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A TWISTED HOPF ALGEBRA OF FINITE TOPOLOGICAL QUANDLES

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Abstract. This paper describes some algebraic properties of the species of finite topological quandles. We construct two twisted bialgebra structures on this species, one of the first kind and one of the second kind. The obstruction for the structure to match the double twisted bialgebra axioms is explicitly described.

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1. Introduction

A quandle is a set Q with a binary operation $\triangleleft : Q \times Q \longrightarrow Q$ satisfying the three axioms

(i) for every $a \in Q$, we have $a \triangleleft a = a$,
(ii) for every pair $a, b \in Q$ there is a unique $c \in Q$ such that $a = c \triangleleft b$, and
(iii) for every $a, b, c \in Q$, we have $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

These three conditions that define a quandle originate from the axiomatization of the three Reidemeister moves on knot diagrams. Quandles are algebraic structures that have various applications in knot theory and related fields. Two typical examples of quandles are the conjugation quandle and the core quandle. The conjugation quandle is derived from any group $(G, \circ)$, where the binary operation is given by conjugation, i.e. $x \triangleleft y = y^{-1} \circ x \circ y$. The core quandle, on the other hand, is another quandle derived from any group $(G, \circ)$, with the binary operation defined by $x \triangleleft y = x \circ y^{-1} \circ x$. Both of these quandles are of great importance in knot theory and have been studied extensively. For more on quandles, see [5,6,12,14,19].

By Alexandroff's theorem [1,17], for any finite set X, there is a bijection between topologies on X and quasi-orders on X, where a quasi-order $\leq$ in X is a reflexive and transitive relation, not necessarily antisymmetric. For any $x, y \in X$, we write $x \leq_{\mathcal{T}} y$ whenever any $\mathcal{T}$-open subset containing x also contains y, and we note

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$x \sim_{\mathcal{T}} y$ whenever both $x \leq_{\mathcal{T}} y$ and $y \leq_{\mathcal{T}} x$ hold. More on finite topological spaces can be found in [2, 3, 16, 18].

Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on a finite set X, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$, denoted by $\mathcal{T}' \prec \mathcal{T}$, if every open subset of $\mathcal{T}$ is also an open subset of $\mathcal{T}'$. This is equivalent to saying that for any $x, y \in X$, if $x \leq_{\mathcal{T}'} y$, then $x \leq_{\mathcal{T}} y$. The quotient $\mathcal{T}/\mathcal{T}'$ of these two topologies is defined as follows: the associated quasi-order relation, denoted by $\leq_{\mathcal{T}/\mathcal{T}'}$, is the transitive closure of the relation $\mathcal{R}$, which is defined by $x\mathcal{R}y$ if and only if $x \leq_{\mathcal{T}} y$ or $y \leq_{\mathcal{T}'} x$. In [7], F. Fauvet, L. Foissy and the second author define a relation noted $\circledleq$ on the set of topologies in X as follows: $\mathcal{T}' \circledleq \mathcal{T}$ if and only if

- $\mathcal{T}' \prec \mathcal{T}$,
- $\mathcal{T}'_{|Y} = \mathcal{T}_{|Y}$ for any subset $Y \subset X$ connected for the topology $\mathcal{T}'$,
- for any $x, y \in X$,

$$x \sim_{\mathcal{T}/\mathcal{T}'} y \iff x \sim_{\mathcal{T}'/\mathcal{T}'} y.$$

Now let $(Q, \leq)$ be a topological space equipped with a continuous map $\mu : Q \times Q \longrightarrow Q$, denoted by $\mu(a, b) = a \lhd b$, such that for every $b \in Q$ the mapping $R_b : a \mapsto a \lhd b$ is a homeomorphism of $(Q, \leq)$. The space Q (together with the map μ) is called a topological quandle [15] if it satisfies for all $a, b, c \in Q$

(i) $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$,
(ii) $a \lhd a = a$.

A finite topological quandle [9, 13] is a topological quandle with a finite underlying set. The study of finite topological quandles is important because finite quandles arise naturally in the study of knots and links, and topological quandles provide a way to study the geometry of these structures [4].

The species formalism, due to A. Joyal [10, 11], is an important tool in combinatorics. The idea of a species is to formalize the notion of "combinatorial equivalence" between objects of a given type, so that one can study the properties of the objects without worrying about their particular representations. To be precise, a linear species is a contravariant functor from the category of finite sets with bijections to the category of vector spaces over a given field $\mathbf{k}$. Specifically, a linear species $\mathbb{E}$ assigns to any finite set X a vector space $\mathbb{E}_X$ over $\mathbf{k}$, and assigns to any bijection $\sigma : X \to Y$ a linear isomorphism $\mathbb{E}_\sigma : \mathbb{E}_Y \to \mathbb{E}_X$, such that the awaited functorial properties hold[1]. One important operation on linear species is the Cauchy tensor product, denoted by $\otimes$, which takes two species $\mathbb{E}$ and $\mathbb{F}$ and produces a new species $\mathbb{E} \otimes \mathbb{F}$ defined by

[1]Contravariance is a matter of choice, some authors prefer covariant species.

- $(\mathbb{E}\otimes\mathbb{F})_A = \bigoplus_{I\subseteq A} \mathbb{E}_I \otimes \mathbb{F}_{A\setminus I}$,
- for any bijection $\sigma : B \longrightarrow A$,

$$(\mathbb{E}\otimes\mathbb{F})(\sigma) : \begin{cases} \bigoplus_{J\subseteq B} \mathbb{E}_J \otimes \mathbb{F}_{B\setminus J} \longrightarrow \bigoplus_{I\subseteq B} \mathbb{E}_I \otimes \mathbb{F}_{B\setminus I} \\ \qquad x\otimes y \longmapsto \mathbb{E}(\sigma_{|I})(x)\otimes \mathbb{F}(\sigma_{|B\setminus I})(y). \end{cases}$$

We also recall that, for any two linear species $\mathbb{E}$ and $\mathbb{F}$, their Hadamard tensor product is defined by [8]:

- $(\mathbb{E}\odot\mathbb{F})(A) = \mathbb{E}(A)\otimes\mathbb{F}(A)$,
- for bijection $\sigma : B \longrightarrow A$, $(\mathbb{E}\odot\mathbb{F})(\sigma) = \mathbb{E}(\sigma)\otimes\mathbb{F}(\sigma)$.

