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RESEARCH ARTICLE

A note on vanishing elements and co-degrees of strongly monolithic characters of finite groups

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ABSTRACT
Character theory of finite groups have an important role in understanding the structure of finite groups. A number of previously unresolved problems related to the structure of finite groups have been solved with the development of representation and character theory. There are many articles in the literature on the relationships between the structure of finite groups and their irreducible characters. Today, many researchers continue to study these relationships. Our purpose in this paper is to prove that for determining some properties of the structure of a finite group G, it is enough to consider only strongly monolithic characters of G instead of all irreducible characters of G. We give relationships between the structure of G and the vanishing elements, co-degrees of strongly monolithic characters of G.

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1. INTRODUCTION

Let G be a finite group and $\chi \in \mathrm{Irr}(G)$, where $\mathrm{Irr}(G)$ denotes the set of irreducible complex characters of G. An irreducible character χ of G is called a monolithic character of G if $G/\ker\chi$ has only one minimal normal subgroup. Also, an irreducible character χ of G is said to be monomial if it is induced from a linear character of some subgroup of G. An element $g \in G$ is called a vanishing element if there exists an irreducible character χ of G such that $\chi(g) = 0$. We know from Burnside's theorem (Theorem 3.15) in Isaacs (1976) that a nonlinear irreducible character of a finite group G always vanishes on some conjugacy class of G. An element $g \in G$ is non-vanishing if $\chi(g) \neq 0$ for every irreducible character χ of G. It is known from Isaacs et all. (1999) that if G is solvable and a non-vanishing element x has odd order, then x must lie in the Fitting subgroup $\mathrm{F}(G)$. Later, Dolfi et all. proved in Dolfi et all. (2010) that if x is a non-vanishing element and the order of x is coprime to 6, then $x \in \mathrm{F}(G)$. Erkoç et all. consider in Erkoç et all. (2023) a smaller subset named the set of SM-vanishing conjugacy classes instead of the set of vanishing conjugacy classes of G.

Firstly the co-degree of an irreducible character χ of G was defined as $|G|/\chi(1)$ in Chillag and Herzog (1990). Then it has been given in Qian et all. (2007) as the number $\mathrm{cod}(\chi) = \frac{|G:\ker\chi|}{\chi(1)}$ because it is very useful for inductive proofs of theorems giving information about the structure of G. In Chen and Yang (2020), authors consider the co-degrees of monolithic, monomial irreducible characters.

Motivated by above papers, we give some results about the relationships between the structure of a finite group and its strongly monolitic characters.

2. PRELIMINARIES

In this paper, all groups under consideration are finite and all characters are complex characters. We use the standard notations such as in Isaacs (1976). The definition of strongly monolithic character of a group have been first given in Erkoç et all. (2023).

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It is known from Proposition 2.3 in Erkoç et all. (2023) that linear characters of a group are not strongly monolithic. Thus, abelian groups do not have strongly monolithic characters. However, a nonabelian group have at least one strongly monolithic character. Also, every nonabelian solvable group has at least one monomial strongly monolithic character. The definition of a strongly monolithic character of a group G is the following:

Definition 2.1. *(Erkoç et all. 2023, Definition 2.2)* Let G be a group. An irreducible character χ of G is called a monolithic character if $G/\ker\chi$ has only one minimal normal subgroup. A monolithic character χ of G is called a strongly monolithic character if one of the following conditions is satisfied:

(i) $Z(\chi) = \ker\chi$, where $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$,
(ii) $G/\ker\chi$ is a p-group whose commutator subgroup is its unique minimal normal subgroup.

Definition 2.2. *(Erkoç et all. 2023, Definition 2.2)* Let G be a group. An element g in G is called an SM-vanishing element of G if there exists a strongly monolithic character χ of G such that $\chi(g) = 0$. The conjugacy class of such an element is called an SM-vanishing conjugacy class of G. If χ is a monomial strongly monolithic character of G, then the conjugacy class of such an element is called an MSM-vanishing conjugacy class of G.

