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RESEARCH ARTICLE

Parseval-Goldstein type theorems for integral transforms in a general setting

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ABSTRACT
This research paper explores Parseval-Goldstein type relations concerning general integral operators. It investigates the continuity properties of these operators and their adjoints over Lebesgue spaces. Through rigorous analysis, the study elucidates the intricate connections between these operators and sheds light on their behaviour within functional spaces. By exploring the convergence and stability of these relations, the paper contributes to a deeper understanding of integral operators behaviour and their implications in various mathematical contexts. The paper also examines specific cases of the main index transforms, including the Kontorovich-Lebedev transform, the Mehler-Fock transform of general order, the index ${}_2F_1$-transform, the Lebedev-Skalskaya transforms and the index Whittaker transform, as well as operators with complex Gaussian kernels, contributing valuable insights into their behaviour and applications.

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1. INTRODUCTION

We consider the integral operator given by

$$(\mathcal{F} f)(y) = \int_I f(x)K(x,y)dx,\ \ y \in I, \tag{1}$$

where K is a measurable complex-valued function $K : I \times I \longrightarrow \mathbb{C}$ (I denoting some open interval in $\mathbb{R}$, possibly unbounded) over the spaces $L^p(I, \tilde{K}(x)dx)$, $1 \le p < \infty$, and $L^\infty(I)$, being $\tilde{K}(x)$ a measurable function on I which satisfies $|K(x, y)| \le \tilde{K}(x)$, for all $x, y \in I$.
We also consider the integral operator

$$(\mathcal{F}^* g)(x) = \int_I g(y)K(x,y)dy,\ \ x \in I, \tag{2}$$

over the space $L^1(I)$. In 1989, Yürekli Yürekli (1989) introduced a Parseval-Goldstein type theorem, elucidating the interconnection between Laplace and Stieltjes transforms, and subsequently explored its ramifications. In 1992, Yürekli extended this investigation to encompass the generalized Stieltjes transform Yürekli (1992). Building upon this foundation, various researchers have delved into analogous connections amonxxg diverse integral transforms, leveraging Parseval-Goldstein type theorems, as evidenced by works from several authors Albayrak and Dernek (2021); Albayrak (2024); Karataş et al. (2020). Parseval's and Plancherel's theorems stand as cornerstone results in mathematics, establishing pivotal relationships between original functions and their transforms, showcasing the preservation of energy or inner products under transformation Dernek et al. (2008, 2007); Yürekli (1989).

The Parseval-Goldstein relations for integral transforms establish a crucial link between norms in the original domain and their transformed counterparts, shedding light on the energy-preserving characteristics and inter-domain consistency of these transforms. This profound analysis significantly contributes to understanding the fundamental properties and applications of integral transforms in mathematical analysis Yürekli (1989, 1992); Albayrak and Dernek (2021); Albayrak (2024); Karataş et al. (2020); Srivastava and Yürekli (1995). The present article delves into the study of Parseval-Goldstein type relations for integral

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operators over Lebesgue spaces.
The $C_c^k(\mathbb{R}_+)$, $k \in \mathbb{N}$, denotes as it is usual the space of compactly supported functions on $\mathbb{R}_+$ which are k-times differentiable with continuity. The article is structured as follows: Section 1 presents an introduction about the general setting. Sections 2 and 3 delve into the continuity properties over Lebesgue spaces of both the integral operators and their adjoints. Section 4 establishes Parseval-Goldstein type relations for these operators. Section 5 examines integral transforms characterized by kernels satisfying specific conditions. Finally, Section 6 offers concluding remarks.