The species $\mathbb{QT}$ of finite topological quandles describes finite topological quandles up to combinatorial equivalence. Specifically, the species $\mathbb{QT}$ is a contravariant functor from the category of finite sets with bijections to the category of vector spaces, which associates to each finite set S the linear span of all finite topological quandles with underlying set S, i.e., the species $\mathbb{QT}$ is defined by:

- for any finite set A, $\mathbb{QT}_A$ is the vector space freely generated by the topological quandle stuctures on A, i.e., $\mathbb{QT}_A = \mathrm{span}(A, \lhd, \leq)$, where $(A,\lhd)$ is a quandle and $\leq$ is a quasi-order compatible with $(A,\lhd)$,
- for any bijection $\sigma : B \longrightarrow A$, $\mathbb{Q}_\sigma$ sends the topological quandle $Q = (A,\lhd,\leq)$ to the topological quandle $\mathbb{Q}_\sigma(Q) = (B, \blacktriangleleft, \leq')$, where $\blacktriangleleft$ and $\leq'$ are defined by relabeling.

The present article is organized as follows: in Section 2, we revisit some important results related to finite quandles. Specifically, we remind the reader of the method developed by B. Ho and S. Nelson in [9] to describe finite quandles with at most 5 elements. Section 3 contains our main results: we construct an external coproduct Δ defined for all $(X,\mathcal{T},\lhd)\in\mathbb{QT}_X$ (where X is a finite set) by:

$$\begin{aligned} \Delta : \mathbb{QT}_X &\longrightarrow (\mathbb{QT}\otimes\mathbb{QT})_X = \bigoplus_{Y\subset X} \mathbb{QT}_{X\setminus Y}\otimes\mathbb{QT}_Y \\ (X,\mathcal{T},\lhd) &\longmapsto \sum_{Y\subseteq X} (X\setminus Y, \mathcal{T}_{|_{X\setminus Y}}, \lhd^{X\setminus Y}) \otimes (Y, \mathcal{T}_{|_Y}, \lhd^Y), \end{aligned}$$

with an explicit quandle structure $\lhd^Z$ on any subset Z of a finite quandle X. We moreover define for any finite set X an internal coproduct $\Gamma : \mathbb{QT}_X \longrightarrow (\mathbb{QT}\odot\mathbb{QT})_X = \mathbb{QT}_X\otimes\mathbb{QT}_X$ by, for all $(X,\mathcal{T},\lhd)\in\mathbb{QT}_X$:

$$\Gamma(X,\mathcal{T},\lhd) = \sum_{\substack{\mathcal{T}'\circledcirc\mathcal{T} \\ \mathcal{T}' \text{ is Q-compatible}}} (X,\mathcal{T}',\lhd)\otimes(X,\mathcal{T}/\mathcal{T}',\lhd).$$

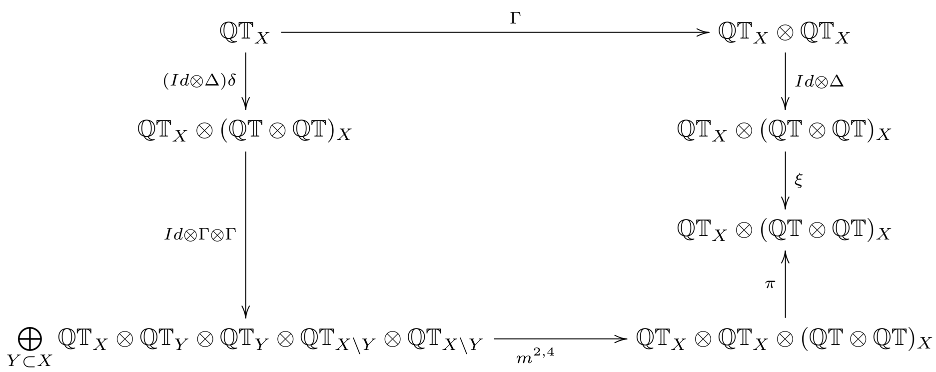
It indeed turns out that the quandle structure is compatible with both topologies $\mathcal{T}'$ and $\mathcal{T}/\mathcal{T}'$. The associative product m of two topological quandle structures on X and Y respectively is given by the disjoint union of the topological spaces and the quandle structures involved: the action of elements of X on Y (and vice-versa) is trivial. We prove that $(\mathbb{QT}, m, \Delta)$ is a commutative connected twisted bialgebra and $(\mathbb{QT}, m, \Gamma)$ is a commutative connected twisted bialgebra on the second kind [8]. Finally, we define a map

$$\xi : \mathbb{QT}_X \otimes (\mathbb{QT} \otimes \mathbb{QT})_X \longrightarrow \mathbb{QT}_X \otimes (\mathbb{QT} \otimes \mathbb{QT})_X$$

by:

$$\xi\big((X, \mathcal{T}, \lhd) \otimes (Y, \mathcal{T}_1, \lhd_1) \otimes (X \backslash Y, \mathcal{T}_2, \lhd_2)\big) = (X, \mathcal{T}, \widetilde{\lhd}) \otimes (Y, \mathcal{T}_1, \lhd_1) \otimes (X \backslash Y, \mathcal{T}_2, \lhd_2)$$

where the new quandle structure $\widetilde{\lhd}$ is explicitly given, such that the coproduct Γ and the map ξ make the following diagram commute:



i.e.,

$$\xi \circ (\mathrm{Id} \otimes \Delta) \circ \Gamma = \pi \circ m^{2,4} \circ (Id \otimes \Gamma \otimes \Gamma) \circ (Id \otimes \Delta) \circ \delta,$$

where δ is the diagonal map $(X, \mathcal{T}, \lhd) \mapsto (X, \mathcal{T}, \lhd) \otimes (X, \mathcal{T}, \lhd)$, and where π is a linear map which will be described below. In other words, $\mathbb{QT}$ is nearly a twisted double bialgebra in the sense of [8], the defect being precisely described by the maps ξ and π.

2. Review of finite quandles

Let $Q = \{x_1, x_2, ..., x_n\}$ be a finite quandle with n elements. B. Ho and S. Nelson in [9] defined the matrix of Q, denoted M_Q, to be the matrix whose entry in row i column j is $x_i \lhd x_j$:

$$M_Q = \begin{bmatrix} x_1 \lhd x_1 & x_1 \lhd x_2 & ... & x_1 \lhd x_n \\ x_2 \lhd x_1 & x_2 \lhd x_2 & ... & x_2 \lhd x_n \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ x_n \lhd x_1 & x_n \lhd x_2 & ... & x_n \lhd x_n \end{bmatrix}$$

Example 2.1. [9] For $Q = \{a, b, c\}$, the quandle matrices for quandles of order 3 are, up to permutations of the underlying three-element set:

$$\begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix}, \qquad \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}, \qquad \begin{bmatrix} a & a & a \\ c & b & b \\ b & c & c \end{bmatrix}.$$

Definition 2.2. Let Q be a quandle. A subquandle $X \subset Q$ is a subset of Q which is itself a quandle under $\lhd$. Let Q be a quandle and $X \subset Q$ a subquandle. We say that X is complemented in Q or Q-complemented if $Q \backslash X$ is a subquandle of Q.