Let $\mathrm{Van}_{\mathrm{sm}}(G)$ be the set of SM-vanishing elements of G, that is,

$$\mathrm{Van}_{\mathrm{sm}}(G) = \{g \in G \mid \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}_{\mathrm{sm}}(G)\},$$

where $\mathrm{Irr}_{\mathrm{sm}}(G)$ is the set of all strongly monolithic characters of G.

Let g be an element of a finite group G. If $\chi(g) \neq 0$ for every strongly monolithic character χ of G, then the element g is called an SM-nonvanishing element. If $\chi(g) \neq 0$ for every monomial strongly monolithic character χ of G, then the element g is called an MSM-nonvanishing element.

The following lemma and Theorem 2.4 will be useful when we prove Theorem 3.2. Actually, we know from Lemma 2.3 of Isaacs et all. (1999) that if x is a nonvanishing element in a finite group G, then x fixes some member of each orbit of the action of G on $\mathrm{Irr}(N)$ where $N \lhd G$.

Lemma 2.3. *Let G be a solvable group with a unique minimal normal subgroup M and $\Phi(G) = 1$. Assume that $x \in G$ is an MSM-nonvanishing element of G. Then x fixes an element in every G-orbit on Irr(M).*

Proof. $1_M \neq \lambda \in \mathrm{Irr}(M)$ and $T = I_G(\lambda)$, where $I_G(\lambda)$ is the inertia group of λ in G. Since $\Phi(G) = 1$, there is a subgroup H of G such that $G = MH$ and $M \cap H = 1$. We know from Problem 6.18 in Isaacs (1976) that there exists a linear character $\theta \in \mathrm{Irr}(T)$ such that $\theta_M = \lambda$. Let $\chi = \theta^G$. Then χ is a faithful irreducible character of G. Otherwise, we would have that $M \leq \ker\chi = \cap_{g\in G}(\ker\theta)^g \leq \ker\theta$, which is a contradiction that $\theta_M = \lambda = 1$. On the other hand, it is clear that $\mathrm{Z}(G) = 1$ since $\Phi(G) = 1$. This implies that $\chi \in \mathrm{Irr}(G)$ is a monomial strongly monolithic character of G. Since $x \in G$ is an MSM-nonvanishing element of G, we get that $\chi(x) \neq 0$. By the definition of the induced character θ^G, there exists an element g of G such that $x^g \in T$. Then x stabilizes $\lambda^{g^{-1}}$, and the proof is complete.

Theorem 2.4. (Isaacs et all. 1999, Theorem 4.2) *Let G act faithfully and irredicibly on a finite vector space V. Let $x \in F(G)$ fix an element in each orbit of G on V. Then $x^2 = 1$.*

3. MAIN RESULTS

It is known that an irreducible character of a group G is called to be of q-defect zero if q does not divide $|G|/\chi(1)$, where q is a prime number. We know from Theorem 8.17 in Isaacs (1976) that if χ is an irreducible character of q-defect zero of G, then $\chi(g) = 0$ whenever q divides the order of $g \in G$.

Let $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$ such that $N \leq \ker\chi$. It is well-known that there exists a one-to-one correspondence between irreducible characters of G/N and irreducible characters of G with kernel containing N. Thus, it is easy to see that χ is a strongly monolithic character of G if and only if χ is a strongly monolithic character of G/N. In the following theorem, we use the notation x^G to denote the conjugacy class of G containing $x \in G$.

