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RESEARCH ARTICLE

Some characterizations of hyperbolic Ricci solitons on nearly cosymplectic manifolds with respect to the Tanaka-Webster connection

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ABSTRACT

It is known that a hyperbolic Ricci soliton is one of the generalization of the Ricci solitons and it is a Riemannian manifold (M, g) furnished with a differentiable vector field U on M and two real numbers λ and μ ensuring $Ric + \lambda L_U g + \frac{1}{2} L_U(L_U g) = \mu g$, where L_U denotes the Lie derivative with respect to the vector field X on M. Furthermore, hyperbolic Ricci solitons yield similar solutions to hyperbolic Ricci flow. In this paper, we study hyperbolic Ricci solitons on nearly cosymplectic manifolds endowed with the Tanaka-Webster connection. We give some results for these manifolds when the potential vector field is a pointwise collinear with the Reeb vector field and a concircular vector field.

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1. INTRODUCTION

The notion of hyperbolic Ricci flow was introduced in Kong and Liu (2007). Let $g_{ij}(t)$ be a family of Riemannian metrics on a Riemannian manifold (M_n, g_0). The hyperbolic Ricci flow is defined by

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}$$

with $g(0) = g_0$, $\frac{\partial g_{ij}}{\partial t} = k_{ij}$, where k_{ij} is a symmetric$(0, 2)-$type tensor field. A self-similar solution $g(t)$ of the hyperbolic Ricci flow on M_n is a hyperbolic Ricci soliton if there exists a 1-parameter family of diffeomorphisms $\rho(t) : M \to M$ and a positive function $\sigma(t)$ such that

$$g(t) = \sigma(t)\rho(t)^*(g_0).$$

If we differentiate above equation twice, we get

$$-2Ric(g(t)) = \sigma''(t)\rho(t)^*(g_0) + 2\sigma'(t)\rho(t)^*(L_X g_0) + \sigma(t)\rho(t)^*(L_X L_X g_0)),$$

where Ric is the Ricci curvature on M, X is the time-dependent vector field and L is the Lie derivative. The family of metrics are said to be expanding, steady or shrinking if σ' is positive, zero or negative, respectively. Substituting $\sigma''(0) = -2\mu$, $\sigma(0) = 1$ and $\sigma'(0) = \lambda$ in the above equation, we get

$$Ric(g_0) + \lambda L_X g_0 + \frac{1}{2} L_X L_X g_0 = \mu g_0$$

for some real constants λ and μ. According to this equation, a hyperbolic Ricci soliton on a Riemannian manifold (M, g) is defined by

$$Ric + \lambda L_X g + \frac{1}{2} L_X(L_X g) = \mu g. \tag{1}$$

A hyperbolic Ricci soliton is called expanding, steady or shrinking if μ is negative, zero or positive, respectively. For recent papers about hyperbolic Ricci solitons see Azami and Fasihi (2023), Azami and Fasihi (2024), Blaga and Özgür (2023), Faraji et al. (2023).

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In this paper, we investigate hyperbolic Ricci solitons on nearly cosymplectic manifolds. The manifolds will be considered with the Tanaka-Webster connection. The paper is organized as follows: In Section 2, we give some fundamental information about nearly cosymplectic manifolds. In Section 3, we express some properties of cosymplectic manifolds satisfying Tanaka-Webster connection. In the final section, we give our main results.

2. NEARLY COSYMPLECTIC MANIFOLDS

An $n = (2k+1)-$dimensional smooth manifold M is called an almost contact metric manifold if it admits a $(1,1)-$tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Riemannian metric g which fulfill, Blair (1976)

$$\phi^2(U) = -U + \eta(U)\xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta(\phi U) = 0, \tag{2}$$

$$\begin{aligned} g(\phi U, \phi V) &= g(U,V) - \eta(U)\eta(V), \ g(\phi U, V) = -g(U, \phi V), \\ g(U, \xi) &= \eta(U), \ \forall U, V \in \chi(M). \end{aligned} \tag{3}$$