Notation. Let $(Q, \lhd)$ be a finite quandle, for $x' \in Q$, we note

$$\begin{aligned} R_{x'} : Q &\longrightarrow Q \\ x &\longmapsto x \lhd x', \end{aligned} \qquad \text{and} \qquad \begin{aligned} L_{x'} : Q &\longrightarrow Q \\ x &\longmapsto x' \lhd x. \end{aligned}$$

Remark 2.3. Due to the fact that any bijection from a finite set onto itself if of finite order, any continuous bijection from a finite topological space onto itself is a homeomorphism. The following statements are therefore equivalent for any finite quandle Q:

- $(Q, \mathcal{T})$ is a topological quandle,
- For any $x \in Q$, R_x and L_x are continuous maps,
- for all $x, y, x', y' \in X$, if $x \leq x'$ and $y \leq y'$, we have $x \lhd y \leq x' \lhd y'$.

3. Algebraic structure of the linear species of finite topological quandles

Let $Q = (X, \lhd)$ be a finite quandle, and Z be any subset of X. Let

$$\lhd^Y : Y \times Y \longrightarrow Y$$

be defined by $a \lhd^Y b = R_b^{\alpha^Y}(a)$, where $\alpha^Y = \inf\{\alpha, R_b^{\alpha}|_{X \backslash Y} = \mathrm{Id}_{X \backslash Y}$ for any $b \in Y\}$. We may shorten the notation to α when the context is clear enough.

Proposition 3.1. *Let $Q = (X, \lhd)$ be a finite quandle. For any subset Y of Q, the pair $(Y, \lhd^Y)$ is a quandle.*

Proof. It is clear that, for all $a \in Y$, equality $a \lhd^Y a = a$ holds. Moreover for all $b \in Y$, the map $R_b^\alpha : Y \to Y$ is a bijection. For all $a, b, c \in Y$, we have

$$(a \lhd^Y c) \lhd^Y (b \lhd^Y c) = R_c^\alpha(a) \lhd^Y R_c^\alpha(b) = R^\alpha_{R_c^\alpha(b)} \circ R_c^\alpha(a) \tag{1}$$

and

$$(a \lhd^Y b) \lhd^Y c = R_c^\alpha \circ R_b^\alpha(a). \tag{2}$$

Using $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$, i.e., $R_c \circ R_b(a) = R_{R_c(b)} \circ R_c(a)$, then for all $n \in \mathbb{N}$ we have,

$$\begin{aligned}
R_c^n \circ R_b &= R_c^{n-1} \circ R_c \circ R_b \\
&= R_c^{n-1} \circ R_{R_c(b)} \circ R_c \\
&= R_c^{n-2} \circ R_c \circ R_{R_c(b)} \circ R_c \\
&= R_c^{n-2} \circ R_{R_c^2(b)} \circ R_c^2 \\
&\;. \\
&\;. \\
&\;. \\
&= R_c \circ R_{R_c^{n-1}(b)} \circ R_c^{n-1} \\
&= R_{R_c^n(b)} \circ R_c^n,
\end{aligned}$$

and for all $m \in \mathbb{N}$ we have,

$$\begin{aligned}
R_c \circ R_b^m &= R_c \circ R_b \circ R_b^{m-1} \\
&= R_{R_c(b)} \circ R_c \circ R_b \circ R_b^{m-2} \\
&= R_{R_c(b)} \circ R_{R_c(b)} \circ R_c \circ R_b^{m-2} \\
&= R^2_{R_c(b)} \circ R_c \circ R_b^{m-2} \\
&\;. \\
&\;. \\
&\;. \\
&= R^{m-1}_{R_c(b)} \circ R_c \circ R_b. \\
&= R^m_{R_c(b)} \circ R_c.
\end{aligned}$$

Hence, for all $n, m \in \mathbb{N}$ we have

$$R_c^n \circ R_b^m = R^m_{R_c^n(b)} \circ R_c^n, \tag{3}$$

or equivalently $R^m_{R^n_c(b)} = R^n_c \circ R^m_b \circ R^{-n}_c$. Applying (3) to $m = n = \alpha$, we therefore get self-distributivity in view of (1) and (2), which proves Proposition 3.1. □

Lemma 3.2. *For any finite quandle $(X, \lhd)$ and for any $Z \subseteq Y \subseteq X$ we have*

$$z_1 (\lhd^Y)^Z z_2 = z_1 \lhd^Z z_2$$

for any $z_1, z_2 \in Z$.

Proof. It is clear that α^Y divides α^Z, and we have for any $z_1, z_2 \in Z$:

$$z_1 (\lhd^Y)^Z z_2 = \left(R^{\alpha^Y}_{z_2}\right)^{\frac{\alpha^Z}{\alpha^Y}} z_1 = z_1 \lhd^Z z_2. \qquad \square$$

Let X be any finite set and $\mathbb{Q}_X = \mathrm{span}(X, \lhd)$ the vector space of quandles in X. We define the external coproduct Δ by:

$$\begin{aligned} \Delta : \mathbb{Q}_X &\longrightarrow (\mathbb{Q} \otimes \mathbb{Q})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{Q}_Y \otimes \mathbb{Q}_Z \\ (X, \lhd) &\longmapsto \sum_{Y \sqcup Z = X} (Y, \lhd^Y) \otimes (Z, \lhd^Z), \end{aligned} \tag{4}$$

and we define an associative product m in $\mathbb{Q}$ by $m : \mathbb{Q}_{X_1} \otimes \mathbb{Q}_{X_2} \longrightarrow \mathbb{Q}_{X_1 \sqcup X_2}$, defined for all $Q_1 = (X_1, \lhd_1) \in \mathbb{Q}_{X_1}$, $Q_2 = (X_2, \lhd_2) \in \mathbb{Q}_{X_2}$, $m(Q_1 \otimes Q_2) = (X_1 \sqcup X_2, \widetilde{\lhd})$, where

- $a \widetilde{\lhd} b = a \lhd_1 b$, for all $a, b \in X_1$,
- $a \widetilde{\lhd} b = a \lhd_2 b$, for all $a, b \in X_2$,
- $a \widetilde{\lhd} b = a$, for all $a \in X_i, b \in X_j$, $i \neq j$.

The product m endows the disjoint union with a quandle structure. Indeed:

(i) For any $a \in X_1 \sqcup X_2$, we obviously have $a \widetilde{\lhd} a = a$.

(ii) Let $x \in X_i, i = 1, 2$, then $R_x(x') = x'$ for all $x' \in X_j, j \neq i$ and furthermore, for every $z \in X_i$, there exists a unique $x' \in X_i$ such that $z = R_x(x')$ (because R_x is a bijection from X_i on itself). Hence R_x is a bijection from $X_1 \sqcup X_2$ on itself.