2. THE $\mathcal{F}$ TRANSFORM OVER THE SPACES $L^p(I, \tilde{K}(x)dx)$, $1 \le p < \infty$

Proposition 2.1. *The next results hold*
(i) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^1(I, \tilde{K}(x)dx)$ into $L^\infty(I)$. If $f \in L^1(I, \tilde{K}(x)dx)$ then*

$$\|\mathcal{F} f\|_{L^\infty(I)} \le \|f\|_{L^1(I,\tilde{K}(x)dx)},$$

furthermore if $K(x,\cdot)$ is continuous for each $x \in I$ then $\mathcal{F} f$ is a continuous function on I. Moreover, the operator $\mathcal{F}$ is a continuous map from $L^1(I, \tilde{K}(x)dx)$ to the Banach space of bounded continuous functions on I.
(ii) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^p(I, \tilde{K}(x)dx)$ into $L^\infty(I)$, $1 < p < \infty$, whenever $\int_I \tilde{K}(x)dx < \infty$. Also if $f \in L^p(I, \tilde{K}(x)dx)$, $1 < p < \infty$, then*

$$\|\mathcal{F} f\|_{L^\infty(I)} \le M\|f\|_{L^p(I,\tilde{K}(x)dx)}, \text{ for some } M > 0,$$

furthermore if $K(x,\cdot)$ is continuous for each $x \in I$ then $\mathcal{F} f$ is a continuous function on I. Moreover, the operator $\mathcal{F}$ is a continuous map from $L^p(I, \tilde{K}(x)dx)$ to the Banach space of bounded continuous functions on I.
(iii) *The integral operator $\mathcal{F}$ given by (1) is a bounded linear operator from $L^\infty(I)$ into $L^\infty(I)$ whenever $\int_I \tilde{K}(x)dx < \infty$. Also if $f \in L^\infty(I)$ then*

$$\|\mathcal{F} f\|_{L^\infty(I)} \le M\|f\|_{L^\infty(I)}, \text{ for some } M > 0,$$

furthermore if $K(x,\cdot)$ is continuous for each $x \in I$ then $\mathcal{F} f$ is a continuous function on I. Moreover, the operator $\mathcal{F}$ is a continuous map from $L^\infty(I)$ to the Banach space of bounded continuous functions on I.

Proof. (i) Let $y_0 \in I$ be arbitrary. Since the map $y \longrightarrow K(x,y)$ is continuous for each fixed $x \in I$, we have

$$K(x,y) \longrightarrow K(x,y_0) \text{ as } y \longrightarrow y_0.$$

Further, we have that $|K(x,y) - K(x,y_0)|\,|f(x)|$ is dominated by the integrable function $2\tilde{K}(x)|f(x)|$. Therefore, by using dominated convergence theorem, we get

$$|(\mathcal{F} f)(y) - (\mathcal{F} f)(y_0)| \le \int_I |f(x)|\,|K(x,y) - K(x,y_0)|\,dx \longrightarrow 0, \ as\ y \longrightarrow y_0.$$

Thus, $\mathcal{F} f$ is a continuous function on I.
Since for each $y \in I$

$$\begin{aligned} |(\mathcal{F} f)(y)| &\le \int_I |f(x)|\,|K(x,y)|\,dx \\ &\le \int_I |f(x)|\tilde{K}(x)dx = \|f\|_{L^1(I,\tilde{K}(x)dx)}, \end{aligned} \tag{3}$$

one has that $\mathcal{F} f$ is a bounded function.
The linearity of the integral operator implies that the $\mathcal{F}$ integral operator is linear. Also from (3) we get that $\|\mathcal{F} f\|_{L^\infty(I)} \le \|f\|_{L^1(I,\tilde{K}(x)dx)}$ and hence $\mathcal{F} : L^1(I, \tilde{K}(x)dx) \longrightarrow L^\infty(I)$ is a continuous linear map.
(ii) Observe that using Hölder's inequality we have for $y \in I$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$,

$$\begin{aligned} |(\mathcal{F} f)(y)| &\le \int_I |f(x)|\,|K(x,y)|\,dx \\ &\le \int_I |f(x)|\tilde{K}(x)dx \\ &= \int_I |f(x)|\tilde{K}(x)^{\frac{1}{p}}\tilde{K}(x)^{\frac{1}{p'}}dx \\ &\le \left(\int_I |f(x)|^p\tilde{K}(x)dx\right)^{\frac{1}{p}}\left(\int_I \tilde{K}(x)dx\right)^{\frac{1}{p'}} \\ &= \|f\|_{L^p(I,\tilde{K}(x)dx)}\left(\int_I \tilde{K}(x)dx\right)^{\frac{1}{p'}}. \end{aligned} \tag{4}$$

