes: *i*) to obtain mean mass, mean luminosity, and mean radius, *ii*) to estimate M and R of single stars. By the time of the break, the number of solutions from detached double-lined eclipsing binaries (DDEB) giving accurate M and R within a few percent levels are increased. Parameters from very close, semi-detached, and contact binaries were excluded for refinement, however, MLR and MRR diagrams were found insufficient to derive MLR and MRR functions because the dispersions are not only due to random observational errors but also due to chemical composition and age differences. Then, a new trend was adopted by replacing classical MLR and MRR with empirical M and R predicting relations. Thus, the purpose one was suppressed also because the new trend found a fruitful application in determining M and R of exoplanet hosting single stars.

Keywords: Stars: fundamental parameters; Stars: luminosity and mass functions; Galaxies: luminosity function and mass functions; Cosmology: miscellaneous

1. INTRODUCTION

Accuracy and precision of observational parameters of stars are crucial not only for improving stellar structure and evolution theories but also for fundamental astrophysics, Galactic and extragalactic studies, and ultimately even for cosmological models because stars and galaxies are the primary building blocks of the universe. Fundamental and statistical relations are essential for understanding physical events occurring in various parts of the universe. The fundamental relations are the relations like Stefan-Boltzmann law ($L = 4\pi R^2 \sigma T_{\text{eff}}^4$) which are characterised by at least two properties; *i*) how various properties of stars are related, e.g. how the luminosity of a star is related to its radius and effective temperature; *ii*) validity is not limited by certain conditions, that is, it applies to all stars and radiating surfaces as long as the source of radiation is thermal. The statistical relations, on the other hand, may not work in all possible cases; appropriate statistical conditions are required.

A good example is the kinetic temperature. A temperature is a physical quantity that is measured in the macroworld but does not exist in the microworld. This is because temperature is a statistically defined quantity implying an average kinetic energy per particle in a substance, which could be a solid, liquid, or gas. The definition of kinetic temperature indicates that a single particle or insignificant number of particles cannot be associated with any temperature. For example, when we talk about air temperature, it indicates the average kinetic energy per air particle, which can be written as:

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT \quad (1)$$

where m and $\langle v^2 \rangle$ are the mean mass and root-mean-square (rms) speed of particles. They are not physical quantities but average values. Increasing the speed of the substance itself by moving it faster, e.g., putting it on a quick aeroplane, does

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not change its temperature. Therefore, equation (1) cannot be written for a randomly chosen particle (or a few particles), i.e., the right-hand side does not exist except for a group of particles satisfying the implied statistic.

Similarly, the mass–luminosity relation (MLR) in the form ($L \propto M^\alpha$), which is defined so far for main-sequence stars, is a statistical relation similar to equation (1), which indicates how main-sequence luminosities are related to main-sequence masses. Because it is not valid for non-main sequence stars, depending upon expected accuracy, it may also not be valid for an individual star. Therefore, it is the relation devised for estimating a typical mass from a typical luminosity, or vice versa.

Consequently, inter-related mass-luminosity (MLR), mass-radius (MRR), and mass-effective temperature relations (MTR) of Eker et al. (2018) are all statistical relations, which should not be treated like fundamental relations. While the bolometric correction – effective temperature relation (BC- T_{eff}) of Flower (1996); BC- T_{eff} and BC-mass relations of Eker et al. (2020, 2021b); Bakış & Eker (2022); Eker & Bakış (2023) are all statistical relations that should not be confused by fundamental relations; otherwise, unexpected results or upsetting errors in the computed quantities become unavoidable.

One cannot say “statistical relations are less valuable than fundamental relations”. On the contrary, a statistical relation could be more valuable; even more practical, or easier to use, e.g., calculating the total energy of a gas as the mean energy per particle multiplied by the number of particles in the gas. Otherwise, the probability of each particle having a certain kinetic energy and the number of particles having this energy are required before integrating them over all possible kinetic energies. Similarly, using an MLR is more practical for determining the masses and luminosity of galaxies, which are the key parameters for determining dark matter in galaxies. Otherwise, to obtain the total mass and total luminosity of the galaxy in question, one would have to add up the individual masses and luminosities of the stars in the galaxy, which is impractical.

Moreover, there could be various astrophysical studies demanding MLR, MRR, and MTR in addition to stellar astrophysics. Live examples such as cometary research (Wysoczańska et al. 2020a,b), Oort clouds (Baxter et al. 2018), heliophysics and planetary habitability (Schrijver et al. 2019), exoplanet investigations (Berger et al. 2020; Arora & Hasegawa 2021; Burt et al. 2021; Caballero et al. 2022; Dattilo et al. 2023), planetary nebula (Munday et al. 2020; Aller et al. 2020), open clusters (Ilin et al. 2021; Akbulut et al. 2021; Yontan et al. 2021, 2023; Yontan 2023), dark matter searches (Garani & Palomares-Ruiz 2022; Peled & Volansky 2022) and quasars (Albert et al. 2021), neutron stars (Yuan et al. 2022), black holes (Gomel et al. 2021a,b), general relativity (Lalremruati & Kalita 2021), gravitational lensing (Ramesh et al. 2022; Pietroni & Bozza 2022) and even search for extra-terrestrial intelligence (SETI, Kerins 2021; Kerins et al. 2023), which all used at least

one of the statistical relations MLR, MRR and MTR of Eker et al. (2018).

Apparently, differentiation between fundamental and statistical relations is important from an astrophysical perspective. Recognizing statistical relations is even more important; otherwise, using them as fundamental relations would be misleading. Unfortunately, some authors such Malkov (2003, 2007), Henry (2004) and Gafeira et al. (2012) including Eker et al. (2015) did not hesitate to call MLR “fundamental law”, “sufficiently fundamental to be applicable to many areas of astronomy”, “one of the most famous empirical law” and “one of the fundamental secrets of the cosmos”. On the contrary, Andersen (1991) and Torres et al. (2010) preferred not to define a MLR and preferred displaying the $\log M - \log L$ diagram without a function (MLR) fitting to the data because the scatter on the diagram is not due to observational random errors but most likely due to abundance and evolutionary effects. Andersen (1991) claims “...departures from a unique relation is real”. If there is no unique function to represent the data on the diagram, why bother to defining one?

Therefore, this review article is dedicated to investigating the evolution of the statistical functions MLR, MRR, and MTR, starting from their discovery until today, and attempting to explain why such quarrels occurred and whether it is possible to resolve conflicts by identifying the nature of the relation. Moreover, the presentation of their conceptual advances and realising whether they were perceived as fundamental laws or statistical relations will benefit to the astronomical community. Developing such a conscious analysis would result in a better understanding of their previous usages and future studies.

2. OVERVIEW

Calibrations of MLR, MRR, and MTR require predetermined accurate stellar parameters such as mass (M), luminosity (L), radius (R), and effective temperature (T_{eff}). The most critical parameter among them is L because it is not an observable quantity; that is, neither a telescope nor a detector to observe the total radiation of a star at all frequencies (Bakış & Eker 2022; Eker & Bakış 2023). Fortunately, there are only one direct and two indirect methods to calculate L of a star (Eker et al. 2021c). The first method is a direct one because it uses independently determined observational R and T_{eff} of a star to calculate its L directly from its radiating surface area ($4\pi R^2$) and bolometric flux (σT_{eff}^4), which is commonly known as the Stefan-Boltzmann law ($L = 4\pi R^2 \sigma T_{\text{eff}}^4$). The other two methods are indirect because the first method, which provides L of a star from its M , requires a predetermined classical MLR in the form of $L \propto M^\alpha$, while the second method, which supplies L of the star from its apparent brightness and distance requires a pre-determined BC- T_{eff} relation. Unfortunately, the indirect methods are useless without their predetermined relations, and thus, one has no other choice but to use the Stefan-Boltzmann

law to produce the earliest sample of L values for the first calibration of MLR, MRR, and MTR.

Assuming that the sample L values are ready, the next most critical parameter is stellar M to establish not only MLR but also MRR and MTR because sequentially required R and T_{eff} should have been already used in computing L of the sample stars. However, like L of a star, M of a star is also not observable. This fact makes the parameter M even more critical because only the masses of binaries (or multiple systems) could be calculated using Kepler's Third Law from the observed orbital semi-major axis and the orbital period, which could be deduced from the observed orbits of visual binaries (or multiple systems) or from the radial velocity curves of double-lined eclipsing spectroscopic binaries. Stellar spectra do not provide orbital inclinations; thus, spectroscopic binaries without eclipses cannot provide component masses unless orbital inclinations are available independently.

Eclipsing binaries are ideal objects for collecting observed radii of stars, whereas the single stars are null, except the ones close in distance and large enough where interferometry could be useful. Although binarity adds complications to the estimation of component effective temperatures, the eclipsing binaries are still advantageous for revealing the most accurate effective temperatures and temperature ratios from the depths of minima if the effective temperature for one of the components is estimated correctly.

Visual binaries (multiple systems too) and double-lined eclipsing binary systems are the only objects provided M from Kepler's third law, with R from eclipses. On the other hand, there is almost no other way to obtain accurate masses and radii of single stars except seismic analysis where mass and radii data for solar-like pulsating stars (e.g. [Gaulme et al. 2016](#); [Bellinger et al. 2019](#)) are obtained. Considering some basic information and data knowledge about this research topic, we must now start reviewing the statistical relations starting from the most prominent one: MLR.

2.1. Revisiting MLR

The famous stellar mass-luminosity relation (MLR) was discovered empirically in the middle of the first half of the 20th century by [Hertzsprung \(1923\)](#) and [Russell et al. \(1923\)](#) independently using masses and absolute brightness of a very limited number of visual binaries. Eclipsing binaries were included later in the statistics. In his MLR, [Eddington \(1926\)](#) was able to use 13 eclipsing binaries together with 29 visual binaries and five Cepheids, which were available to him at that time, while [McLaughlin \(1927\)](#) increased the number of eclipsing systems to 41 in his plots.

By the time of the mass-luminosity relation was discovered, there were only three kinds of brightness measurements: 1) Visual magnitudes (m_v) from the brightness observations of stars by eye. 2) Photographic magnitudes (m_p) measured from the sizes of star images on photographic plates. 3) Bolometric

magnitudes (m_b) as heat measurements, which could be coming from galvanometric, bolometric, or radiometric observations of stars. It was thought that bolometric magnitudes coming from the heat measurements of stars represent the total radiation output (luminosity) best. Therefore, assuming that stars are hot spheres of gases, [Eddington \(1926\)](#) formulated a theoretical relation between the mass and absolute bolometric magnitude of a star. Furthermore, he continued to confirm his theoretical relation using the existing data [Eddington \(1926, page 154\)](#) from binaries. [Gabovits \(1938\)](#) agreed [Eddington \(1926\)](#) after twelve years by re-examining the mass- M_{Bol} data again and declared "We conclude that, as revealed by our selected first-class data, the stars (chiefly of the main-sequence) probably follow a strict mass-luminosity law". On the other hand, having larger data sets including visual binaries, spectroscopic binaries, Hyades and Trumpler stars, and white dwarfs, [Kuiper \(1938\)](#) was rather suspicious of accepting the mass-luminosity relation as a law because he commented "It is doubtful whether this mean relation has any physical significance" after discussing it on a $\log M - M_{\text{Bol}}$ diagram.