(iii) Let $x, y, z \in X_1 \sqcup X_2$. Four possible cases may arise:

- Either $x, y, z \in X_i$, it is obvious that $(x \widetilde{\lhd} y) \widetilde{\lhd} z = (x \widetilde{\lhd} z) \widetilde{\lhd} (y \widetilde{\lhd} z)$.
- Or $x \in X_i, y, z \in X_j, i \neq j$, we have;
  $(x \widetilde{\lhd} y) \widetilde{\lhd} z = x \widetilde{\lhd} z = x$ and $(x \widetilde{\lhd} z) \widetilde{\lhd} (y \widetilde{\lhd} z) = x \widetilde{\lhd} y = x$.
- Or $x, y \in X_i, z \in X_j, i \neq j$, we have;
  $(x \widetilde{\lhd} y) \widetilde{\lhd} z = x \widetilde{\lhd} y = x \lhd_i y$ and $(x \widetilde{\lhd} z) \widetilde{\lhd} (y \widetilde{\lhd} z) = x \widetilde{\lhd} y = x \lhd_i y$.
- Or $x, z \in X_i, y \in X_j, i \neq j$, we have;
  $(x \widetilde{\lhd} y) \widetilde{\lhd} z = x \widetilde{\lhd} z = x \lhd_i z$ and $(x \widetilde{\lhd} z) \widetilde{\lhd} (y \widetilde{\lhd} z) = (x \lhd_i z) \widetilde{\lhd} y = x \lhd_i z$.

The self-distributivity

$$(x \widetilde{\lhd} y) \widetilde{\lhd} z = (x \widetilde{\lhd} z) \widetilde{\lhd} (y \widetilde{\lhd} z)$$

is therefore always verified, hence $(X_1 \sqcup X_2, \widetilde{\lhd})$ is a quandle.

Example 3.3.

$$m\left(\begin{bmatrix} c & c & c \\ e & d & d \\ d & e & e \end{bmatrix} \otimes \begin{bmatrix} a & b \\ b & b \end{bmatrix}\right) = \begin{bmatrix} c & c & c & c & c \\ e & d & d & d & d \\ d & e & e & e & e \\ a & a & a & a & b \\ b & b & b & b & b \end{bmatrix},$$

$$m\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix} \otimes \begin{bmatrix} c & e & d \\ e & d & c \\ d & c & e \end{bmatrix}\right) = \begin{bmatrix} a & b & a & a & a \\ a & b & b & b & b \\ c & c & c & e & d \\ d & d & e & d & c \\ e & e & d & c & e \end{bmatrix},$$

$$\begin{aligned}\Delta\left(\begin{bmatrix} a & a & a \\ c & b & b \\ b & c & c \end{bmatrix}\right) &= \begin{bmatrix} a & a & a \\ c & b & b \\ b & c & c \end{bmatrix} \otimes \mathbf{1} + \mathbf{1} \otimes \begin{bmatrix} a & a & a \\ c & b & b \\ b & c & c \end{bmatrix} \\ &+ \begin{bmatrix} a \end{bmatrix} \otimes \begin{bmatrix} b & b \\ c & c \end{bmatrix} + \begin{bmatrix} b \end{bmatrix} \otimes \begin{bmatrix} a & a \\ c & c \end{bmatrix} + \begin{bmatrix} c \end{bmatrix} \otimes \begin{bmatrix} a & a \\ b & b \end{bmatrix} \\ &+ \begin{bmatrix} b & b \\ c & c \end{bmatrix} \otimes \begin{bmatrix} a \end{bmatrix} + \begin{bmatrix} a & a \\ c & c \end{bmatrix} \otimes \begin{bmatrix} b \end{bmatrix} + \begin{bmatrix} a & a \\ b & b \end{bmatrix} \otimes \begin{bmatrix} c \end{bmatrix},\end{aligned}$$

$$\begin{aligned}\Delta\left(\begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}\right) &= \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix} \otimes \mathbf{1} + \mathbf{1} \otimes \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix} \\ &+ \begin{bmatrix} a \end{bmatrix} \otimes \begin{bmatrix} b & b \\ c & c \end{bmatrix} + \begin{bmatrix} b \end{bmatrix} \otimes \begin{bmatrix} a & a \\ c & c \end{bmatrix} + \begin{bmatrix} c \end{bmatrix} \otimes \begin{bmatrix} a & a \\ b & b \end{bmatrix} \\ &+ \begin{bmatrix} b & b \\ c & c \end{bmatrix} \otimes \begin{bmatrix} a \end{bmatrix} + \begin{bmatrix} a & a \\ c & c \end{bmatrix} \otimes \begin{bmatrix} b \end{bmatrix} + \begin{bmatrix} a & a \\ b & b \end{bmatrix} \otimes \begin{bmatrix} c \end{bmatrix}.\end{aligned}$$

Theorem 3.4. $(\mathbb{Q}, m, \Delta, u, \varepsilon)$ *is a commutative cocommutative connected twisted bialgebra.*

Proof. Here the unit u is given by $u(1) = \mathbf{1}$ where $\mathbf{1}$ is the empty quandle, and the counit ε is given by $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(Q, \lhd) = 0$ for any nonempty finite quandle. We start by showing associativity of the disjoint union product m. Let $Q_1 = (X_1, \lhd_1) \in \mathbb{Q}_{X_1}$, $Q_2 = (X_2, \lhd_2) \in \mathbb{Q}_{X_2}$ and $Q_3 = (X_3, \lhd_3) \in \mathbb{Q}_{X_3}$, we have $m(m \otimes Id)(Q_1 \otimes Q_2 \otimes Q_3) = (X_1 \sqcup X_2 \sqcup X_3, \lhd)$, where:

- $a \lhd b = a \lhd_1 b$, for all $a, b \in X_1$,
- $a \lhd b = a \lhd_2 b$, for all $a, b \in X_2$,
- $a \lhd b = b$, for all $a \in X_1, b \in X_2$,
- $a \lhd b = a$, for all $a \in X_2, b \in X_1$,
- $a \lhd b = a \lhd_3 b$, for all $a, b \in X_3$,
- $a \lhd b = b$, for all $a \in X_1 \sqcup X_2, b \in X_3$,
- $a \lhd b = a$, for all $a \in X_3, b \in X_1 \sqcup X_2$.

On the other hand,
$m(m \otimes Id)(Q_1 \otimes Q_2 \otimes Q_3) = (X_1 \sqcup X_2 \sqcup X_3, \overline{\lhd})$, where:

- $a \overline{\lhd} b = a \lhd_1 b$, for all $a, b \in X_1$,
- $a \overline{\lhd} b = a \lhd_2 b$, for all $a, b \in X_2$,
- $a \overline{\lhd} b = a \lhd_3 b$, for all $a, b \in X_3$,
- $a \overline{\lhd} b = a$, for all $a \in X_2, b \in X_3$,
- $a \overline{\lhd} b = a$, for all $a \in X_3, b \in X_2$,
- $a \overline{\lhd} b = b$, for all $a \in X_1, b \in X_2 \sqcup X_3$,
- $a \overline{\lhd} b = a$, for all $a \in X_2 \sqcup X_3, b \in X_1$.