Theorem 3.1. *Let G be a finite group. If the set of SM-vanishing elements of G are the union of at most three conjugacy classes of G, then G is solvable.*

Proof. Let G be a counterexample to the theorem with minimum possible order. Suppose that G has two distinct minimal normal subgroups M_1 and M_2. It is easy to see that the hypotheses of theorem are inherited by factor groups. Thus, both of G/M_1 and

G/M_2 are solvable groups by induction. Since G is isomorphic to a subgroup of $G/M_1 \times G/M_2$, we have a contradiction that G is solvable. This implies that G cannot have two distinct minimal normal subgroups. Now, let M be the unique minimal normal subgroup of G. Since G is a counterexample, M must be nonabelian and $Z(G) = 1$. Therefore, there exists a nonabelian simple group S such that $M = S_1 \times \cdots \times S_k$ where $k \geq 1$ and $S_i \cong S$ for every i. First assume that S has irreducible characters of q-defect zero for every prime q dividing the order of S. Thus, if θ is an irreducible character of q-defect zero of S, then $\psi := \theta \times \cdots \times \theta$ is an irreducible character of q-defect zero of M. It follows from Lemma 2.4 in Erkoç et all. (2023) that every element of M of order divisible by q is an SM-vanishing element of G. We know that $2||M|$, because M is a nonsolvable group. Also, there exist distinct primes p and q such that $p, q \geq 3$ and $p, q \in \pi(M)$. Hence, there exist x, y and z elements of S such that $|x| = 2$, $|y| = p$ and $|z| = q$. Since $x, y, z \in \mathrm{Van}_{\mathrm{sm}}(G)$ and the set of SM-vanishing elements of G are the union of at most three conjugacy classes of G, we get that $\pi(M) = \{2, p, q\}$ and $\mathrm{Van}_{\mathrm{sm}}(G) = x^G \cup y^G \cup z^G$. Then, M must be a simple group. Otherwise, we would have $k \geq 2$. Without loss of generality, we may assume that $y \in S_1$ and $z \in S_2$. Thus, we would have that $|yz| = pq$. But this contradicts with the hypothesis of theorem because $(yz)^G \notin \{x^G, y^G, z^G\}$ and $yz \in \mathrm{Van}_{\mathrm{sm}}(G)$. Since M is non-cyclic simple group of order divisible by exactly three primes, we obtain from Theorem 1 in Herzog (1968) that $M \in \{PSL(2,5),\ PSL(2,8),\ PSL(2,17),\ PSL(2,7),\ PSL(2,9),\ PSL(3,3),\ U_3(3),\ U_4(2)\}$. Using the Atlas Conway et all. (1985), we obtain the following table containing $x_i \in M$ of distinct orders for $1 \leq i \leq 4$.

M	$\lvert x_1\rvert$	$\lvert x_2\rvert$	$\lvert x_3\rvert$	$\lvert x_4\rvert$
$PSL(2,7)$	$2A$	$3A$	$4A$	$7A$
$PSL(2,9)$	$2A$	$3A$	$4A$	$5A$
$PSL(2,8)$	$2A$	$3A$	$7A$	$9A$
$PSL(2,17)$	$2A$	$3A$	$4A$	$17A$
$PSL(3,3)$	$2A$	$3A$	$4A$	$13A$
$U_3(3)$	$2A$	$3A$	$4A$	$7A$
$U_4(2)$	$2A$	$3C$	$4A$	$5A$

Therefore, M cannot be groups in the list. Since $C_G(M) = 1$, we know that G is almost simple group. Therefore, we get that $G \cong A_5$ or $G \cong S_5$. But this is a contradiction because the set of SM-vanishing elements of A_5 or S_5 are union of more than three conjugacy classes of the group. Therefore, there exists a prime number q dividing the order of S such that S does not have any irreducible character of q-defect zero. It follows from Lemma 2.3 in Robati (2019) that there exist irreducible characters θ_1, θ_2, θ_3, θ_4 of S which extend to $\mathrm{Aut}(S)$ and elements x_1, x_2, x_3, x_4 of distinct order such that $\theta_i(x_i) = 0$ for $1 \leq i \leq 4$. Also, we have from Lemma 5 in Bianchi et all. (2007) that $\theta_i \times \cdots \times \theta_i \in \mathrm{Irr}(M)$ extends to G for $1 \leq i \leq 4$. Now, let $\psi_i \in \mathrm{Irr}(G)$ such that $(\psi_i)_M = \theta_i \times \cdots \times \theta_i$ for $1 \leq i \leq 4$. It is clear that ψ_i is a faithful irreducible character of G for $1 \leq i \leq 4$. Otherwise, we would have that $M \leq \ker\psi_i \cap M = \ker(\psi_i)_M = \ker(\theta_i \times \cdots \times \theta_i) = 1$ for $1 \leq i \leq 4$, which is a contradiction. Therefore, ψ_i is a strongly monolithic character of G and $\psi_i(x_i) = 0$ for $1 \leq i \leq 4$. Since the elements x_i are of distinct orders, x_i elements lie in distinct conjugacy classes of G for $1 \leq i \leq 4$ and so, the set of SM-vanishing elements of G are the union of at least four conjugacy classes of G, which is a contradiction. This contradiction completes the proof. $\square$