MLR has been updated and revised many times until a major break occurred at the very beginning of the last decade of the 20th century on the issue of whether it is a statistical relation or a fundamental law. Looking at the developments before this break would be useful to understand it better. [Petrie \(1950a,b\)](#) used 93 spectroscopic binary systems, [Strand & Hall \(1954\)](#) studied 23 visual binaries, [Eggen \(1956\)](#) investigated 34 visual binaries, [McCluskey & Kondo \(1972\)](#) considered 40 visual binaries and 35 eclipsing systems, [Cester et al. \(1983\)](#) gathered 45 visual and 40 spectroscopic binaries, [Griffiths et al. \(1988\)](#) analysed 72 detached main-sequence binaries, 25 detached OB, 6 resolved binaries and 23 visual binaries when revising the MLR relations. [Demircan & Kahraman \(1991\)](#) preferred to study masses and luminosities of 70 eclipsing binaries (140 stars) only, including the main sequence components of detached and semi-detached binaries as well as the components of OB-type contact and near contact binaries. [Karetnikov \(1991\)](#) used 303 eclipsing systems of different types. At this point, we must keep in mind that the observational data was very limited, therefore, the authors combined different kinds of binaries, whether they are eclipsing or not, without differentiating between detached, semi-detached, and contact systems. This was done to increase the statistical reliability of MLRs calibrated.

Those early generation relations, including the very earliest ones, were demonstrated mass-absolute magnitude diagrams mostly, some with the best fitting function, and some without a fitting curve. Considering the classical form $L \propto M^\alpha$ of MLR, first [Eggen \(1956\)](#) intended to define the power of mass (α) so he expressed $L = \mu^{3.1}$, where μ is the total mass of a double-star system defined as $\mu = a^3/P^2\pi^3$ from the Kepler's Harmonic Law. Then, [McCluskey & Kondo \(1972\)](#) preferred to use a relation in the form $M \propto L^\beta$, where M and L are the masses and luminosities of the components while α and β are the constants to be determined by the data on various mass-

absolute bolometric magnitude diagrams. Accordingly, Cester et al. (1983); Griffiths et al. (1988) and Demircan & Kahraman (1991) preferred to study MLR on a mass-luminosity diagram for defining unknown constants on the classical form ($L \propto M^\alpha$) of MLR either by fitting a curve to all data or dividing the mass range into two regions as low- and high-mass stars, or three regions as the high, intermediate or Solar, and low-mass stars, in order to determine the inclinations (power of M) and zero point constants of the linear MLRs on the $\log M - \log L$ diagram.

2.1.1. Is MLR a Fundamental Relation?

The major break on the concept and splitting practises on the purpose of calibrating an MLR was triggered by Andersen (1991) who collected 45 detached double-lined eclipsing binary (DDEB) systems (90 stars) having both masses and radii accurate within 2%, which were the most accurate stellar data of the time. Selecting only DDEB stars was not intended to reject the reliability of the other sources for determining masses and radii; only because M and R from DDEB were considered the most reliable.

Andersen (1991) rejected calibrating any form of MLR function to fit data on the $\log M - \log L$ diagram. The diagram displayed by him is without a fitting curve (MLR) because the scatter from the curve is certainly not only due to random observational errors but also due to abundance and evolutionary effects. He exclaimed: "At first glance, the mass-luminosity diagram shows a tight, well-defined mean relation. Closer inspection, taking individual uncertainties into account, reveals that departures from a unique relation are real". Then, if there is no unique curve expressible by a function to represent data, why bother drawing one?

In very early times, especially after its discovery, MLR was claimed to be one of the most prominent empirical laws of nature by Eddington (1926) and Gabovits (1938) and commonly used by researchers either reckoning L of a star from its M or estimating M of a star from its L , at least until the middle of the 20th century, and perhaps until Andersen (1991). This practice was extremely useful for single stars because there was no other observational way to directly access their masses, but the absolute bolometric magnitudes of the ones with known parallaxes were rather easy to obtain using a proper bolometric correction (BC), and then their masses were estimated from a predetermined bolometric magnitude-mass relation. The method was even helpful in double-checking trigonometric parallaxes of single stars and even eclipsing binaries with light curve solutions giving R and T_{eff} of the components, thus L values are compatible by the Stefan-Boltzmann law. Those very early times were a period in astrophysics when nuclear reactions were not fully established and understood correctly. The discovery of nuclear reactions in the cores of stars is attributed to Sir Arthur Eddington (Bahcall 2000) a few years before the discovery of MLR by Hertzsprung (1923) and Russell et al. (1923).

Astronomers waited until 1932 for the discovery of neutrons

(Chadwick 1933) to study and fully understand hydrogen fusion as a main source of stellar energy. Only after the CNO cycle was established by Hans Bethe and Von Weizsacher (Clayton 1968), and only a year later, the p-p chain reactions were suggested by Bethe (1939), there were the solutions of stellar structure equations with the nuclear energy, which placed our theoretical understanding of the evolution of stars with evolutionary tracks on a solid ground (Clayton 1968). The empirically discovered MLR was confirmed later theoretically in the sense that mass (M) is the prime parameter that determines the internal structure, size (R), and luminosity (L) of a star not only for the time span of the main sequence but also throughout the star's lifetime until its death, where its initial chemical composition can cause little variation. Therefore, the scatter on a $\log M - \log L$ diagram for field main-sequence stars is not only due to observational errors but also due to various ages and chemical compositions.

Additionally, at least until the middle of the 20th century, maybe until about the middle of its second half, but certainly not until Andersen (1991), the observational accuracy was not high enough to differentiate theoretical (true) L and M of stars, which are marked on a Hertzsprung Russell (H-R) diagram to form evolutionary tracks. As long as the observational accuracies of L and M are much lower than the uncertainty of error bars covering the full thickness of the main sequence, astronomers did not suspect inconsistency between the predicted and observed quantities using an MLR. It was normal for them to be satisfied, as in the case of fundamental law.

By the time of Andersen (1991), who collected the most accurate M and L data from DDEB, there were sufficiently accurate masses and luminosities; thus, one can deduce that the scatter on the $\log M - \log L$ diagram is not only due to observational errors of M and L but also due to abundance and evolutionary effects. Andersen (1991) felt something was not right, that is, there must be a problem in treating classical MLR in the form $L \propto M^\alpha$ as a fundamental relation. Andersen thought that such a fundamental relation must not only fit the data uniquely but also must contain the other parameters involving the chemical composition and evolution.

Being influenced by the common usage, a kind of paradigm not easy to be free off, Andersen (1991) was expecting, like the others before him, to obtain the mass of a single star from its luminosity within acceptable accuracy [ideally $\pm 5\%$ see Andersen (1991) page 93]. However, with a mean MLR in the form $L \propto M^\alpha$ for the main-sequence stars, this was not possible anymore due to improved observational techniques with higher accuracy in both L and M values available to him. Nevertheless, he succeeded in this aim about two decades later by replacing $\log L$ with a function including observational parameters T_{eff} , surface gravity ($\log g$), and relative iron abundance $[\text{Fe}/\text{H}]$ to incorporate evolution (age) and chemical composition as

$$\log M = a_1 + a_2 X + a_3 X^2 + a_4 X^3 + a_5 (\log g)^2 + a_6 (\log g)^3 + a_7 [\text{Fe}/\text{H}], \quad (2)$$

in a new review paper, in which [Torres et al. \(2010\)](#) was the leading author; where $X \equiv \log T_{\text{eff}} - 4.1$ as a parameter that contains the most effective variable, T_{eff} , related to the luminosity with its fourth power in the Stefan-Boltzmann law, and a 's which are the calibrated coefficients $a_1 = 1.5689 \pm 0.058$, $a_2 = 1.3787 \pm 0.029$, $a_3 = 0.4243 \pm 0.029$, $a_4 = 1.139 \pm 0.240$, $a_5 = -0.1425 \pm 0.011$, $a_6 = 0.01969 \pm 0.0019$, and $a_7 = 0.1010 \pm 0.058$. This is a function providing 6.4% accuracy for main-sequence and evolved stars above $0.6M_{\odot}$. It is obvious that such a function cannot be drawn on a $\log M - \log L$ diagrams but provides stellar M as the classical MLR but is much more accurate not only for main-sequence stars but also for giants and sub-giants. Thus, both [Andersen \(1991\)](#) and [Torres et al. \(2010\)](#) presented their $\log M - \log L$ diagrams without a fitting function.

Incorrect diagnoses are natural to be continued by incorrect treatments. First, classical MLR in the form of $L \propto M^{\alpha}$ is a statistical relation devised for estimating a typical mass from a typical luminosity, or vice versa, and not for estimating M or R of a single star, even if it was once used to estimate M of a star from its L . Therefore, It is not right to diagnose the classical MLR as one of the fundamental relations to calculate the mass of a star from its luminosity. It is not right to look for a relation that gives the mass of a star from other stellar observational parameters for replacing MLR. It is not right to call this relation MLR if it provides M of a star from the other parameters even if L is included in the right-hand side of the equal sign, except if the right-hand side contains only L as a variable. It is not right to claim that there is no uniquely fitting function to the data on $\log M - \log L$ the diagram despite a tight, mean relation between M and L as if the least squares method would fail to produce one.

Nevertheless, [Andersen \(1991\)](#)'s exclamation had a noticeable consequence in stellar astrophysics. It appears as if a main cause of deviation in understanding, definition, and usage of newly defined MLR functions from the main path, which was continued by [Ibanoglu et al. \(2006\)](#) and [Eker et al. \(2015, 2018\)](#) where statistical relation between masses and luminosities of main-sequence stars was kept in the form $L \propto M^{\alpha}$.

2.1.2. Mass Predicting Relations of the Deviated Path

[Andersen \(1991\)](#)'s objection was very effective in the literature that some authors ([Malkov 2003, 2007](#); [Torres et al. 2010](#)) also presented their empirical mass-luminosity diagrams without a curve fitting to the data. [Gafeira et al. \(2012\)](#) just gave mass-luminosity relations for main sequence FGK stars without displaying them on a mass-luminosity diagram. [Fernandes et al. \(2021\)](#), which compared results to [Torres et al. \(2010\)](#), could be considered as a new improved version of [Gafeira et al. \(2012\)](#). [Fernandes et al. \(2021\)](#), thus, did not display mass-luminosity relations on the mass-luminosity diagram.

Various kinds of mass-predicting empirical relations were calibrated, and most of them were erroneously called MLR.