So, $m(m \otimes \mathrm{Id})(Q_1 \otimes Q_2 \otimes Q_3) = (X_1 \sqcup X_2 \sqcup X_3, \lhd) = (X_1 \sqcup X_2 \sqcup X_3, \overline{\lhd}) = m(m \otimes \mathrm{Id})(Q_1 \otimes Q_2 \otimes Q_3)$.

Cocommutativity of the coproduct is clear, let us look at coassociativity. We have for any finite quandle $Q = (X, \lhd)$:

$$(\Delta \otimes \mathrm{Id})\Delta(X, \lhd) = \sum_{X = A \sqcup B \sqcup C} \big(A, (\lhd^{A \sqcup B})^A\big) \otimes \big(B, (\lhd^{A \sqcup B})^B\big) \otimes (C, \lhd^C).$$

On the other hand, we have

$$(\mathrm{Id} \otimes \Delta)\Delta(X, \lhd) = \sum_{X = A \sqcup B \sqcup C} (A, \lhd^A) \otimes \big(B, (\lhd^{B \sqcup C})^B\big) \otimes \big(C, (\lhd^{B \sqcup C})^C\big).$$

From Lemma 3.2 we therefore get

$$(\Delta \otimes \mathrm{Id})\Delta(X, \lhd) = (\mathrm{Id} \otimes \Delta)\Delta(X, \lhd) = \sum_{X = A \sqcup B \sqcup C} (A, \lhd^A) \otimes (B, \lhd^B) \otimes (C, \lhd^C),$$

which proves coassociativity of Δ. Finally, we show immediately that

$$\Delta \circ m\big((X_1, \lhd_1) \otimes (X_2, \lhd_2)\big) = m^{2,3}\big(\Delta(X_1, \lhd_1) \otimes \Delta(X_2, \lhd_2)\big). \qquad \square$$

The following result is a topological version of Proposition 3.1:

Proposition 3.5. *For any finite topological quandle $(X, \mathcal{T}, \lhd)$ and for any subset $Y \subseteq X$, the triple $(Y, \mathcal{T}_{|_Y}, \lhd^Y)$ is a topological quandle.*

Proof. Let $y_1, y_1', y_2, y_2' \in Y$ with $y_1 \leq_{\mathcal{T}} y_1'$ and $y_2 \leq_{\mathcal{T}} y_2'$. From compatibility between the topology and the quandle structure, we have $R^n_{y_2} y_1 \leq_{\mathcal{T}} R^n_{y_2'} y_1'$ for any nonnegative integer n. In particular,

$$y_1 \lhd^Y y_2 = R^{\alpha}_{y_2} y_1 \leq_{\mathcal{T}} R^{\alpha}_{y_2'} y_1' = y_1' \lhd^Y y_2'.$$ □

Theorem 3.6. *Let X be any finite set and $\mathbb{QT}_X = span(X, \mathcal{T}, \lhd)$, where $(X, \lhd)$ is a quandle and $\mathcal{T}$ is a topology compatible with $(X, \lhd)$. Let m the product defined in $\mathbb{QT}$ by*

$$m\big((X_1, \mathcal{T}_1, \lhd_1) \otimes (X_2, \mathcal{T}_2, \lhd_2)\big) = (X_1 \sqcup X_2, \mathcal{T}_1\mathcal{T}_2, \widetilde{\lhd})$$

and let Δ the coproduct defined by

$$\Delta(X, \mathcal{T}, \lhd) = \sum_{y \in \mathcal{T}} \left(X \setminus Y, \mathcal{T}\big|_{X\setminus Y}, \lhd^{X\setminus Y}\right) \otimes \left(Y, \mathcal{T}\big|_{Y}, \lhd^{Y}\right).$$

Then $(\mathbb{QT}, m, \Delta, u, \varepsilon)$ is a commutative connected twisted bialgebra.

Proof. It suffices to show the coassociativity of coproduct Δ in the species of topological quandles $\mathbb{QT}$. Let X be a finite set and $Q = (X, \mathcal{T}, \lhd) \in \mathbb{QT}_X$, we have in view of Lemma 3.2:

$$(\Delta \otimes \mathrm{Id})\Delta(X, \mathcal{T}, \lhd) = \sum_{Z, Y \in \mathcal{T},\, Z \subseteq Y} (X\setminus Y, \mathcal{T}_{|X\setminus Y}, \lhd^{X\setminus Y}) \otimes (Y\setminus Z, \mathcal{T}_{|Y\setminus Z}, \lhd^{Y\setminus Z}) \otimes (Z, \mathcal{T}_{|Z}, \lhd^{Z})$$

and

$$(\mathrm{Id} \otimes \Delta)\Delta(X, \mathcal{T}, \lhd) = \sum_{Z, Y \in \mathcal{T},\, Z \subseteq Y} (X\setminus Y, \mathcal{T}_{|X\setminus Y}, \lhd^{X\setminus Y}) \otimes (Y\setminus Z, \mathcal{T}_{|Y\setminus Z}, \lhd^{Y\setminus Z}) \otimes (Z, \mathcal{T}_{|Z}, \lhd^{Z}),$$

which proves Theorem 3.6. The unit and counit axioms, as well as the compatibility with the disjoint union product, are straightforward. □

Remark 3.7. Note that this coproduct on the species of finite topological quandles is not cocommutative, contrarily to the coproduct defined by (4) on the species of finite quandles.

Lemma 3.8. [7, Proposition 2.7] *Let $\mathcal{T}$ and $\mathcal{T}''$ be two topologies on X. If $\mathcal{T}'' \lessdot \mathcal{T}$, then $\mathcal{T}' \longmapsto \mathcal{T}'/\mathcal{T}''$ is a bijection from the set of topologies $\mathcal{T}'$ on X such that $\mathcal{T}'' \lessdot \mathcal{T}' \lessdot \mathcal{T}$, onto the set of topologies $\mathcal{U}$ on X such that $\mathcal{U} \lessdot \mathcal{T}/\mathcal{T}''$.*

Theorem 3.9. *Let $Q = (X, \lhd)$ be a finite quandle and let $\mathcal{T}$ be a topology on X. For any $\mathcal{T}' \lessdot \mathcal{T}$ we have:*

1. *if $\mathcal{T}$ and $\mathcal{T}'$ are Q-compatible, then $\mathcal{T}/\mathcal{T}'$ is Q-compatible,*
2. *if $\mathcal{T}$ and $\mathcal{T}/\mathcal{T}'$ are Q-compatible, then $\mathcal{T}'$ is Q-compatible.*

Proof. Let $Q = (X, \lhd)$ be a finite quandle and let $\mathcal{T}' \circleddash \mathcal{T}$.