Proceeding as in (i) one obtains (ii).
(iii) Observe that for $y \in I$

$$\begin{aligned} |(\mathcal{F}f)(y)| &\leq \int_I |f(x)|\tilde{K}(x)dx \\ &\leq \text{esssup}_{x\in I}|f(x)| \cdot \int_I \tilde{K}(x)dx \\ &= \|f\|_{L^\infty(I)} \cdot \int_I \tilde{K}(x)dx. \end{aligned} \tag{5}$$

Thus similar to (i) one obtains (iii).

Proposition 2.2. *The next results hold*
(i) *The integral operator $\mathcal{F}$ given by* (1) *is a bounded linear operator from $L^1(I, \tilde{K}(x)dx)$ into $L^q(I, w(x)dx)$, $0 < q < \infty$, when w is a measurable function on I such that $w > 0$ a.e. on I and $\int_I w(x)dx < \infty$.*
(ii) *The integral operator $\mathcal{F}$ given by* (1) *is a bounded linear operator from $L^p(I, \tilde{K}(x)dx)$, $1 < p < \infty$, into $L^q(I, w(x)dx)$, $0 < q < \infty$, when $\int_I \tilde{K}(x)dx < \infty$, $\int_I w(x)dx < \infty$, being w a measurable function on I such that $w > 0$ a.e. on I.*
(iii) *The integral operator $\mathcal{F}$ given by* (1) *is a bounded linear operator from $L^\infty(I)$ into $L^q(I, w(x)dx)$, $0 < q < \infty$, when $\int_I \tilde{K}(x)dx < \infty$, $\int_I w(x)dx < \infty$, being w a measurable function on I such that $w > 0$ a.e. on I.*

Proof. (i) Observe that from (3) for each $y \in I$ one has

$$|(\mathcal{F}f)(y)| \leq \|f\|_{L^1(I,\tilde{K}(x)dx)}.$$

Then, for $0 < q < \infty$, one has

$$\left(\int_I |(\mathcal{F}f)(x)|^q w(x)dx\right)^{\frac{1}{q}} \leq \|f\|_{L^1(I,\tilde{K}(x)dx)} \left(\int_I w(x)dx\right)^{\frac{1}{q}} < \infty.$$

(ii) The proof is similar to (i) when one make use of (4) instead of (3).
(iii) The proof is similar to (i) when one make use of (5) instead of (3).

3. THE TRANSFORM $\mathcal{F}^*$ OVER THE SPACES $L^1(I)$

Proposition 3.1. *The integral operator $\mathcal{F}^*$ given by* (2) *is a bounded linear operator from $L^1(I)$ into $L^q(I, w(x)dx)$, $0 < q < \infty$, when w is a measurable function on I such that $w > 0$ a.e. on I and $\tilde{K} \in L^q(I, w(x)dx)$.*

Proof. Observe that for each $x \in I$

$$\begin{aligned} |(\mathcal{F}^* f)(x)| &\leq \int_I |f(y)|\,|K(x,y)|\,dy \\ &\leq \int_I |f(y)|dy \cdot \tilde{K}(x). \end{aligned}$$

Then, for $0 < q < \infty$, one has

$$\left(\int_I |(\mathcal{F}^* f)(x)|^q\, w(x)dx\right)^{\frac{1}{q}} \leq \|f\|_{L^1(I)} \left(\int_I (\tilde{K}(x))^q\, w(x)dx\right)^{\frac{1}{q}} < \infty.$$