Now, we consider the semidirect product $G := He_3 \rtimes C_2$ (SmallGroup $(54, 8)$ in GAP) where C_2 acts faithfully on He_3. The notations C_2 and He_3 denote a cyclic group of order 2 and a nonabelian group of order 27 of exponent 3, respectively. Since $1 < Z(G)$, all faithful irreducible characters of G are not strongly monolithic. G has only four strongly monolithic characters of degree 2. While the set of SM-vanishing elements of G are the union of three conjugacy classes of G, the set of vanishing elements of G are the union of seven conjugacy classes of G. Thus, Theorem 3.1 generalizes [Robati (2019), Theorem 2.8].

Theorem 3.2. *Let G be a solvable group and x be an element of odd order of G. If $\chi(x) \neq 0$ for all monomial strongly monolithic character χ of G, then $x \in F(G)$.*

Proof. Let G be a counterexample to the theorem with minimum possible order. By induction, $xN \in \mathrm{F}(G/N)$ for every nontrivial normal subgroup N of G because $2 \nmid |xN|$ and $\theta(xN) \neq 0$ for every monomial strongly monolithic character θ of G/N. Suppose that G has two distinct minimal normal subgroups M_1 and M_2. Then we know that $\varphi : G \longrightarrow G/M_1 \times G/M_2$, defined by $\varphi(g) = (gM_1, gM_2)$ for $g \in G$, is an injective homomorphism. Hence, we get that $\varphi(x) \in \mathrm{F}(G/M_1) \times \mathrm{F}(G/M_2) = \mathrm{F}(G/M_1 \times G/M_2)$ and so, $\varphi(x) \in \varphi(G) \cap \mathrm{F}(G/M_1 \times G/M_2) \leq \mathrm{F}(\varphi(G))$. Thus, we obtain that $x \in \mathrm{F}(G)$, which is a contradiction. This implies that G cannot have two distinct minimal normal subgroups. Let M be be the unique minimal normal subgroup of G. It is clear that $\Phi(G) = 1$ because $\mathrm{F}(G/\Phi(G)) = \mathrm{F}(G)/\Phi(G)$. It follows from Gaschütz Theorem (III, 4.5 in Huppert (1967)) that $\mathrm{F}(G) = M$ and so $C_G(M) = M$. Now, let V be the group of irreducible characters of M. Then, V is faithful and irreducible G/M-module. Also, we know from Lemma 2.3 that the element xM fixes some element of each orbit of G/M on V. On the other hand, we see that $xM \in \mathrm{F}(G/M)$ by the induction. Hence, we have from Teorem 2.4 that $(xM)^2 = x^2M = M$ and so, we obtain that $x^2 \in M$.

Therefore, we conlude that $x \in M = \mathrm{F}(G)$ because x is an element of odd order of G, which is a contradiction. This contradiction completes the proof. □

Let G be a finite group and $g \in G$. In Pang et all. (2016), authors prove that if the order of $gG' \in G/G'$ does not divide $|\mathrm{Irr}_m(G)|$, then there exists χ in $\mathrm{Irr}_m(G)$ such that $\chi(g) = 0$ where $\mathrm{Irr}_m(G)$ is the set of all irreducible monomial characters of G. Similarly, we give the following theorem.