(1) If $\mathcal{T}$ and $\mathcal{T}'$ are Q-compatible, then: for $x, x', y, y' \in X$, $x \leq_{\mathcal{T}/\mathcal{T}'} x'$ and $y \leq_{\mathcal{T}/\mathcal{T}'} y'$ imply that there exist $t_1, \ldots, t_n, s_1, \ldots, s_m \in X$, such that $x\mathcal{R}t_1\mathcal{R}t_2 \ldots \mathcal{R}t_n\mathcal{R}y$ and $x'\mathcal{R}s_1\mathcal{R}s_2 \ldots \mathcal{R}s_m\mathcal{R}y'$. Recall that $a\mathcal{R}b$ means ($a \leq_{\mathcal{T}} b$ or $a \geq_{\mathcal{T}'} b$). First, we prove that $x \lhd y\mathcal{R}t_1 \lhd s_1$ or $x \lhd y\mathcal{R}x \lhd s_1\mathcal{R}t_1 \lhd s_1$. For $x\mathcal{R}t_1$, and $y\mathcal{R}s_1$, we have four possible cases:

- First case; $x \leq_{\mathcal{T}} t_1$, and $y \leq_{\mathcal{T}} s_1$. Since $\mathcal{T}$ is Q-compatible, we have $x \lhd y \leq_{\mathcal{T}} t_1 \lhd s_1$, hence $x \lhd y\mathcal{R}t_1 \lhd s_1$.
- Second case; $x \geq_{\mathcal{T}'} t_1$, and $y \geq_{\mathcal{T}'} s_1$. Since $\mathcal{T}'$ is Q-compatible, we have $x \lhd y \geq_{\mathcal{T}'} t_1 \lhd s_1$, hence $x \lhd y\mathcal{R}t_1 \lhd s_1$.
- Third case; $x \leq_{\mathcal{T}} t_1$, and $y \geq_{\mathcal{T}'} s_1$. Since R_{s_1} is continuous, we have $R_{s_1}(x) \leq_{\mathcal{T}} R_{s_1}(t_1)$, so $x \lhd s_1\mathcal{R}t_1 \lhd s_1$ and since L_x is continuous, $L_x(y) \geq_{\mathcal{T}'} L_x(s_1)$, so $x \lhd y \geq_{\mathcal{T}'} x \lhd s_1$. Therefore $(x \lhd y)\mathcal{R}(x \lhd s_1)\mathcal{R}(t_1 \lhd s_1)$.
- Fourth case; $x \geq_{\mathcal{T}'} t_1$, and $y \leq_{\mathcal{T}} s_1$. Since R_{s_1} is continuous, we have $R_{s_1}(x) \geq_{\mathcal{T}'} R_{s_1}(t_1)$, so $x \lhd s_1\mathcal{R}t_1 \lhd s_1$ and since L_x is continuous, $L_x(y) \leq_{\mathcal{T}} L_x(s_1)$, so $(x \lhd y)\mathcal{R}(x \lhd s_1)$. Therefore $(x \lhd y)\mathcal{R}(x \lhd s_1)\mathcal{R}(t_1 \lhd s_1)$. By induction we find that:

$$(x \lhd y)\mathcal{R}(x \lhd s_1)\mathcal{R}(t_1 \lhd s_1)\mathcal{R}(t_1 \lhd s_2)\mathcal{R}(t_2 \lhd s_2)\mathcal{R}(t_2 \lhd s_3)\mathcal{R} \ldots$$

$$\ldots \mathcal{R}(t_{n-1} \lhd s_{n-1})\mathcal{R}(t_{n-1} \lhd s_n)\mathcal{R}(t_n \lhd s_n)\mathcal{R}(t_n \lhd y')\mathcal{R}(x' \lhd y').$$

Therefore $\mathcal{T}/\mathcal{T}'$ is Q-compatible.

(2) If $\mathcal{T}$ and $\mathcal{T}/\mathcal{T}'$ are Q-compatible, then: for $x, x', y, y' \in X$, using that $\mathcal{T}' \circleddash \mathcal{T}$, we get that ($x \leq_{\mathcal{T}'} x'$ and $y \leq_{\mathcal{T}'} y'$) implies ($x \leq_{\mathcal{T}} x'$ and $y \leq_{\mathcal{T}} y'$). Using that $\mathcal{T}$ is Q-compatible, then $x \lhd y \leq_{\mathcal{T}} x' \lhd y'$. On the other hand, ($x \leq_{\mathcal{T}'} x'$ and $y \leq_{\mathcal{T}'} y'$) implies ($x \sim_{\mathcal{T}/\mathcal{T}'} x'$ and $y \sim_{\mathcal{T}/\mathcal{T}'} y'$). Using that $\mathcal{T}/\mathcal{T}'$ is Q-compatible, then $x \lhd y \sim_{\mathcal{T}/\mathcal{T}'} x' \lhd y'$. So $x \lhd y$ and $x' \lhd y'$ are in the same connected component for the topology $\mathcal{T}'$, then: $x \lhd y \leq_{\mathcal{T}} x' \lhd y'$ implies that $x \lhd y \sim_{\mathcal{T}'} x' \lhd y'$ (because $\mathcal{T}'_{|Y} = \mathcal{T}_{|Y}$ for any subset $Y \subset X$ connected for the topology $\mathcal{T}'$). This proves that $\mathcal{T}'$ is Q-compatible. □

We define the internal coproduct Γ for all $(X, \mathcal{T}, \lhd) \in \mathbb{QT}_X$ by:

$$\Gamma(X, \mathcal{T}, \lhd) = \sum_{\substack{\mathcal{T}' \circleddash \mathcal{T} \\ \mathcal{T}' \text{ is Q-compatible}}} (X, \mathcal{T}', \lhd) \otimes (X, \mathcal{T}/\mathcal{T}', \lhd).$$

Theorem 3.10. $(\mathbb{QT}, m, \Gamma)$ *is a commutative twisted bialgebra of the second kind.*

Proof. Let X be a finite set, for $(X, \mathcal{T}, \lhd) \in \mathbb{QT}_X$, we have

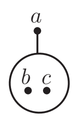
$$(\Gamma \otimes \mathrm{Id})\Gamma(X, \mathcal{T}, \lhd) = \sum_{\substack{\mathcal{T}'' \lessdot \mathcal{T}' \lessdot \mathcal{T} \\ \mathcal{T}'', \mathcal{T}' \text{ are Q-compatible}}} (X, \mathcal{T}'', \lhd) \otimes (X, \mathcal{T}'/\mathcal{T}'', \lhd) \otimes (X, \mathcal{T}/\mathcal{T}', \lhd).$$