[Gorda & Svechnikov \(1998\)](#) choose the form $M_{\text{Bol}} = a + b \log M$, where M_{Bol} is the absolute bolometric magnitude and M is the mass. [Henry & McCarthy \(1993\)](#) preferred $\log M = aM_{\xi} + b$ for infrared colours, where M_{ξ} indicates absolute magnitudes at the J , H , and K bands, and $\log M = aM_V^2 + bM_V + c$ for the V band to express various MLR with unknown coefficients a , b , and c to be determined by the data on various diagrams. The former relation in the form $M_{\text{Bol}} = a + b \log M$ could be justified to be named mass-luminosity relation since M_{Bol} of a star is directly related to its luminosity, but a relation in the form $\log M = aM_{\xi} + b$, is definitely not a mass-luminosity relation. Such relations should be called mass-absolute brightness relations to avoid confusion.

The style of expressing V -band mass-absolute brightness relation as second-degree polynomials covering masses $0.6 M_{\odot}$ to $22.89 M_{\odot}$ is continued by [Xia & Fu \(2010\)](#) who calibrated it within two regions: one for $M_V < 1.05$ mag ($2.31 < M/M_{\odot} < 22.89$) and other for $M_V > 1.05$ mag ($0.60 < M/M_{\odot} < 2.31$) using the dynamical masses and V -band absolute magnitudes of 203 main-sequence stars, but the relation is still called MLR. The accuracy of the predicted masses is estimated to be within 5%. The style of displaying mass (M/M_{\odot}) versus absolute magnitude data in which the absolute magnitudes are in the visual and the K -bands is continued by [Benedict et al. \(2016\)](#) for the low end of the main sequence ($M < 0.6M_{\odot}$) but in different functional forms predicting masses as accurate as $\pm 0.035M_{\odot}$ in the region $M = 0.2M_{\odot}$. Similarly, both direct (absolute magnitude for a given mass) and inverse (mass for a given absolute magnitude) relations of [Benedict et al. \(2016\)](#) are better known as empirical mass-brightness relations rather than mass-luminosity relations. The empirical data of [Benedict et al. \(2016\)](#) are also compared to the theoretical mass-absolute brightness curves of age 1 Gyr of [Baraffe et al. \(2015\)](#) and [Dotter \(2016\)](#). K -band data fit better than the V -band data.

Mass-Absolute visual magnitude [$M_V - (\log m)$], mass-luminosity [$\log L - (\log m)$], mass-temperature [$\log T_{\text{eff}} - (\log m)$], and mass-radius [$\log R - (\log m)$] relations and corresponding inverse relations are calibrated for the intermediate-mass stars in the mass range $1.4 < M/M_{\odot} < 12$ by [Malkov \(2007\)](#) in forms of polynomials with empirically determined coefficients from the fundamental parameters of stars collected from DDEB and visual binaries. Similar empirical relations were studied later from a wider perspective by [Moya et al. \(2018\)](#) using a set of observational parameters M , R , T_{eff} , g (surface gravity), ρ (mass density), and $[\text{Fe}/\text{H}]$ of 934 stars of eclipsing binaries and single stars observed by asteroseismology and interferometry where two thirds of the stars are on the main sequence. A total of 576 linear combinations of T_{eff} , L , g , ρ , and $[\text{Fe}/\text{H}]$ (and logarithms) were used as independent variables to estimate M and R , but [Moya et al. \(2018\)](#) presented only 38 of them with regression statistics $\text{adj-}R^2$ higher than 0.85. Accuracy better than 10% was achieved in almost all cases of 38 equations. The term ‘‘empirical relations’’ used by [Moya et al. \(2018\)](#), and the names of relations given by [Malkov \(2007\)](#)

are proper and consistent in comparison to the names used for the mass predicting relations in the previous paragraph.

Using $\log L/L_\odot$, $[\text{Fe}/\text{H}]$, and star age/age $_\odot$ as free variables, [Gafeira et al. \(2012\)](#) suggested three different equations so that one can choose one according to the availability of data. The simplest is first, which is a third-degree polynomial without a constant term, having X , where $X \equiv \log L/L_\odot$. The second equation adds $[\text{Fe}/\text{H}]$ as a new variable in another third-degree polynomial into the equation in addition to the first. The third equation combines three third-degree polynomials, where the third uses the relative age (age/age $_\odot$) of the star as a variable. With all the parameters, the third relation looks like equation (2) given above by [Torres et al. \(2010\)](#). That is, the part $a_1 + a_2X + a_3X^2 + a_4X^3$ is replaced by $0.0219(\pm 0.023) \log L/L_\odot + 0.063(\pm 0.060)(\log L/L_\odot)^2 - 0.119(\pm 0.112)(\log L/L_\odot)^3$ while the part $a_7[\text{Fe}/\text{H}]$ is replaced by $+0.079(\pm 0.031)[\text{Fe}/\text{H}] - 0.122(\pm 0.119)[\text{Fe}/\text{H}]^2 - 0.145(\pm 0.234)[\text{Fe}/\text{H}]^3$ at last the part $+a_5(\log g)^2 + a_6(\log g)^3$ is replaced by $+0.144(\pm 0.062)(\text{age}/\text{age}_\odot) - 0.224(\pm 0.104)(\text{age}/\text{age}_\odot)^2 - 0.076(\pm 0.045)(\text{age}/\text{age}_\odot)^3$. Adding age and metallicity improved the mass estimation (15% to 5%) for FGK stars ([Gafeira et al. 2012](#)). However, only the first equation could be called MLR, but not the other two, which are better to be called mass-luminosity-metallicity relation and mass-luminosity-metallicity-age equation, respectively. Nevertheless, [Fernandes et al. \(2021\)](#) corrected this by changing the name “the mass and radius-luminosity-metallicity-age relations” in the new version, which was calibrated by 56 stars with metallicity and mass in the ranges $-0.34 < [\text{Fe}/\text{H}] (\text{dex}) < 0.27$ and $0.66 < M/M_\odot < 1.8$ from the DEBCat catalogue¹. The estimated accuracy in the new version is 3.5% and 5.9% in predicting M and R of single stars, respectively. [Serenelli et al. \(2021\)](#) commented: “[Gafeira et al. \(2012\)](#) provided three relations for the stellar mass, but only two of them can be easily applied”.

[Serenelli et al. \(2021\)](#) studied mass determination methods from a wider perspective, including all existing techniques older and newer; involving spectroscopy and/or photometry; theoretical or observational and at last summarised them in a figure (Figure 16 of [Serenelli et al. 2021](#)) where one can see their applicable ranges of stellar mass as well as the accuracy and precision of the predicted masses. Despite deceiving examples as summarised in the three paragraphs above, [Moya et al. \(2018\)](#) were careful not to use the word “MLR” for naming their “empirical relations”, which are established to estimate M and R only. While [Fernandes et al. \(2021\)](#) constantly called their similar functions MLR and claimed: “for single nearby Solar-type stars, the luminosity can be obtained observationally, but not the mass”, opposing the fact that the luminosity of a star is actually not an observable parameter because there is no telescope/detector to observe at all wavelengths. Call-

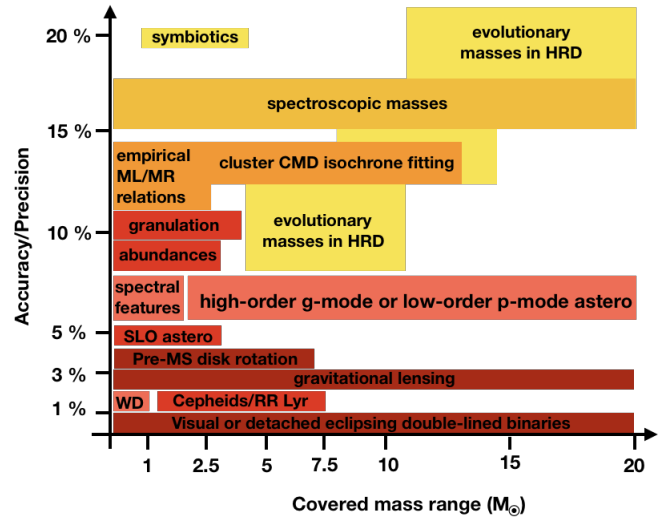


Figure 1. Mass ladder summarising the capacity of various methods to obtain stellar masses (credit to [Serenelli et al. 2021](#)).

ing these M or R predicting relations ML/MR would be a main source of confusion among readers. For example, “empirical ML/MR relations” marked in Figure 16 of [Serenelli et al. \(2021\)](#) at an accuracy/precision in between 10% and 15% for stars $M < 2.5M_\odot$, wherein the caption “ML/MR for mass-luminosity and mass-radius relations” is written clearly, could be confusing to a careful reader.

This is because: Is it possible for [Serenelli et al. \(2021\)](#) using ML/MR to indicate both the empirical relations of [Moya et al. \(2018\)](#) and the classical mass-luminosity/mass-radius relations (MLR/MRR) of [Eker et al. \(2018\)](#)? According to Figure 1 (Figure 16 of [Serenelli et al. 2021](#)) the answer would be “yes” because the relations of [Moya et al. \(2018\)](#) are not marked as “empirical relations” on the figure, and it is not possible to forget them because they cover one of the very important sections of the review (Section 4.4), where the M predicting relations of other authors are summarised and compared to their counterparts in [Moya et al. \(2018\)](#) (see Table 1). However, [Serenelli et al. \(2021\)](#) declared “All the relations except two (those with the largest number of dimensions) have precision better than 5%”. If the answer is “yes”, then, another problem arises: “If ML/MR in Figure 1 implies both classical MLR/MRR and the new M/R predicting relations, why the precision for the ML/MR is marked to be from 10% to 15% while M/R predicting relations are said to have precisions better than 5%?”.

It is possible that ML/MR in Figure 1 were actually marked from 3% to 15%, but this region on the figure is shadowed by various other methods of predicting stellar mass. Then another conceptual problem comes out: Only similar quantities can be compared. it is not right to compare the accuracy/precision of the empirical relations of [Moya et al. \(2018\)](#), which are solely devised for predicting masses of single stars, to the accuracy/precision of the classical MLR, which is primarily devised to establish as a statistical relation between typical masses and luminosities of main-sequence stars. It is inconsistent and

¹ <https://www.astro.keele.ac.uk/jkt/debcats/>

Table 1. Mass predicting empirical relations of other authors are compared with their counterparts in Moya et al. (2018) (credit to Serenelli et al. 2021).