An almost contact metric manifold (M, g, η, ξ, ϕ) is called a contact metric manifold if

$$g(U, \phi V) = d\eta(U, V).$$

An almost contact metric manifold (M, g, η, ξ, ϕ) is said to be a nearly cosymplectic manifold if

$$(\nabla_U \phi)V + (\nabla_V \phi)U = 0, \ \forall U, V \in \chi(M).$$

For a nearly cosymplectic manifold, we have

$$\nabla_\xi \xi = 0 \text{ and } \nabla_\xi \eta = 0.$$

On the other hand, for a $(1,1)-$type tensor field H which is defined as

$$\nabla_U \xi = HU. \tag{4}$$

It is known that H is skew symmetric and anti-commutative with ϕ. Moreover, H satisfies $H\xi = 0$ and $\eta \circ H = 0$ and fulfills the following situations, Nicola et al. (2018):

$$(\nabla_\xi \phi)U = \phi HU = \frac{1}{3}(\nabla_\xi \phi)U \ ,$$

$$g((\nabla_U \phi)V, HW) = \eta(V)g(H^2U, \phi W) - \eta(U)g(H^2V, \phi W),$$

$$(\nabla_U H)V = g(H^2U \ , V)\xi - \eta(V)H^2U,$$

$$tr(H^2) = \text{constant},$$

$$R(V,W)\xi = \eta(V)H^2W - \eta(W)H^2V,$$

$$S(\xi, W) = -\eta(W)tr(H^2),$$

$$S(\phi V, W) = S(V, \phi W), \ \ \phi Q = Q\phi,$$

$$S(\phi V, \phi W) = S(V,W) + \eta(V)\eta(W)tr(H^2).$$

3. NEARLY COSYMPLECTIC MANIFOLDS ADMITTING TANAKA-WEBSTER CONNECTION

Let (M, g, η, ξ, ϕ) be an almost contact metric manifold. The Tanaka-Webster connection $\bar{\nabla}$ with respect to the Levi-Civita connection ∇ is defined by

$$\bar{\nabla}_U V = \nabla_U V + (\nabla_U \eta)(V)\xi - \eta(V)\nabla_U \xi - \eta(U)\phi V, \tag{5}$$

for all $U, V \in \chi(M)$, Tanno (1969). Using (3) and (4), we rewrite equation (5) as

$$\bar{\nabla}_U V = \nabla_U V + g(\nabla_U \xi, V)\xi - \eta(V)HU - \eta(U)\phi V. \tag{6}$$

Putting $V = \xi$ in (6) and using (2) and (4), we obtain

$$\bar{\nabla}_U \xi = 0. \tag{7}$$

Using (6), the Riemannian curvature tensor $\bar{R}$ of the connection $\bar{\nabla}$ is given by

$$\begin{aligned} \bar{R}(U,V)W &= R(U,V)W - g(W,HU)HV - g(H^2V,W)\eta(U)\xi - 2g(V,HU)\phi W\eta(U)\eta(W)\phi HV \\ &+ g(H^2U,W)\eta(V)\xi - \eta(V)(\nabla_U\phi)W - \eta(V)g(HU,\phi W)\xi + g(W,HV)HU \\ &+ \eta(W)\eta(U)H^2V - \eta(W)\eta(V)H^2U - \eta(V)\eta(W)\phi HU \\ &+ \eta(U)(\nabla_V\phi)W + \eta(U)g(HV,\phi W)\xi. \end{aligned} \tag{8}$$

Taking contraction in (8), the Ricci tensor $\overline{Ric}$ of the connection $\bar{\nabla}$ is given by

$$\begin{aligned} \overline{Ric}(V,W) &= Ric(V,W) + 2g(HV,\phi W) - \eta(V)div(\phi)W + g(W,HV)tr(H) \\ &- \eta(W)\eta(V)tr(H^2) - \eta(V)\eta(W)tr(\phi H) + 2g(HW,HV), \end{aligned} \tag{9}$$

where Ric denotes the Ricci tensor of the Levi-Civita connection ∇. Contracting in (9), the scalar curvature $\bar{r}$ is obtained as

$$\bar{r} = r - tr(H^2)(2k+1),$$

where r is the scalar curvature of the Levi-Civita connection ∇, Ayar (2022).