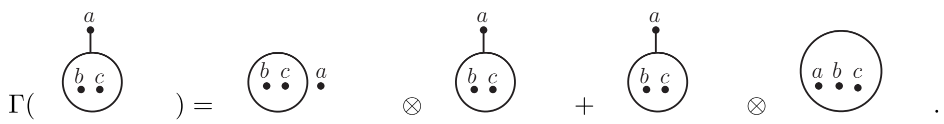
On the other hand, we have

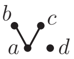
$$(\mathrm{Id} \otimes \Gamma)\Gamma(X, \mathcal{T}, \lhd) = \sum_{\substack{\mathcal{T}'' \lessdot \mathcal{T},\, \mathcal{U} \lessdot \mathcal{T}/\mathcal{T}'' \\ \mathcal{T}'', \mathcal{U} \text{ are Q-compatible}}} (X, \mathcal{T}'', \lhd) \otimes (X, \mathcal{U}, \lhd) \otimes \big(X, (\mathcal{T}/\mathcal{T}'')/\mathcal{U}, \lhd\big).$$

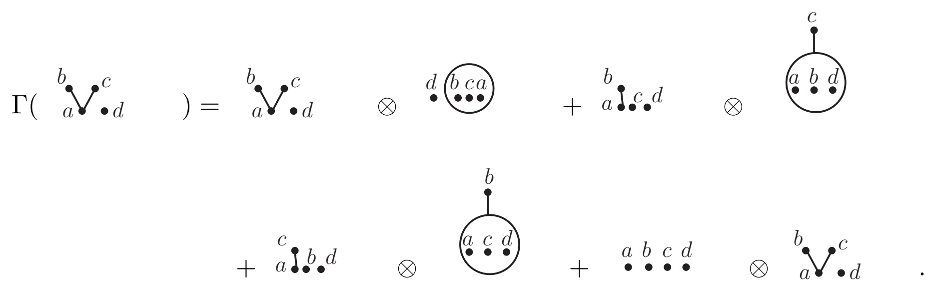
The result then comes from Lemma 3.8 and Theorem 3.9. Hence, $(\Gamma \otimes \mathrm{Id})\Gamma = (\mathrm{Id} \otimes \Gamma)\Gamma$, and consequently Γ is coassociative. Finally we have directly:

$$\Gamma\big((X_1, \mathcal{T}_1, \lhd_1)(X_2, \mathcal{T}_2, \lhd_2)\big) = \Gamma(X_1, \mathcal{T}_1, \lhd_1)\Gamma(X_2, \mathcal{T}_2, \lhd_2). \qquad \square$$

Example 3.11. For $(X, \lhd) = \begin{bmatrix} a & a & a \\ c & b & b \\ b & c & c \end{bmatrix}$ and $\mathcal{T} =$  , then $(X, \mathcal{T}, \lhd)$ is a topological quandle and



For $(X, \lhd) = \begin{bmatrix} a & a & a & a \\ b & b & b & c \\ c & c & c & b \\ d & d & d & d \end{bmatrix}$ and $\mathcal{T} =$  , $(X, \mathcal{T}, \lhd)$ is a topological quandle and



Theorem 3.12. *For any finite set X, let*

$$\xi : \mathbb{QT}_X \otimes (\mathbb{QT} \otimes \mathbb{QT})_X \longrightarrow \mathbb{QT}_X \otimes (\mathbb{QT} \otimes \mathbb{QT})_X$$

be the map defined by:

$$\xi\big((X,\mathcal{T},\lhd)\otimes(Y,\mathcal{T}_1,\lhd_1)\otimes(X\backslash Y,\mathcal{T}_2,\lhd_2)\big)=\begin{cases}(X,\mathcal{T},\widetilde{\lhd})\otimes(Y,\mathcal{T}_1,\lhd_1)\otimes(X\backslash Y,\mathcal{T}_2,\lhd_2)\\ \textit{if } \mathcal{T} \textit{ is } \widetilde{\lhd}\textit{-compatible,}\\ 0 \;\textit{else,}\end{cases}\tag{5}$$

where the quandle $(X,\widetilde{\lhd})$ *is the disjoint union product of the two quandles* $(Y,\lhd_1)$ *and* $(X\backslash Y,\lhd_2)$*. The following diagram commutes:*

$$\begin{array}{ccc}
\mathbb{QT}_X & \xrightarrow{\quad\Gamma\quad} & \mathbb{QT}_X\otimes\mathbb{QT}_X\\
\downarrow{\scriptstyle (Id\otimes\Delta)\delta} & & \downarrow{\scriptstyle Id\otimes\Delta}\\
\mathbb{QT}_X\otimes(\mathbb{QT}\otimes\mathbb{QT})_X & & \mathbb{QT}_X\otimes(\mathbb{QT}\otimes\mathbb{QT})_X\\
 & & \downarrow{\scriptstyle \xi}\\
\downarrow{\scriptstyle Id\otimes\Gamma\otimes\Gamma} & & \mathbb{QT}_X\otimes(\mathbb{QT}\otimes\mathbb{QT})_X\\
 & & \uparrow{\scriptstyle \pi}\\
\bigoplus_{Y\subset X}\mathbb{QT}_X\otimes\mathbb{QT}_Y\otimes\mathbb{QT}_Y\otimes\mathbb{QT}_{X\backslash Y}\otimes\mathbb{QT}_{X\backslash Y} & \xrightarrow[m^{2,4}]{} & \mathbb{QT}_X\otimes\mathbb{QT}_X\otimes(\mathbb{QT}\otimes\mathbb{QT})_X
\end{array}$$

i.e.,

$$\xi\circ(\mathrm{Id}\otimes\Delta)\circ\Gamma=\pi\circ m^{2,4}\circ(Id\otimes\Gamma\otimes\Gamma)\circ(Id\otimes\Delta)\circ\delta,$$

where δ *is the diagonal map* $(X,\mathcal{T},\lhd)\mapsto(X,\mathcal{T},\lhd)\otimes(X,\mathcal{T},\lhd)$*, and where* π *is a linear map which will be described below.*

Proof. Let X be a finite set and $(X,\mathcal{T},\lhd)\in\mathbb{TQ}_X$, we have

$$\xi\circ(\mathrm{Id}\otimes\Delta)\circ\Gamma(X,\mathcal{T},\lhd)=\xi\circ(\mathrm{Id}\otimes\Delta)\left(\sum_{\substack{\mathcal{T}'\lessdot\mathcal{T}\\ \mathcal{T}' \text{ is } \lhd\text{-compatible}}}(X,\mathcal{T}',\lhd)\otimes(X,\mathcal{T}/\mathcal{T}',\lhd)\right)$$

$$=\xi\left(\sum_{\substack{\mathcal{T}'\lessdot\mathcal{T},\ \mathcal{T}' \ \lhd\text{-compatible}\\ Y\in\mathcal{T}/\mathcal{T}'}}(X,\mathcal{T}',\lhd)\otimes\big(X\backslash Y,(\mathcal{T}/\mathcal{T}')\big|_{X\backslash Y},\lhd^{X\backslash Y}\big)\otimes\big(Y,(\mathcal{T}/\mathcal{T}')\big|_{Y},\lhd^{Y}\big)\right)$$

$$=\sum_{\substack{\mathcal{T}'\lessdot\mathcal{T},\ \mathcal{T}' \ \lhd\text{-compatible}\\ Y\in\mathcal{T}/\mathcal{T}',\ \mathcal{T}' \ \widetilde{\lhd}\text{-compatible}}}(X,\mathcal{T}',\widetilde{\lhd})\otimes\big(X\backslash Y,(\mathcal{T}/\mathcal{T}')\big|_{X\backslash Y},\lhd^{X\backslash Y}\big)\otimes\big(Y,(\mathcal{T}/\mathcal{T}')\big|_{Y},\lhd^{Y}\big).$$