Ref.	Relation	Acc/Prec	Ref.	Corresponding relation	Acc/Prec
T10	$M = f(X, X^2, X^3, \log^2 g, \log^3 g, [\text{Fe}/\text{H}])$	7.4/52.9	M18	$M = f(T_{\text{eff}}, \log g, [\text{Fe}/\text{H}])$	7.5/3.4
G12	$M = f(\log L, \log^2 L, \log^3 L)$	14.0/0.6	M18	$\log M = f(\log L)$	10.1/0.1
G12	$M = f(\log L, \log^2 L, \log^3 L, [\text{Fe}/\text{H}], [\text{Fe}/\text{H}]^2, [\text{Fe}/\text{H}]^3)$	8.9/0.8	M18	$\log M = f(\log L, [\text{Fe}/\text{H}])$	9.9/0.9
M07	$M = f(\log L, \log^2 L)$	11.2/—	M18	$\log M = f(\log L)$	10.08/0.13
E18	$\log L = f(\log M)$	33.3/6.9	M18	$\log L = f(\log M)$	31.9/0.6

References: T10 (Torres et al. 2010), G12 (Gafeira et al. 2012), M07 (Malkov 2007), E18 (Eker et al. 2018), M18 (Moya et al. 2018).

meaningless to compare E18 with M18 in Table 9 of Serenelli et al. (2021) (see Table 1). Similarly, it is scientifically inconsistent if empirical ML/MR in Figure 1 indicates classical MLR/MRR as well, rather than the empirical relations of Moya et al. (2018). This is because Eker et al. (2018) declared: “(classical) MRR and MTR functions, as well as the MLR functions, are needed by the astronomical community for practical purposes. These include the need to be able to estimate a typical luminosity, radius, and T_{eff} for main-sequence stars of a given mass”. Which is definitely different from the purpose of predicting M or R of single stars all over the H-R diagram?

After having M and R predicting relations, further calculations are required for a meaningful comparison. First, calculate M , R , and then L (if not used as input variable) for main-sequence stars only. Then, plot $\log M - \log L$ and $\log M - \log R$ diagrams and fit the preferred functions (ML or MR) using the least squares method. Finally, ML/MR becomes comparable to MLR/MRR of Eker et al. (2018). Why bother doing that, if there are many reliable direct methods of obtaining M , R , and L of main-sequence stars within accuracy 1% and a few % respectfully from radial velocity and light curves of DDEB stars, which are marked in the bottom of Figure 1 covering the full ranges of stellar masses. Thus, ML/MR is incompatible with classical MLR/MRR (Figure 1).

2.1.3. Recognition of statistical MLRs

The classical form of MLR ($L \propto M^\alpha$) appreciated by Cester et al. (1983), Griffiths et al. (1988), Karetnikov (1991) and Demircan & Kahraman (1991) before the break initiated by Andersen (1991). This classical form continued by Ibanoglu et al. (2006) when comparing mass-luminosity relations for detached and semidetached Algols despite Andersen (1991)’s exclamation. The most recent examples are by Eker et al. (2015, 2018). The classical or any other form reducible to the classical ($L \propto M^\alpha$) has the advantage of being easy to interpret with a value given to the power of M , which is known to change by the energy generation rate per star mass at the cores of stars. Thus, the derivative of MLR function (dL/dM), the inclination of a line, on the $\log M - \log L$ diagram is the value of alpha. On the other hand, the real advantage is not only that it works in both directions (M from L , or L from M), but also because it permits

one to relate typical masses and luminosities of main-sequence stars in general.

Obviously, the statistically determined relationship between M and L for the main-sequence stars is not true for a single star. This is because L , R , and T_{eff} of a star change with time, while M of the star stays constant (an evolutionary effect) because mass loss of main-sequence stars (especially for the ones cooler than B spectral types) is too small; thus, mass loss is usually ignored (Daszyńska-Daszkiewicz & Miszuda 2019; Bressan et al. 2012). Moreover, there is a metallicity effect that also changes L , R , and T_{eff} slightly, therefore, stellar structure and evolution models use chemical composition and M as the two basic free parameters for computing L , R , and T_{eff} , which stands for the output of an internal structure model to be confirmed externally. Finally, it can be concluded that any kind of relation between M and L of a single star cannot easily be deduced from evolutionary tracks or isochrones. However, the distribution of available stellar luminosities on H-R diagrams or on $\log M - \log L$ diagrams clearly shows that there must exist, at least a statistical, one-to-one relation between a typical mass and typical luminosity of main-sequence stars. However, if L and M were totally independent, identification of main-sequence stars on the H-R diagram would not be possible. In fact, the main-sequence stars were first recognised just according to their extraordinary positional appearance, even on very primitive H-R diagrams. Because of their distinct positions, they were named main-sequence stars, which are still actively used. After all, it is obvious to everyone now that as soon as main-sequence luminosities are placed on a $\log M - \log L$ diagram, a one-to-one relation between M and L shows itself clearly with high-level statistical significance.

What has actually happened so far is that; the statistical relation between stellar M and L for main-sequence stars is so strong and obvious that it was discovered even before the stellar structure and evolution theory was fully established. That was the first reason why it had been evaluated as one of the fundamental secrets of the cosmos, similar to the H-R diagram itself (why are stars not distributed evenly on the surface of the H-R diagram but mostly gathered on the main sequence?). The second reason was that the observational accuracy of those early years of astrophysics was not sufficient to distinguish it as

a statistical relation; therefore, it was treated as a fundamental relation even though some serious objections mentioned above have occurred. Nevertheless, no one has yet clearly declared that MLR is just a statistical relation.

Andersen (1991) totally rejected the existence of a real relation between M and L of the main-sequence stars because, according to him, a unique relation (like a Planck law, keeping T_{eff} constant, the deviations from the Planck curve are only due to random errors of observed intensities) does not exist since deviations from a fitting curve (MLR) is not just only due to random observational errors of L and M , but also due to chemical composition and age differences. Eker et al. (2015, 2018) defended the existence of an MLR relation by claiming that any sample of data could be expressed by a unique relation because a unique fit of a curve on sample data is guaranteed by the least squares method.

However, this defence occurred intuitively because the distinction between MLR as a fundamental or statistical relation was not yet fully established. Even Eker et al. (2015) said “One of the fundamental secrets of the cosmos, the famous stellar mass–luminosity relation (MLR), was discovered empirically ...”. Only later, Eker et al. (2018) declared “The main-sequence MLR is one of the fundamentally confirmed and universally recognised astronomical relations”.

How to solve non-uniqueness problems attributed to the light curves of spotted stars, which is a type of problem Andersen (1991) pointed out against MLR, was discussed first by Eker (1999). The same principles adopted by Eker et al. (2018) for defending the uniqueness of MLR, MRR, and MTR functions from the most accurate masses and luminosities of 509 main-sequence stars as the components of DDEB in the Solar neighbourhood of the Milky Way. There could be three types of non-uniqueness problems. According to Eker et al. (2018), there is no non-uniqueness problem of type I because the preferred function expressing MLR is a power law ($L \propto M^\alpha$). There should not be a non-uniqueness problem of type II also because there exist many methods like the least squares for one to achieve a unique fit and determine its coefficients uniquely. Finally, the problem of type III does not exist either because the parameter space of MLR is so simple that there is always one correlation between L and M , that is, there is only one L value for a given M and vice versa in the case of inverse MLR. The most general approach to non-uniqueness problems was recently applied to main sequence BC- T_{eff} relations by Eker et al. (2021b) when discussing the chronic zero-point problems of the BC scale, which is important for obtaining accurate L of single stars from apparent magnitudes if distances are known.

Apparently, there is no non-uniqueness problem associated with classical MLR, but one may still run into problems of obtaining accurate L from a given M or vice versa. Such problems are inevitable because classical MLR is not devised solely for obtaining accurate L from an accurate M or vice versa. The main obstacle for obtaining accurate M of a single star from its L using classical MLR is not because of the absence of a

unique function (Andersen 1991), but because of the degeneracy induced by stellar structure and evolution theory (Eker et al. 2018). This is because, theoretically, there are an infinite number of L values for a main-sequence star of given M depending on its chemical composition and age. Thus, varying chemical composition and ages are not necessary to cause non-uniqueness to an existing MLR. On the contrary, the chemical composition and age of a star break inferred degeneracy to obtain accurate L , M , and R , which appear on classical MLR and MRR functions as only variables. Therefore, one must conduct further investigation on evolutionary tracks. Knowing the mass and chemical composition of a star, the correct track will be chosen. By knowing the age, both L and R will be read on the track. If the age is not known, then either L or R must be known to read the age of the star on the track. Stellar structure and evolution theory is useful for determining the accurate mass of a single star from its accurately determined $[\text{Fe}/\text{H}]$ and L , where the value of R could also be obtained if T_{eff} of the star is known accurately.

Without distinguishing them from a fundamental relation, Eker et al. (2015) calibrated four linear MLR functions covering the mass range $0.38 < M/M_\odot < 32$, using masses and luminosities of 268 main-sequence stars selected from 514 stars as the components of 257 DDEB from the catalogue of Eker et al. (2014). Three distinct break points separating the four distinct mass domains on a $\log M - \log(L/M)$ diagram were identified. The mass domains were named as low mass ($0.38 < M/M_\odot \leq 1.05$), intermediate mass ($1.05 < M/M_\odot \leq 2.4$), high mass ($2.4 < M/M_\odot \leq 7$), and very high mass ($7 < M/M_\odot \leq 32$). The linear MLRs of the four mass domains were compared to linear and quadratic MLR of the full range $0.38 < M/M_\odot < 32$ and the four-piece linear MLRs were found to best represent the data on a $\log M - \log L$ diagram. The break points separating the mass domains were interpreted as abrupt changes in the power of M , most probably due to changes in the type of efficient nuclear reaction operating in the cores of main-sequence stars.

The statistical nature of MLR was sensed but not fully grasped yet by Eker et al. (2018). The referees were against calibrating new MLRs, which are useless in predicting M and R of single stars. The new trend in determining MLRs was to use them to obtain M and R of single stars, as done by Andersen (1991), Henry (2004), Malkov (2007), Torres et al. (2010), and Gafeira et al. (2012). Therefore, the justifications for recalibrating them again after only three years were explained by Eker et al. (2018) as: There are two tables in the handbook of astronomers, commonly known as “Allen’s Astrophysical Quantities” (Cox 2000). The first of the two tables lists calibration of MK spectral types with seven columns spectral type, $M(V)$, $B - V$, $U - B$, $V - R$, $R - I$, T_{eff} and BC (Table 15.7 on page 388). The second table with six columns spectral type, mass, radii, surface gravity, mean mass density, and rotational speed (Table 15.8 on page 389) is actually at continuation of the first table. The columns of the two tables were thus connected

by spectral types. The second table has a notification mark to say “columns containing uncertain values”, which actually indicates statistically determined typical masses, radii, surface gravity, and mean mass for a given spectral type. The rationale of the new paper was obvious that interrelated MLR, MRR, and MTR would be very useful in supplying reliable statistical information to the astronomical community.

After three years, Eker et al. (2018), this time, calibrated a six-piece MLR covering the mass range $0.179 < M/M_{\odot} < 31$, using 509 main-sequence stars selected from 639 stars as the components of 318 pairs (DDEB) and one detached spectroscopic triple. One should not be surprised by the higher increasing rate of the calibrating stars, which appears to be 90% while the number of newly added DDEB binaries is only 24%. This is because stars with less accurate M and R up to 15% were used in the calibrations rather than up to 3% as in the previous study. Thus, Eker et al. (2018) concluded that it is not always good to eliminate less accurate data for a better study because it may mean loss of information rather than gain. Lowering the limiting relative accuracy to 15% was a significant contribution to extending the low mass limit down to $0.179M_{\odot}$ from $0.38M_{\odot}$ and discovery of two more break points on the $\log M - \log L$ diagram. That is, adding new DDEB stars (24%) to the list was not as effective as lowering the limiting accuracy.