4. MAIN RESULTS

Before expressing our main results, we should remind definitions of the nearly quasi-Einstein manifolds and Einstein manifolds.

Definition 4.1. Let (M,g) be a Riemannian manifold. If $Ric = \alpha g + \beta E$ for some functions α and β on M, where E is a non-zero tensor of type $(0,2)$, then the manifold (M,g) is called a nearly quasi-Einstein manifold. If $\beta = 0$, then the manifold (M,g) is said to be an Einstein manifold. Here, Ric denotes the Ricci tensor of the Levi-Civita connection ∇.

Now, we can give our findings.

Theorem 4.2. *Let M be a nearly cosymplectic manifold with the Tanaka-Webster connection admitting a hyperbolic Ricci soliton. If the potential vector field X is a pointwise collinear with ξ, then M is a nearly-quasi Einstein manifold.*

Proof. If the potential vector field X is a pointwise collinear with ξ, then there exists a smooth function b such that $X = b\xi$. Using (7), we have

$$\begin{aligned} (\bar{L}_X g)(U,V) &= g(\bar{\nabla}_U X, V) + g(\bar{\nabla}_V X, U) \\ &= g(U(b)\xi + b\bar{\nabla}_U\xi, V) + g(V(b)\xi + b\bar{\nabla}_V\xi, U) \\ &= U(b)\eta(V) + V(b)\eta(U) \\ &= g(\nabla b, U)\eta(V) + g(\nabla b, V)\eta(U) \end{aligned} \tag{10}$$

for all $U, V \in \chi(M)$, where ∇ denotes the gradient operator. The Lie derivative of (7) is given by

$$\begin{aligned} (\bar{L}_X \circ \bar{L}_X)g(U,V) &= X\bar{L}_X g(U,V) - \bar{L}_X g(\bar{L}_X U, V) - \bar{L}_X g(U, \bar{L}_X V) \\ &= X[g(\nabla b, U)\eta(V) + g(\nabla b, V)\eta(U)] \\ &- [g(\nabla b, \bar{L}_X U)\eta(V) + g(\nabla b, V)\eta(\bar{L}_X U)] \\ &- [g(\nabla b, \bar{L}_X V)\eta(U) + g(\nabla b, U)\eta(\bar{L}_X V)] \\ &= Xg(\nabla b, U)\eta(V) + g(\nabla b, U)X\eta(V) + Xg(\nabla b, V)\eta(U) \\ &+ g(\nabla b, V)X\eta(U) - g(\nabla b, \bar{L}_X U)\eta(V) - g(\nabla b, V)\eta(\bar{L}_X U) \\ &- g(\nabla b, \bar{L}_X V)\eta(U) - g(\nabla b, U)\eta(\bar{L}_X V). \end{aligned} \tag{11}$$

Putting (10) and (11) in (1), we occur

$$\begin{aligned} \overline{Ric}(U,V) &= \mu g(U,V) - \lambda(\bar{L}_X g)(U,V) - \frac{1}{2}(\bar{L}_X \circ \bar{L}_X)g(U,V) \\ &= \mu g(U,V) - \lambda g(\nabla b, U)\eta(V) - \lambda g(\nabla b, V)\eta(U) \\ &- \frac{1}{2}Xg(\nabla b, U)\eta(V) - \frac{1}{2}g(\nabla b, U)X\eta(V) - \frac{1}{2}Xg(\nabla b, V)\eta(U) \\ &- \frac{1}{2}g(\nabla b, V)X\eta(U) + \frac{1}{2}g(\nabla b, \bar{L}_X U)\eta(V) + \frac{1}{2}g(\nabla b, V)\eta(\bar{L}_X U) \\ &+ \frac{1}{2}g(\nabla b, \bar{L}_X V)\eta(U) + \frac{1}{2}g(\nabla b, U)\eta(\bar{L}_X V). \end{aligned} \tag{12}$$