4. PARSEVAL-GOLDSTEIN TYPE THEOREMS

Theorem 4.1. *For $\mathcal{F}$ and $\mathcal{F}^*$ given by* (1) *and* (2)*, respectively, and $g \in L^1(I)$, then the following Parseval-Goldstein type relation holds*

$$\int_I (\mathcal{F}f)(x)g(x)dx = \int_I f(x)(\mathcal{F}^* g)(x)dx, \tag{6}$$

whenever
(i) $f \in L^1(I, \tilde{K}(x)dx)$,
or
(ii) $f \in L^p(I, \tilde{K}(x)dx)$, $1 < p < \infty$, *and* $\int_I \tilde{K}(x)dx < \infty$,
or
(iii) $f \in L^\infty(I)$, *and* $\int_I \tilde{K}(x)dx < \infty$,

where for all cases $\tilde{K}(x)$ satisfies $|K(x,y)| \leq \tilde{K}(x)$, for all $x, y \in I$.

Proof. (i) In fact, from (3) and for each $y \in I$ one has

$$|(\mathcal{F}f)(y)| \leq \|f\|_{L^1(I,\tilde{K}(x)dx)}.$$

Therefore,

$$\int_I |(\mathcal{F}f)(y)||g(y)|dy \leq \|f\|_{L^1(I,\tilde{K}(x)dx)}\|g\|_{L^1(I)}.$$

Also, for each $x \in I$

$$|(\mathcal{F}^*g)(x)| \leq \int_I |g(y)|\,|K(x,y)|\,dy \leq \tilde{K}(x)\|g\|_{L^1(I)}.$$

Then

$$\begin{aligned}\int_I |f(x)||(\mathcal{F}^*g)(x)|dx &\leq \int_I |f(x)|\tilde{K}(x)dx\|g\|_{L^1(I)}\\ &= \|f\|_{L^1(I,\tilde{K}(x)dx)}\|g\|_{L^1(I)}.\end{aligned}$$

Thus, by using Fubini's theorem one obtains the result (6).
(ii) The proof is similar to (i) making use of (4) instead of (3).
(iii) The proof is similar to (i) making use of (5) instead of (3).

Remark 4.2. From this result the operator $\mathcal{F}^*$ becomes the adjoint of the operator $\mathcal{F}$ over $L^p(I,\tilde{K}(x)dx)$, $1 \leq p < \infty$, and $L^\infty(I)$.

Assume that $K(\cdot,y) \in C^n(I)$ for each $y \in I$ and A_x is a n-th differential operator such that

$$A_x\,(K(x,y)) = P(y)K(x,y), \tag{7}$$

for all $x, y \in I$, where P is a polynomial.
For $k \in \mathbb{N}$ and $K(\cdot,y) \in C^{nk}(I)$ for each $y \in I$, then

$$A_x^k\,(K(x,y)) = [P(y)]^k K(x,y),$$

where A_x^k denotes the k-th power of the operator A_x.
Denote $A_x^{'}$ be the adjoint of A_x.
For $f \in C_c^{nk}(I)$ and $K(\cdot,y) \in C^{nk}(I)$ for each $y \in I$, $k \in \mathbb{N}$, one has

$$\left(\mathcal{F}\left(A_x^{'\,k} f\right)\right)(y) = [P(y)]^k(\mathcal{F}f)(y),\ \ y \in I.$$

Thus for Q being a polynomial of degree m and $f \in C_c^{nm}(I)$ and $K(\cdot,y) \in C^{nm}(I)$ for each $y \in I$, $m \in \mathbb{N}$, then

$$\left(\mathcal{F}\left(Q\left(A_x^{'}f\right)\right)\right)(y) = Q\left(P(y)\right)(\mathcal{F}f)(y),\ \ y \in I. \tag{8}$$