The distribution of the luminosities of 509 main-sequence stars on a $\log M - \log L$ diagram is shown in Figure 2, where the vertical lines mark the positions of the break points. Between the break points, there is a linear MLR luminosity of that domain; thus, there exists a six-piece MLR to cover the full range $0.179 < M/M_{\odot} \leq 31$. A six-degree polynomial, shown by a blue dotted line to fit the full range, was found producing a best fit better than the higher and the lower degree polynomials to explain the full range data by a single function. It is obvious in Figure 2 that the six-piece MLRs are even better at representing stellar luminosity than any polynomial of any degree.

The six-mass domains, which are named ultra-low mass ($0.179 < M/M_{\odot} \leq 0.45$), very-low mass ($0.45 < M/M_{\odot} \leq 0.72$), low mass ($0.72 < M/M_{\odot} < 1.05$), intermediate mass ($1.05 < M/M_{\odot} \leq 2.4$), high mass ($2.4 < M/M_{\odot} \leq 7$), and very-high mass ($7 < M/M_{\odot} \leq 31$), and their best fitting MLR functions are shown in Figure 3, where they were also compared to a single MLR function of sixth degree covering the entire mass range $0.179 < M/M_{\odot} \leq 31$. Analytical expressions and statistical parameters N (number of stars in the domain), R^2 (correlation coefficient), σ (standard deviation), and α (the power of M) as the inclination of a linear MLR on a $\log M - \log L$ diagram are listed in Table 2.

2.2. Revisiting MRR

Although it appears to be natural to sense a relation between M and R of main-sequence stars immediately after the discovery of MLR by Hertzsprung (1923) and Russell et al. (1923), the

empirical MRR of main-sequence stars did not appear in the literature for another one and a half decades. Kuiper (1938), who was suspicious about MLR being one of the fundamental relations, also plotted a $\log M - \log R$ diagram but did not discuss it. Only after mid of the 20th century, the studies discussing empirical stellar mass-radius relation (MRR) begin to appear in the literature (McCrea 1950; Plaut 1953; Huang & Struve 1956; Lacy 1977, 1979; Kopal 1978; Patterson 1984; Gimenez & Zamorano 1985; Harmanec 1988; Demircan & Kahraman 1991).

If M and L are related, why not M and R , since L is already known to be related to the square of R according to the Stefan-Boltzmann law? The main obstacle seems to be the difficulty of accessing reliable stellar radii. McCrea (1950) appears to be one of the first founders of the idea for two reasons: *i*) Using the light curve solutions of eclipsing binaries, not necessarily all being detached, that is, including W UMa binaries too, Plaut (1953) has drawn a diagram in the manner proposed by McCrea (1950). *ii*) Readers were referred to McCrea (1950) for theoretical interpretation of the relation in the form $\log(M/R) = a + b \log M$ suggested by Plaut (1953). Using the parameters of 30 systems with both components on the main sequence chosen from 130 eclipsing binaries of all kinds compiled by Plaut (1953), the coefficients of the relation were rectified as $a = -0.058 \pm 0.026$, $b = 0.335 \pm 0.29$ with a dispersion of ± 0.11 according to the least squares.

The masses and radii of the eclipsing stars compiled by Plaut (1953) had been studied by Huang & Struve (1956), who plotted them on two separate spectral type- $\log R$ diagrams, on which both components are smaller, thus not touching, both components are touching, primary is touching but secondary is not, and the secondary is touching but the primary is not touching. The inner contact surfaces were marked by special symbols for each. The two separate diagrams made Huang & Struve (1956) believe that the sub-giant components in such systems could have evolved from early main-sequence stars.

Empirical MRR in the form $\log R(M) = 8.495 - 0.2M_{\text{Bol}}(M) - 2 \log T_{\text{eff}}(M)$ suggested by Hoxie (1973) for the low mass stars ($M < 1M_{\odot}$) in the Solar neighbourhood. Comparing this MRR with low-mass model calculations of stellar R , Hoxie (1973) announced a discordance implying that theoretical radii are 30% or smaller than the observationally derived radii in the mass-radius plane for stars of mass less than $0.5M_{\odot}$. Subsequently, a method for estimating R of nearby stars was declared by Lacy (1977). Based on the Barnes-Evans relation and free of assumptions of spectral types, luminosity class, effective temperature, or bolometric correction, this method is applied to nearby single and double stars with accurate parallaxes and $V - R$ photometry. The double stars (visual binaries) were useful to supply components M and R , where R needed to be compared with predicted R by the new method with Barnes-Evans relation. Additional comparison was also possible to Lacy (1977) using R from non-contact eclipsing binaries with well-determined dimensions selected from Batten (1967)

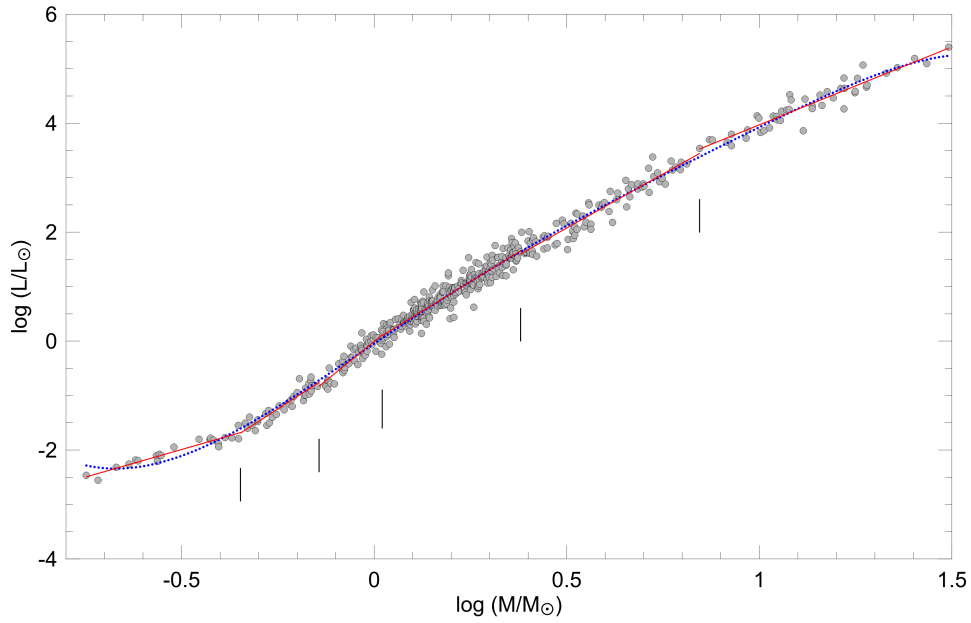


Figure 2. The dotted (blue) line is a sixth-degree polynomial, the solid (red) lines are classical MLRs, and the vertical lines are the break points separating the mass domains where the linear lines were fitted (credit to [Eker et al. 2018](#)).

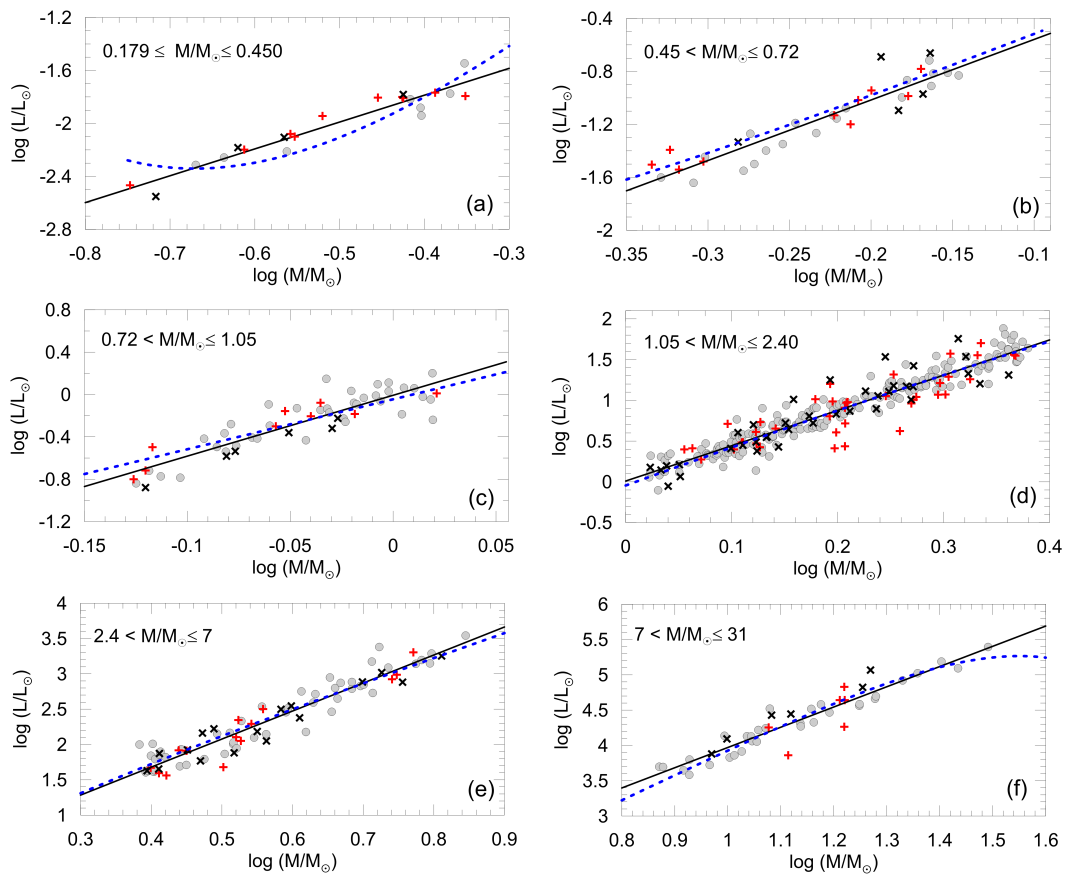


Figure 3. Mass domains and classical MLRs representing stellar masses and luminosities in each domain. 6th-degree polynomial representing all data (same as the dotted blue line in Figure 2) is not as successful as linear MLRs in low-mass domains ($M < 1.05$) and very high-mass domain ($M > 7$). Data accuracy is: (o) very accurate < 3%, (+) accurate (3-6%), and (x) less accurate (6-15%) (credit to [Eker et al. 2018](#)).

Table 2. Classical MLRs for main-sequence stars in various mass domains (credit to Eker et al. 2018).