Theorem 3.3. *Let G be a finite group, χ be a nonlinear irreducible character of G whose kernel is maximal among the kernels of all nonlinear irreducible characters of G and $g \in G$. If the order of gN in G/N does not divide $|Irr_{sm}(G/ker\chi)|$ where $N = G'ker\chi$, then g is an SM-vanishing element of G.*

Proof. Let χ be a nonlinear irreducible character of G whose kernel is maximal among the kernels of all nonlinear irreducible characters of G. We know from Corollary 2.6 in Erkoç et all. (2023) χ is a strongly monolithic character of $G/\ker\chi$. Furthermore for any linear character λ of $G/\ker\chi$, $\chi\lambda$ is a strongly monolithic character of $G/\ker\chi$. Hence, λ permutes $\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)$. We get that

$$\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi) = \{\, \theta\lambda \mid \theta \in \mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)\,\}.$$

This implies that

$$\prod_{\theta\in \mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)} \theta(g) = \prod_{\theta\in \mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)} (\theta\lambda)(g) = \left(\prod_{\theta\in \mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)} \theta(g) \right) \lambda(g)^n,$$

where $n = |\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)|$. If g is an SM-nonvanishing element of G, then by the above equality, $\lambda(g)^n = 1$ for any linear character λ of $G/\ker\chi$. It follows that $g^n\ker\chi \in G'\ker\chi/\ker\chi$. Then, we have that $|gN|$ divides $|\mathrm{Irr}_{\mathrm{sm}}(G/\ker\chi)|$, which contradicts with our hypothesis. This contradiction completes the proof. □

Theorem 3.4. *Let G be a solvable group and let p be a prime divisor of $|G|$. If $cod(\chi)$ is a p'-number for every monomial strongly monolithic character χ of G, then G has a normal p-complement.*

Proof. Let G be a counterexample to the assertion with the minimal possible order. Since the hypotheses of the theorem are inherited by factor groups, G has a unique minimal normal subgroup M. It follows that G/M has a normal p-complement by induction. Since G does not have a normal p-complement, p must divide $|M|$. Thus, M is elementary abelian p-subgroup. Furthermore, we have $\mathrm{Z}(G) = 1$. Otherwise, a Hall p'-subgroup H of G would be normal since $MH \trianglelefteq G$ and H is a characteristic subgroup of MH. Moreover, we have from Lemma 1 (a) in Berkovich and Zhmud' (1997) that $\Phi(G) = 1$. Then, there exists a subgroup K of G such that $G = MK$ and $M \cap K = 1$. Let λ be a nonprincipal character in $\mathrm{Irr}(M)$. Write $T = \mathrm{I}_G(\lambda)$ as the inertia group of λ in G. Notice that M is complemented in G and so is in T. We get that $T = M\mathrm{I}_K(\lambda)$. It follows from Problem 6.18 in Isaacs (1976) that λ extends to T and so there exists a linear character $\theta \in \mathrm{Irr}(T)$ such that $\theta_M = \lambda$. This implies that $\chi = \theta^G$ is a monomial irreducible character of G. Thus, χ is a faithful irreducible character of G. Otherwise, we get that $M \leq \ker\chi = \bigcap_{g\in G} (\ker\theta)^g \leq \ker\theta$. But this contradicts with $\theta_M = \lambda \neq 1$. Hence χ is a monomial strongly monolithic character of G, since $\mathrm{Z}(G) = 1$. By the assumption, we have that

$$\mathrm{cod}(\chi) = \frac{|G:\ker\chi|}{\chi(1)} = \frac{|G|}{\theta^G(1)} = \frac{|G|}{|G:T|} = |T| = |M|.|\mathrm{I}_K(\lambda)|$$

is a p'-number. This contradicts with the fact that M is a p-group. The proof is complete. □

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