Theorem 4.3. *Set A_x a n-th differential operator satisfying the equality (7) and denote by $A_x^{'}$ its adjoint. Let Q be a polynomial of degree m and $f \in C_c^{nm}(I)$, $K(\cdot,y) \in C^{nm}(I)$ for each $y \in I$. Then, for any $g \in L^1(I)$, the following Parseval-Goldstein relation holds*

$$\int_I (\mathcal{F}f)\,(x)g(x)Q\left(P(x)\right)dx = \int_I \left(Q\left(A_x^{'}f\right)\right)(x)(\mathcal{F}^*g)(x)dx.$$

Proof. The proof is an immediate consequence of relation (8) and (i) of Theorem 4.1, having into account that $C_c^{nm}(I) \subseteq L^1(I,\tilde{K}(x)dx)$.

5. PARTICULAR CASES: THE MAIN INDEX TRANSFORMS AND THE OPERATORS WITH COMPLEX GAUSSIAN KERNELS

In this section, we explore a range of integral transforms characterized by kernels that fulfill specific conditions. These conditions play a crucial role in the properties and applications of these transforms, making them particularly noteworthy in studying Parseval-Goldstein type relations in mathematical analysis. Below, we present examples of integral transforms with kernels satisfying the condition $|K(x,y)| \leq \tilde{K}(x)$, for all $x, y \in I$ and $\int_I \tilde{K}(x)dx < \infty$.
(i) For the Kontorovich-Lebedev transform González and Negrín (2019); Naylor (1990); Prasad A. and Mandal (2018); Srivastava

et al. (2016); Yakubovich (2012); Maan and Negrín (2024) $I = (0, \infty)$, $\tilde{K}(x) = K_0(x)$, where $K_0(x)$ is the modified Bessel function of the third kind (or the Macdonald function) defined by (Erdélyi et al. 1953, p. 5, section 7.2.2., Entry 13) one has that $\int_I \tilde{K}(x)dx < \infty$ and the differential operator is given by $A_x = x^2 D_x^2 + xD_x - x^2$.
(ii) For the Mehler-Fock transform of general order μ with $\Re(\mu) > \frac{-1}{2}$ González and Negrín (2019, 2017); Lebedev (1949); Srivastava et al. (2016); Yakubovich (2012); Maan and Negrín (2024) $I = (0, \infty)$, $\tilde{K}(x) = P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)$, where $P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)$ is the associated Legendre function of the first kind (Erdélyi et al. 1953, p. 122, section 3.2., Entry 3) one has that $\int_I \tilde{K}(x)dx < \infty$ and $A_x = (\sinh x)^{-\mu-1} D_x (\sinh x)^{2\mu+1} D_x (\sinh x)^{-\mu}$ as the differential operator.
(iii) For the index ${}_2F_1$-transform Hayek et al. (1992); Hayek and González (1993, 1994, 1997); Maan and Negrín (2024); Maan et al. (2023) $I = (0, \infty)$, $\Re(\mu) > -1/2$, $\alpha \in \mathbb{C}$, $\tilde{K}(x) = x^{\Re(\alpha)} {}_2F_1\left(\Re(\mu) + \frac{1}{2}, \Re(\mu) + \frac{1}{2}; \Re(\mu) + 1; -x\right)$, where ${}_2F_1(a, b; c; z)$ represents the Gauss hypergeometric function (Erdélyi et al. 1953, p. 56, section 2.1.1., Entry 2). Observe that $\int_I \tilde{K}(x)dx < \infty$ for $-1 < \Re(\alpha) < -1 + \Re(\mu) + \frac{1}{2}$ and the differential operator as $A_x = x^{\alpha-\mu}(x+1)^{-\mu} D_x x^{\mu+1}(x+1)^{\mu+1} D_x x^{-\alpha}$.
(iv) For the Lebedev-Skalskaya transforms Mandal and Prasad (2022); Mandal et al. (2022); Maan and Negrín (2024) $I = (0, \infty)$, $\tilde{K}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}}$ which $\int_I \tilde{K}(x)dx < \infty$ and $A_x = x^2 D_x^2 + 2xD_x - x(x-1)$ as the differential operator for the Lebedev-Skalskaya transform $(\Re f)(y) = \int_0^\infty \Re K_{\frac{1}{2}+iy}(x) f(x)dx$, and $A_x = x^2 D_x^2 + 2xD_x - x(x+1)$ is the differential operator for the Lebedev-Skalskaya transform $(\Im f)(y) = \int_0^\infty \Im K_{\frac{1}{2}+iy}(x) f(x)dx$. Here $\Re K_{\frac{1}{2}+iy}(x)$ and $\Im K_{\frac{1}{2}+iy}(x)$ are the real and imaginary parts of the Macdonald function $K_{\frac{1}{2}+iy}(x)$, respectively (Erdélyi et al. 1953, p. 5, section 7.2.2., Entry 13).
(v) The operators with complex Gaussian kernels González and Negrín (2019, 2018); Negrín (1995) are given by