Domain	N	Mass Range	Classical MLR	R^2	σ	α
Ultra low mass	22	$0.179 < M/M_{\odot} \leq 0.45$	$\log L = 2.028(135) \times \log M - 0.976(070)$	0.919	0.076	2.028
Very low mass	35	$0.45 < M/M_{\odot} \leq 0.72$	$\log L = 4.572(319) \times \log M - 0.102(076)$	0.857	0.109	4.572
Low mass	53	$0.72 < M/M_{\odot} \leq 1.05$	$\log L = 5.743(413) \times \log M - 0.007(026)$	0.787	0.129	5.743
Intermediate mass	275	$1.05 < M/M_{\odot} \leq 2.40$	$\log L = 4.329(087) \times \log M + 0.010(019)$	0.901	0.140	4.329
High mass	80	$2.4 < M/M_{\odot} \leq 7$	$\log L = 3.967(143) \times \log M + 0.093(083)$	0.907	0.165	3.967
Very high mass	44	$7 < M/M_{\odot} \leq 31$	$\log L = 2.865(155) \times \log M + 1.105(176)$	0.888	0.152	2.865

and Koch et al. (1970). Having a sufficient number of accurate masses and radii (both 5% or better) from the visual systems and non-contact eclipsing binaries, Lacy (1977) was also able to produce the most reliable $\log M - \log R$ diagram so far, where the empirical relation of Hoxie (1973) was plotted with a special symbol different from the symbols of visual binaries, eclipsing systems that have undergone mass exchange and without any mass exchange.

Therefore, Lacy (1977) suggested $\log R = 0.640 \log M + 0.011$ for the region $0.12 \leq \log M \leq 1.30$ and $\log R = 0.917 \log M - 0.020$ for the region $-1.00 \leq \log M \leq 1.30$ as two MRR functions to indicate the zero-age main sequence (ZAMS), where R and M are in Solar units. Lacy (1977) interpreted the break point as a signal crossing over from the region of the p-p chain to the C-N-O cycle. The terminal age main sequence (TAMS) line was deduced from the models of Iben (1967) and Paczyński (1970) marked on the $\log M - \log R$ as a dashed line, while ZAMS is shown by a continuous line with a break at $1.3 \pm 1M_{\odot}$. Theory and observation were found to be in good agreement for stars $M \gtrsim 1M_{\odot}$, but the models of M dwarfs having 25% smaller radii than real stars. Another line of development used the Catalogue of the Elements of Eclipsing Binaries (Kopal & Shapley 1956), where Kopal (1959) obtained a good statistical definition of the MRR. He found a linear relation, in logarithmic scales, for the values of individual stars irrespective of being primary or secondary, but the slope is different for massive and less massive stars with a transition at $\sim 2M_{\odot}$ (Kopal & Shapley 1956; Kopal 1978). Popovici & Dumitrescu (1974), who compiled data from Kopal & Shapley (1956) and Svetchnikoff (1969)², were mainly interested in the radius-luminosity diagram. Five mean empirical mass-radius relations had been constructed by Habets & Heintze (1981). The number of MRR is five because it is calibrated not only for main-sequence stars but also for luminosity class IV (subgiants), III (giants), V_{\odot} (ZAMS) and EV_{\odot} (EZAMS) using visual and eclipsing binaries which are at the same time double-lined spectroscopic systems collected by themselves. Patterson (1984) was unsatisfied by the MRRs suggested before him, thus he proposed the empirical ZAMS mass-radius relation in the form $R = \alpha M^{\beta}$, where M and R in Solar units

and $\alpha = 1$, $\beta = 0.88 \pm 0.02$ for the region $0.1 \leq M/M_{\odot} \leq 0.8$ and $\alpha = 0.98$, $\beta = 1.00$ for the region $0.8 \leq M/M_{\odot} \leq 1.4$ by a quotation “poorly defined for $M \leq 0.4M_{\odot}$ ” for himself to apply it to cataclysmic variables. Gimenez & Zamorano (1985) preferred using the Catalogue of Stellar Masses and Radii published by Popper (1980) for studying the classical MRR on $\log M - \log R$ diagram. Using reliable M and R from OB and B6-M detached eclipsing and visual binaries and resolved spectroscopic binaries, Gimenez & Zamorano (1985) determined $a = 0.041 \pm 0.011$ and $b = 0.749 \pm 0.011$ for an MRR in the form $\log R = a + b \log M$ with a correlation coefficient 0.96.

It must be noted that Lacy (1977, 1979), Gimenez & Zamorano (1985) and Demircan & Kahraman (1991) presented their $\log M - \log R$ diagrams without fitting an MRR function to the data, but with a ZAMS line as the lower limit and a TAMS line as the upper limit of main-sequence stars, which are estimated with the help of theoretical stellar structure and evolution models. The most recent example is by Eker et al. (2018) is shown in Figure 4, where it is clear that the statistical MRR does not appear as strong as the classical MLR expressed on $\log M - \log L$ diagram (Figure 3). Considering the difference in data distribution on both diagrams $\log M - \log L$ and $\log M - \log R$, it is not difficult to understand why $\log M - \log R$ diagram is presented without a fitting curve like Figure 3.

Because of the very large scatter caused by the evolution of stars $M \gtrsim 1.3M_{\odot}$ ($\log M > 0.15$), data of main-sequence stars on a $\log M - \log R$ diagram are not suitable for expression by a function or a curve. Obviously, plotting a ZAMS MLR on a $\log M - \log R$ diagram permits one to see the evolution of stellar radii for main-sequence stars $M \gtrsim 1.3M_{\odot}$.

Despite its hassle-free appearance and tight and narrow distribution of stellar L on a $\log M - \log L$ diagram for the full mass range of main-sequence stars, different forms of MLR functions causing confusion were suggested, calibrated, and shown together with the data. Looking at the already suggested main sequence MLR and MRR functions so far, one should see another noticeable difference. Contrary to the different forms of MLR suggested by different study groups, the main sequence MRR in the form $\log R = a + b \log M$ were common. The form $R \propto M^{\beta}$ suggest by Patterson (1984), Demircan & Kahraman (1991) and Karetnikov (1991), appears different, but it is not. The same form of function is expressed once in logarithmic form, while the other is a power law.

² Catalog Orbitalnii Elementov, Mass i Svetimostii Tesnii Dvornik Svezd, Sverdlovsk

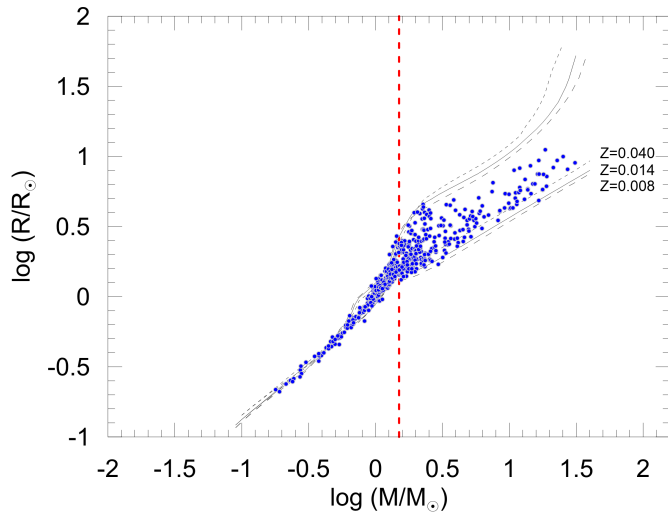


Figure 4. Main sequence mass-radius diagram of DDEB stars. ZAMS and TAMS lines are from the PARSEC models of Bressan et al. (2012) (credit to Eker et al. 2018). The vertical line is a division at $M = 1.5M_{\odot}$ ($\log M = 0.176$).

Despite such obvious differences (appearance on a diagram, formal expression), similarities between MLR and MRR studies also exist. Andersen (1991)’s exclamation against classical MLR also affected MRR studies; thus, modern MRR also appears to have deviated from the main path. Older MRR studies aimed to calibrate an MRR relation for two practical purposes: *i*) to get an answer to the question: how mean (or typical) M and R of main-sequence stars are related? *ii*) to estimate R of a star for a given M , or vice versa. The former is useful for making models that require mean M and R , and the latter is useful for guessing the R of a star from its M , or vice versa. Models using mean M and R as well as mean L , have a larger application area not only in stellar astrophysics but also in Galactic and extra-galactic studies.

The exclamation of Andersen (1991), unfortunately, caused the first aim to be neglected. As explained above, in the new trend of calibrating MLR, there is only one aim: to obtain the most accurate M and R together from the other observed parameters of the star. Thus, Malkov (2007), Torres et al. (2010), Moya et al. (2018), and Fernandes et al. (2021) have not only calibrated a single relation for predicting R of a single star but also at least one more relation useful to deliver M of the same star. Thus, the new trend of MRR cannot be considered independent of the new trend of MLR appearing after Andersen (1991).

2.3. Revisiting MTR

If M and L are related, why not M and T_{eff} since L is known to be equal to the surface area times the surface flux, where the flux is proportional to the fourth power of the effective temperature. A relation even stronger than mass-radius is expected because the power of T_{eff} is two times greater than the power of R in the Stefan-Boltzmann law. Its calibration and usage

also appear easier than the calibration and usage of the MRR relation. Regardless, except for the spectral type-effective temperature (horizontal axis of the H-R diagram) relationship, it took almost sixty years for a MTR to appear in the literature.

First empirical mass- T_{eff} relation is studied by Habets & Heintze (1981) and shown on a $\log M - \log T_{\text{eff}}$ diagram for main sequence-stars after studying newly calibrated mass-spectral type and spectral type- T_{eff} relations from the main-sequence components of eclipsing binaries. The empirical relation was shown together with two theoretical mass- T_{eff} relations of Stothers (1974, for a metallicity $Z = 0.02$ with $Y = 0.49$ and 0.25) and the one by Demarque & Gisler (1975, for $Z = 0.02$ and $Y = 0.25$). The form of the relation was not given. Karetnikov (1991) later determined coefficients of mass- T_{eff} relation in the form $\log T_{\text{eff}} = a + b \log M$ for six different types of eclipsing systems using absolute parameters of 303 eclipsing binaries of different types with varying and constant orbital periods without showing them on diagrams. The relations were determined for primaries and secondaries separately; thus, coefficients of 24 relations were determined and listed without showing them on $\log M - \log T_{\text{eff}}$ diagrams.

Apparently, MTR relation had also been affected by Andersen (1991)’s objection to MLR because Malkov (2007) announced a direct $[\log T_{\text{eff}} - (\log M)]$ and an inverse $[\log M - (\log T_{\text{eff}})]$ mass-effective temperature relations together with his M and R predicting relations and their inverse functions. Like $\log M - \log L$ diagram without a fitting curve, his M versus $\log T_{\text{eff}}$ diagram is too presented without a fitting curve. Being not interested in calibrating an MTR, Moya et al. (2018), on the other hand, used T_{eff} as one of the free parameters for predicting M and R of single stars. Moya et al. (2018), calibrated 38 relations: 18 for M and 20 for R . This is very much similar to Torres et al. (2010) who suggested only two relations using T_{eff} , $\log g$ and $[\text{Fe}/\text{H}]$ as free parameters, one for obtaining M one for obtaining R , to replace classical MLR claimed inadequate to provide stellar M by two reasons: *i*) It is a mean relation, thus M from L is very inaccurate. *ii*) Scatter on $\log M - \log L$ diagram is not only due to random observational errors of M and L but also due to stellar age and chemical composition differences. As if, a reliable (or true) MLR function must contain all the parameters that introduce scattering. Such an MLR function, however, cannot be drawn on a $\log M - \log L$ diagram.