Taking a non-vanishing $(0,2)-$type tensor E as

$$\begin{aligned} E(U,V) &= -\lambda g(\nabla b,U)\eta(V) - \lambda g(\nabla b,V)\eta(U) \\ &\quad -\frac{1}{2}[Hess(b)(X,U)\eta(V) - Hess(b)(X,V)\eta(U) + (\overline{\nabla}_U X)(b)\eta(V) \\ &\quad +(\overline{\nabla}_V X)(b)\eta(U) + V(b)g(\overline{\nabla}_U X,\xi) + U(b)g(\overline{\nabla}_U X,\xi)]. \end{aligned} \tag{13}$$

Equation (12) becomes

$$\overline{Ric}(U,V) = \mu g(U,V) + E(U,V).$$

This shows that M is a nearly quasi-Einstein manifold with respect to the Tanaka-Webster connection $\bar{\nabla}$.

Proposition 4.3. *Let M be a nearly cosymplectic manifold with the Tanaka-Webster connection admitting a hyperbolic Ricci soliton. If the potential vector field is the Reeb vector field ξ, then M is an Einstein manifold.*

Proof. Taking $b = 1$ in (13) shows that $\overline{Ric}(U,V) = \mu g(U,V)$. This gives us M is an Einstein manifold.

Theorem 4.4. *Let M be a nearly cosymplectic manifold with the Tanaka-Webster connection admitting a hyperbolic Ricci soliton. If the potential vector field is a concircular vector field X, then*

$$\mu = -2tr(H^2) - tr(H) + 2f^2 + 2\lambda f.$$

Proof. It is known that if X is concircular vector field on M, then there exists a smooth function f such that

$$\nabla_U X = fU \tag{14}$$

for all $U \in \chi(M)$. Using (14), we obtain

$$\begin{aligned} (\bar{L}_X g)(U,V) &= g(\bar{\nabla}_U X,V) + g(\bar{\nabla}_V X,U) \\ &= g(fU,V) + g(U,fV) \\ &= 2fg(U,V). \end{aligned} \tag{15}$$

Using equation (15), we get

$$\begin{aligned} (\bar{L}_X \circ \bar{L}_X)g(U,V) &= X\bar{L}_X g(U,V) - \bar{L}_X g(\bar{L}_X U,V) - \bar{L}_X g(U,\bar{L}_X V) \\ &= X(2fg(U,V)) - 2fg(\bar{L}_X U,V) - 2fg(U,\bar{L}_X V) \\ &= 2(Xf)g(U,V) + 2fg(\bar{\nabla}_X U,V) + 2fg(U,\bar{\nabla}_X V) \\ &\quad -2fg(\bar{\nabla}_U X,V) + 2fg(\bar{\nabla}_U X,V) - 2fg(U,\bar{\nabla}_X V) + 2fg(U,\bar{\nabla}_V X) \\ &= 2(Xf)g(U,V) + 2fg(\bar{\nabla}_U X,V) + 2fg(U,\bar{\nabla}_V X) \\ &= 2(Xf)g(U,V) + 4f^2 g(U,V). \end{aligned} \tag{16}$$

Putting (15) and (16) in (1), we deduce

$$Ric(U,V) + (Xf)g(U,V) + 2f^2 g(U,V) + 2\lambda fg(U,V) = \mu g(U,V).$$

Substituting $U = V = \xi$ in (9), we obtain $\mu = -2tr(H^2) - tr(H) + 2f^2 + 2\lambda f$.

5. CONCLUSION

In this paper, we study hyperbolic Ricci solitons on nearly cosymplectic manifolds with respect to the Tanaka-Webster connection by considering the potential vector field as a pointwise collinear with the Reeb vector field and a concircular vector field. Our results in the present work may provide an insight for further studies on hyperbolic Ricci solitons with respect to some other connections.

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