$$(\mathcal{F} f)(y) = \int_{-\infty}^{\infty} f(x) \exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\} dx, \ \ y \in \mathbb{R}, \epsilon, \beta, \delta, \gamma, \zeta \in \mathbb{C}.$$

In this case $I = (-\infty, \infty)$ and $K(x, y) = \exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\}$. Observe that: $|K(x, y)| \leq \exp\left\{-\Re\epsilon x^2 - \Re\beta y^2 + 2\Re\delta xy + \Re\gamma\right.$
And so, for $\Re\beta \geq 0$ and $\Re\delta = \Re\zeta = 0$ one has $|K(x, y)| \leq \exp\left\{-\Re\epsilon x^2 + \Re\gamma x\right\} = \tilde{K}(x)$.
Thus for (i) one works in $L^1(I, \tilde{K}(x)dx)$ for $\Re\beta \geq 0$, $\Re\delta = \Re\zeta = 0$.
For the cases (ii) and (iii) and being $\Re\beta \geq 0$, $\Re\delta = \Re\zeta = 0$, one also needs $\int_{-\infty}^{\infty} \tilde{K}(x)dx < \infty$ which holds for $\Re\epsilon > 0$.
Concerning the differential operator for the operators with complex Gaussian kernels observe that:

$$\begin{aligned} D_x (K(x, y)) &= D_x \left(\exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\}\right) \\ &= (-2\epsilon x + 2\delta y + \gamma) \exp\left\{-\epsilon x^2 - \beta y^2 + 2\delta xy + \gamma x + \zeta y\right\} \end{aligned}$$

Then

$$D_x (K(x, y)) + 2\epsilon x K(x, y) = (2\delta y + \gamma) K(x, y)$$

So, for this case we take the differential operator as $A_x = D_x + 2\epsilon x$.

Remark 5.1. In the case of index Whittaker transform Maan and Prasad (2022, 2024); Sousa et al. (2020, 2019) $I = (0, \infty)$, $\tilde{K}(x) = x^a \Psi(a, 1; x) x^{-2a-1} e^{-x}$, $a > 0$, where $\Psi(a, 1; x)$ is known as the Tricomi function Sousa et al. (2019). The convergence of the integral $\int_I \tilde{K}(x)dx$ is not assured for $a > 0$.

6. CONCLUSIONS

The current research article extensively explores the continuity properties across Lebesgue spaces for integral transforms within a general framework, including their adjoints. By placing a significant emphasis on Parseval-Goldstein relations, this study unveils the energy-preserving characteristics and inter-domain consistency inherent in these transforms. Such a comprehensive analysis greatly contributes to our comprehension of the fundamental properties and applications of these integral transforms within mathematical analysis. The findings presented herein offer a systematic examination of various index integral transforms, such as the Kontorovich-Lebedev transform, the Mehler-Fock transform of general order, the ${}_2F_1$-transform, the Lebedev-Skalskaya transforms, and also the operators with complex Gaussian kernels.

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