On the other hand, following the old tradition of looking for mean relations for MRR and MTR, Eker et al. (2015) have compared L , R , and T_{eff} distributions on $\log M - \log L$, $\log M - \log R$ and $\log M - \log T_{\text{eff}}$ diagrams. The first comparison between $\log M - \log L$, $\log M - \log R$ has been commented as: “the appearance of data on the $\log M - \log R$ diagram is very different from the appearance on the $\log M - \log L$ diagram (compare Figures 2 and 4), which rather looks like a band of data expressible by a function; however, with a very narrow distribution of radii for masses $M < 1M_{\odot}$ and a broad band of radii for stars with $M > 1M_{\odot}$, a single function to express a MRR would be odd and meaningless”. Then, a comparison between

$\log M - \log R$, $\log M - \log T_{\text{eff}}$ were commented as: “the temperature evolution within the main-sequence band is not that obvious on the $M - T_{\text{eff}}$ diagram. At first glance, it resembles the MLR”. It is indeed not like MRR, where the main-sequence evolution of R is obvious for $M > 1M_{\odot}$ (compare Figures 2, 4, and 5).

Despite the distribution of data on $\log M - \log T_{\text{eff}}$ diagram (Figure 5) resembles the distribution of data on $\log M - \log L$ diagram, Eker et al. (2015) preferred not to calibrate a mean MTR because it would have been odd or inappropriate to oppose literature where MLR and MRR are many, but MTR is almost absent. Later, Eker et al. (2018) noted: “Stefan–Boltzmann law clearly indicates that stellar luminosity are related to stellar radii and effective temperatures. Having empirically determined the MLR and MRR available, one is not free to determine another independent mass–effective temperature relation (MTR)”. The three independently calibrated MLR, MRR, and MTR functions are not guaranteed to give consistent L , R , and T_{eff} for a given M . The solution to the problem is given in the next subsection.

2.4. Revisiting interrelated MLR, MRR, and MTR

It has been noticed that the distribution of R on the $\log M - \log R$ diagram (Figure 4) on the left of the vertical line ($M \leq 1.5M_{\odot}$) is smooth and tight, that is, it is expressible by a simple function, which is to be called MRR. However, in the high mass region $M > 1.5M_{\odot}$ due to faster evolution, R values are scattered very much, and thus, its band-like appearance does not seem expressible by a curve of a function.

Despite its overall appearance resembling an MLR, the T_{eff} distribution on the $\log M - \log T_{\text{eff}}$ diagram shows almost opposite characteristics when choosing the same mass domain to calibrate an MTR. One can easily notice that the low-mass region $M \leq 1.5M_{\odot}$ (left of the vertical line) in Figure 5 has a tight but wavy distribution, which is rather not possible to fit a simple function. On the contrary, one can easily spot the domain of high-mass stars $M > 1.5M_{\odot}$ (right of the vertical line) in Figure 5 with a sufficiently tight but smoothly varying distribution, which is easily expressed by a function called MTR.

Having six linear MLRs already calibrated (Table 2, Figures 2 and 3) for the main-sequence stars in the full range of observed stellar masses, probable inconsistencies among the mean L , R , and T_{eff} values for a given M will be eliminated if the MRR was calibrated for the low-mass region $M \leq 1.5M_{\odot}$ only and the MTR was calibrated for the high-mass region $M > 1.5M_{\odot}$ only. This is because the vertical line is a dimensionless border between the low- and high-mass regions in both R and T_{eff} distributions as shown in Figures 4 and 5. Otherwise, with independently calibrated MLR, MRR, and MTR functions, we obtain a mean luminosity ($\langle L \rangle$), a mean radius ($\langle R \rangle$), and a mean effective temperature ($\langle T_{\text{eff}} \rangle$) for a given M . No one would know which of the three mean values are wrong because the mean $\langle L \rangle$ will not be equal to the mean

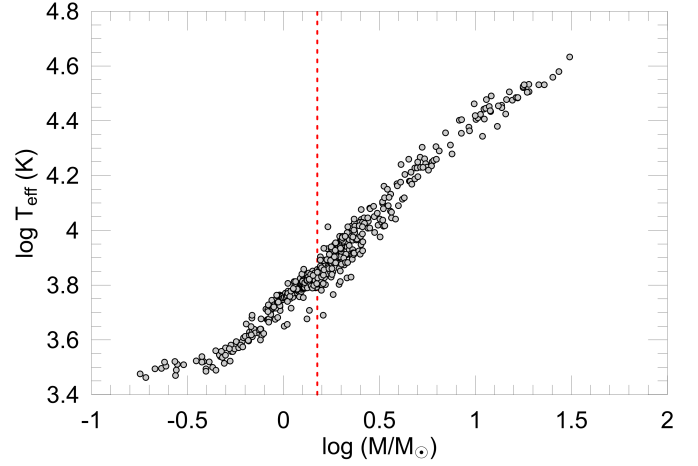


Figure 5. Main sequence mass-effective temperature diagram of DDEB stars (credit to Eker et al. 2018). The vertical line is a division at $M = 1.5M_{\odot}$ ($\log M = 0.176$).

surface area $\langle 4\pi R^2 \rangle = 4\pi(\langle R^2 \rangle)$ multiplied by the mean surface flux $\langle \sigma T^4 \rangle = \sigma(\langle T_{\text{eff}}^4 \rangle)$ for a typical main-sequence star of given M . Choosing the most eligible regions on $\log M - \log R$ and $\log M - \log T_{\text{eff}}$ diagrams as compensating mass domains for covering the full mass range, not only removes a probable inconsistency but also guarantees the most trustable MLR and MRR for the low-mass stars ($M \leq 1.5M_{\odot}$) and the most trustable MLR and MTR for the high-mass stars ($M > 1.5M_{\odot}$). Then, one could calculate consistent $\langle L \rangle$, $\langle R \rangle$ and $\langle T_{\text{eff}} \rangle$ for the full range of main-sequence stars, which occurs in two steps for both mass domains. For low-mass stars: *i*) Use MLR and MRR to calculate $\langle L \rangle$ and $\langle R \rangle$ for a given mass, *ii*) use Stefan-Boltzmann law to calculate $\langle T_{\text{eff}} \rangle$ for the same mass from its already computed $\langle L \rangle$ and $\langle R \rangle$. For high-mass stars: *i*) Use MLR and MTR to calculate $\langle L \rangle$ and $\langle T_{\text{eff}} \rangle$ for a given mass, *ii*) use Stefan-Boltzmann law to calculate $\langle R \rangle$ for the same mass from its already computed $\langle L \rangle$ and $\langle T_{\text{eff}} \rangle$.

Utilising the least squares, Eker et al. (2018) determined an empirical MRR directly from M and R of 233 main-sequence stars for low-mass stars within range $0.179 \leq M/M_{\odot} \leq 1.5$, and an empirical MTR from $\log M$ and $\log T_{\text{eff}}$ of 276 main-sequence stars for high-mass stars within range $1.5 < M/M_{\odot} \leq 31$. Because the table giving the empirical MRR and MTR by Eker et al. (2018) has misprints, the open forms of the functions, correlation coefficients (R^2), and standard deviations (σ) from Eker et al. (2021a) are given here. The empirical MRR in the form of a quadratic equation, where R and M are Solar units, is:

$$R = 0.438(0.098) \times M^2 + 0.479(0.180) \times M + 0.137(0.075), \quad (3)$$

has $R^2 = 0.867$ and $\sigma = 0.176$. The empirical MRR in the form of a quadratic equation, where T_{eff} and M are the Kelvin and Solar units, is:

$$\log T_{\text{eff}} = -0.170(0.026) \times (\log M)^2 + 0.888(0.037) \times \log M + 3.671(0.010), \quad (4)$$

has $R^2 = 0.961$ and $\sigma = 0.042$.

The empirical MRR and MTR by Eker et al. (2018) are

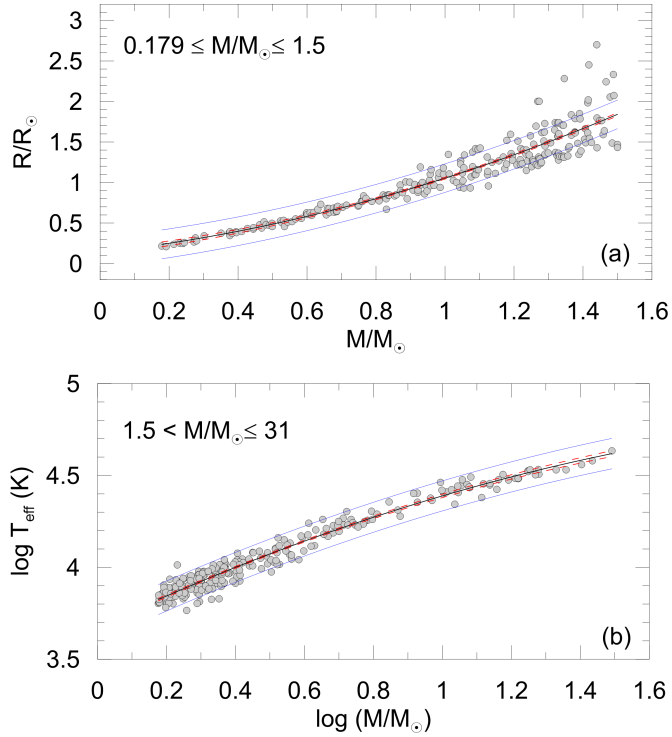


Figure 6. Empirical MRR and MTR calibrated from 509 main-sequence stars for the mass ranges $0.179 \leq M/M_{\odot} \leq 1.5$ and $1.5 \leq M/M_{\odot} \leq 31$ of DDEB stars. Note that MTR is on a logarithmic scale, but MRR is not (credit to Eker et al. 2018).

shown together with data in Figure 6. The division between the low- and high-mass stars at $1.5M_{\odot}$ is a nice coincidence that plots of MRR and MTR appear with the same numbers in the horizontal axis; thus, one must be careful that MTR is shown on a logarithmic scale while MRR is not.

The mean values ($\langle L \rangle$, $\langle R \rangle$, $\langle T_{\text{eff}} \rangle$) computed from the empirical MLR, MRR, and MTR of Eker et al. (2018) are compared to L , R and T_{eff} of 509 main-sequence stars chosen from 318 DDEB and one detached eclipsing triple, as shown in Figure 7. The solid lines mark the mean values calculated directly from MLR, MRR, or MLR, MTR, while the dotted lines mark the mean values $\langle T_{\text{eff}} \rangle$ calculated from $\langle L \rangle$, $\langle R \rangle$ for the low-mass stars and $\langle R \rangle$ calculated from $\langle L \rangle$ and $\langle T_{\text{eff}} \rangle$ for the high-mass stars.

Calculating the mean values ($\langle L \rangle$, $\langle R \rangle$, $\langle T_{\text{eff}} \rangle$) in two steps as described above has the advantage of further checking them and confirming whether the mean values of the first step is consistent or not. Note that, only if $\langle L \rangle$ and $\langle R \rangle$ values of the low-mass stars are consistent, then in the second step consistent $\langle T_{\text{eff}} \rangle$ values could be produced from $\langle L \rangle$ and $\langle R \rangle$; similarly, only if $\langle L \rangle$ and $\langle T_{\text{eff}} \rangle$ values of high-mass stars are consistent, then in the second step, consistent $\langle R \rangle$ values could be produced according to the Stefan-Boltzmann law. The three break points and varying inclinations of MLR before, between, and after the break points in the mass range $M \leq 1.5M_{\odot}$, apparently, are causing the wavy look of the MTR of low-mass stars. What simple function would have produced the current successful

appearance of the fit displayed in Figure 7c by the dotted line? Similar influence of the break points and effect of varying inclinations of MLR before, between, and after the break points for the high-mass stars $M > 1.5M_{\odot}$ are there in the middle panel (Figure 7b) but appear to be lost within the scatter caused by the faster evolution of more massive stars. Therefore, it is a bull's eye to choose the mass region $M \leq 1.5M_{\odot}$ for devising MRR and to choose the mass region $M > 1.5M_{\odot}$ for devising MTR.

The two-step procedure in determining the mean values of $\langle L \rangle$, $\langle R \rangle$, $\langle T_{\text{eff}} \rangle$ for a given M may appear problematic because it introduces extra errors in the error propagation. Users should not be deceived by this illusion. First, this is because consistent mean values of higher uncertainty are better than inconsistent more accurate ones. To avoid inconsistency, one may use a two-step procedure as described above or use MLR and MRR only or MLR and MTR only for the full mass range of main-sequence stars. Thus, in either case, the two-step procedure is inevitable for consistent results. For the high-mass stars $M > 1.5M_{\odot}$ the scatter on R is too large; thus, authors usually prefer not to determine MRR. In the former case, MRR and MTR would

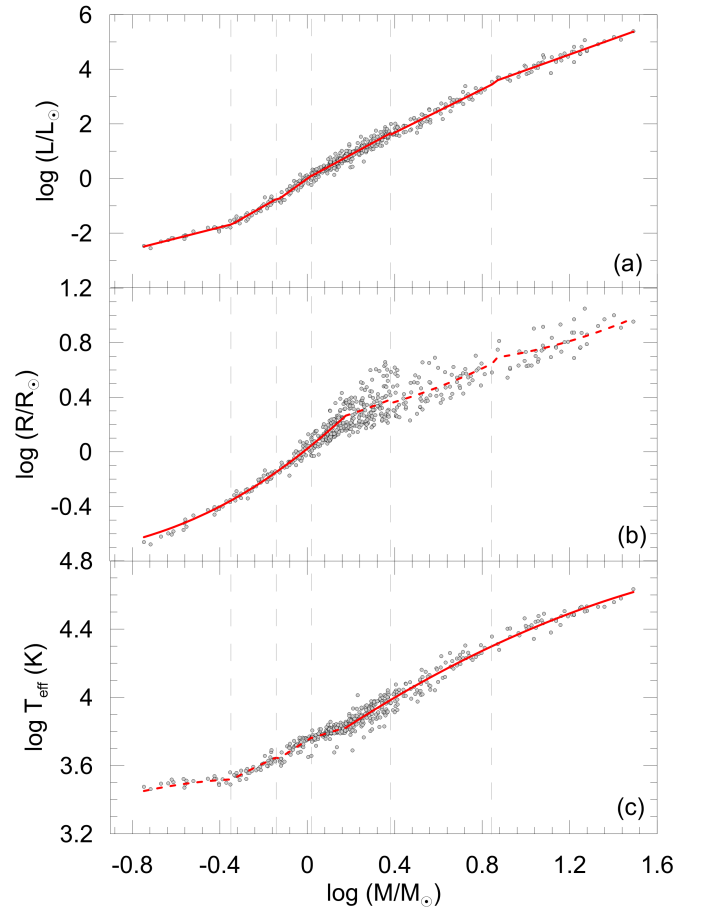


Figure 7. The mean values ($\langle L \rangle$, $\langle R \rangle$, $\langle T_{\text{eff}} \rangle$) computed from empirical MLT, MRR, and MTR are compared to L , R and T_{eff} of 509 main-sequence stars. The mean values taken directly from MLR, MRR or MLR, MTR are solid, The mean values from the Stefan-Boltzmann law are dotted. The dashed vertical lines represent the break points of MLR (credit to Eker et al. 2018).

be missing for high-mass stars, while interrelated MLR, MRR, and MTR would be possible only for low-mass stars. On the other hand, for low-mass stars $M < 1.5M_{\odot}$ it is not possible to find a simple function to fit in determining MTR. Then, interrelated MLR, MRR, and MTR become possible only for high-mass stars since MRR and MTR would be missing for low-mass stars $M < 1.5M_{\odot}$. If one prefers to fit an MRR together with MLR in the full mass region, then he/she will end up having very unreliable $\langle R \rangle$ for the stars $M > 1.5M_{\odot}$. Error propagation, then, would produce similarly unreliable, even worse $\langle T_{\text{eff}} \rangle$ since the independent MTR function is not there. If one prefers to calibrate MLR and MTR in the full range of masses, similar unreliable or erroneous $\langle T_{\text{eff}} \rangle$, $\langle R \rangle$ values emerge for the low mass stars $M < 1.5M_{\odot}$. Therefore, the two-step procedure described in the paragraph before this one in determining the mean values of $\langle L \rangle$, $\langle R \rangle$, $\langle T_{\text{eff}} \rangle$ for a given M is always better and more reliable than the independently determined MLR, MRR and MTR for sure.

3. CONCLUSIONS

Fundamental relations are handy tools to explain simple natural phenomena or starting points for understanding more complex phenomena. The main difference between statistical and fundamental relations is that a statistical relation is valid under certain conditions implied by the data from which it was formulated, whereas a fundamental relation is rather independent of data, that is, data are not there to constrain it but only to confirm or to falsify it. All empirical relations derived from observational or experimental data are statistical in spirit. This does not, however, mean a fundamental relation cannot be derived from observational and/or experimental data. A statistical study could reproduce and may even find a fundamental relationship.

Physical laws are indifferent to fundamental relations. Their scope is wider; for example, the Stefan-Boltzmann law ($L = 4\pi R^2 \sigma T_{\text{eff}}^4$) is valid for all stars radiating thermally and for a hypothetical star representing $\langle L \rangle$, $\langle R \rangle$, and $\langle T_{\text{eff}} \rangle$ implied the main sequence MLR, MRR, and MTR. In contrast, MLR, MRR, and MTR are valid to give mean values for a main-sequence star of a given M within a valid mass range, which was suggested after their calibrations. Because they are not valid for non-main sequence stars, they are also not valid for estimating L , R , or T_{eff} of a main-sequence star with a known M only. This is because, nowadays, many stellar structure and evolution models exist for one to look for L , R , and T_{eff} of a star according to its M , chemical composition (X, Y, Z), and age.

Because of their simplicity, some statistical relations can be mistaken or misused as fundamental relations. A good example of this is the classical MLR. This is because, immediately after its discovery by Hertzsprung (1923) and Russell et al. (1923), for a while, there were no other methods to obtain masses of single stars but estimate them from L or absolute bolometric magnitude (M_{Bol}) by using an MLR in the form of $L \propto M^{\alpha}$ or M_{Bol} - mass diagrams.

Having no alternative is another property of fundamental relations. Indeed, neither the Stefan-Boltzmann law nor the Planck law (spectral energy distribution of a black body with specific temperature) have alternatives. Since there is no alternative to estimate M of a single star, using a classical MLR convinced early astronomers that it could be used as a fundamental relation. Today, there are alternatives to estimating M and R of single stars; thus, there is no excuse to assume any form of MLR as fundamental laws. One might still want to use MLR, MRR, and MTR, as a fundamental law, that is, if he/she is more interested in knowing M of star from its L . This is permissible with a larger uncertainty covering the main sequence at the value of M on a $\log M - \log L$ diagram, simply $\pm 1\sigma$, which is given in Table 2.

On the other hand, statistical relations could be devised to serve a very specific purpose; for example, to obtain M and R of a single star from its other observables. In this respect, a statistical relation may appear to operate like a fundamental relation, but this is an illusion. M and R predicting relations are still statistical relations if their validity depends on the data from which they were calibrated. In addition, because there are many alternatives to provide R or M of a star from its T_{eff} , L , g , ρ , and $[\text{Fe}/\text{H}]$, as expressed by Moya et al. (2018) and as summarised by Serenelli et al. (2021). Malkov (2007), Torres et al. (2010), and Fernandes et al. (2021).

Naming a newly devised or renaming a re-calibrated empirical relation is very important; as important as classifying it as another statistical relation or one of the fundamental laws. Names must be unique to avoid confusion and guide users towards the original purpose of the relation. Repeating the same improper and non-unique names should not be advocated by saying: Let us keep the same name in the past. This will be nothing but insisting on the same error. There are many mis-calling or non-unique naming examples in the past, which are identified with possible correct names in the previous section. Especially after Andersen (1991)'s exclamation that some R and M predicting relations were non-uniquely and incorrectly called MLR. Let us hope, Chevalier et al. (2023), and Malkov et al. (2022) would be the last examples who erroneously named their mass- M_G diagrams, where M_G is absolute brightness in *Gaia* G band, "mass-luminosity" diagram, and their mass - M_G relations "mass-luminosity" relation. Possible unique names for them are the "mass-absolute brightness" diagram or the "mass-absolute brightness" relation at *Gaia* G band.

The empirical relations for predicting M and R of single stars developed as alternatives to classical MLR and MRR after the split, initiated after Andersen (1991)'s exclamation, occurred on the classical path aiming to obtain mean values $\langle L \rangle$, $\langle M \rangle$, and $\langle R \rangle$, and individual L , R , and M of single stars. In this respect, it is not possible for one to say, "the empirical relations giving M and R of single stars are more valuable than the classical MLR and MRR". Claiming the opposite is also not correct. Both schools of thought have useful applications that are valuable concerning their own aims. Empirical M and R

prediction relations have found a fruitful application to explore exoplanets hosting single stars. Estimating M and R of planet-hosting stars is the starting point for exploring hosted exoplanets (Stassun et al. 2017, 2018). Classical MLR, MRR, and MTR, on the other hand, are practical for constructing astrophysical models that need mean values, which are beneficial not only to stellar astrophysics but also to Galactic and extragalactic search, and even to cosmological models.

Further improvements in the prediction accuracy of single star parameters depend mainly on the quantity and quality of the radial velocity and light curve solutions of DDEB. Advances in already established relations or new forms are possible because of the increasing demand for exoplanet investigations. Developments on the classical MLR, MRR, and MTR, on the other hand, are encouraged to include the metallicity effect, as well as to extend them further towards high and low mass limits, not only for the main sequence but also for other luminosity classes.

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