On the other hand,

$$\begin{aligned}
&m^{1,3}\circ(\Gamma\otimes\Gamma)\circ\Delta(X,\mathcal{T},\lhd)\\
&\quad= m^{1,3}\circ(\Gamma\otimes\Gamma)\left(\sum_{Y\in\mathcal{T}}(X\backslash Y,\mathcal{T}_{|_{X\backslash Y}},\lhd^{X\backslash Y})\otimes(Y,\mathcal{T}_{|_{Y}},\lhd^{Y})\right)\\
&\quad= m^{1,3}\Bigg(\sum_{\substack{Y\in\mathcal{T},\,\mathcal{T}_1\circledleq\mathcal{T}_{|_Y},\,\mathcal{T}_2\circledleq\mathcal{T}_{|_{X\backslash Y}}\\ \mathcal{T}_1\text{ is }\lhd^{Y}\text{-comp},\,\mathcal{T}_2\text{ is }\lhd^{X\backslash Y}\text{-comp.}}}(X\setminus Y,\mathcal{T}_2,\lhd^{X\backslash Y})\otimes\\
&\qquad\otimes(X\setminus Y,\mathcal{T}_{|_{X\backslash Y}}/\mathcal{T}_2,\lhd^{X\backslash Y})\otimes(Y,\mathcal{T}_1,\lhd^{Y})\otimes(Y,\mathcal{T}_{|_{Y}}/\mathcal{T}_1,\lhd^{Y})\Bigg)\\
&\quad= \sum_{\substack{Y\in\mathcal{T},\,\mathcal{T}_1\circledleq\mathcal{T}_{|_Y},\,\mathcal{T}_2\circledleq\mathcal{T}_{|_{X\backslash Y}}\\ \mathcal{T}_1\text{ is }\lhd^{Y}\text{-comp},\,\mathcal{T}_2\text{ is }\lhd^{X\backslash Y}\text{-comp.}}}(X,\mathcal{T}_1\mathcal{T}_2,\widetilde{\lhd})\otimes(X\setminus Y,\mathcal{T}_{|_{X\backslash Y}}/\mathcal{T}_2,\lhd^{X\backslash Y})\otimes(Y,\mathcal{T}_{|_{Y}}/\mathcal{T}_1,\lhd^{Y})\\
&\quad= \sum_{\substack{Y\in\mathcal{T},\,\mathcal{T}'\circledleq\mathcal{T}_{|_Y}\sqcup\mathcal{T}_{|_{X\backslash Y}}\\ \mathcal{T}'\text{ is }\widetilde{\lhd}\text{-comp}}}(X,\mathcal{T}',\widetilde{\lhd})\otimes\big(X\setminus Y,(\mathcal{T}/\mathcal{T}')_{|_{X\backslash Y}},\lhd^{X\backslash Y}\big)\otimes\big(Y,(\mathcal{T}/\mathcal{T}')_{|_{Y}},\lhd^{Y}\big)\\
&\quad= \sum_{\substack{\mathcal{T}'\circledleq\mathcal{T}\\ Y\in\mathcal{T}/\mathcal{T}',\,\mathcal{T}'\,\widetilde{\lhd}\text{-compatible}}}(X,\mathcal{T}',\widetilde{\lhd})\otimes\big(X\setminus Y,(\mathcal{T}/\mathcal{T}')_{|_{X\backslash Y}},\lhd^{X\backslash Y}\big)\otimes\big(Y,(\mathcal{T}/\mathcal{T}')_{|_{Y}},\lhd^{Y}\big).
\end{aligned}$$

Theorem 3.12 is therefore verified with π defined by

$$\pi\Big((X,\mathcal{T},\lhd)\otimes(X,\mathcal{T}',\widetilde{\lhd})\otimes\big(X\setminus Y,(\mathcal{T}/\mathcal{T}')_{|_{X\backslash Y}},\lhd^{X\backslash Y}\big)\otimes\big(Y,(\mathcal{T}/\mathcal{T}')_{|_{Y}},\lhd^{Y}\big)\Big)=$$

$$\begin{cases}(X,\mathcal{T}',\widetilde{\lhd})\otimes\big(X\setminus Y,(\mathcal{T}/\mathcal{T}')_{|_{X\backslash Y}},\lhd^{X\backslash Y}\big)\otimes\big(Y,(\mathcal{T}/\mathcal{T}')_{|_{Y}},\lhd^{Y}\big)\\ \text{ if }\mathcal{T}'\text{ is }\lhd\text{-compatible,}\\ 0\text{ else.}\end{cases}$$

□

We therefore notice that Γ and Δ are not compatible, i.e. we do not get a double twisted bialgebra. The maps ξ and π above precisely account for the defect.

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References

[1] P. Alexandroff, *Diskrete raume*, Rec. Math. (Mat. Sbornik) N.S., 2(44)(3) (1937), 501-519.

[2] M. Ayadi, *Twisted pre-Lie algebras of finite topological spaces*, Comm. Algebra, 50(5) (2022), 2115-2138.

[3] M. Ayadi and D. Manchon, *Doubling bialgebras of finite topologies*, Lett. Math. Phys., 111(4) (2021), 102 (23 pp).

[4] J. S. Carter, J. Scott, S. Kamada and M. Saito, Surfaces in 4-Space, Chapter 5, Springer Science and Business Media, 2012.

[5] M. Elhamdadi, *Distributivity in quandles and quasigroups*, in Algebra, geometry and mathematical physics, Springer Proc. Math. Stat. Springer, Heidelberg, 85 (2014), 325-340.

[6] M. Elhamdadi and S. Nelson, Quandles - An Introduction to the Algebra of Knots, Student Mathematical Library 74, Amer. Math. Soc., Providence, 2015.

[7] F. Fauvet, L. Foissy and D. Manchon, *The Hopf algebra of finite topologies and mould composition*, Ann. Inst. Fourier, 67(3) (2017), 911-945.

[8] L. Foissy, *Twisted bialgebras, cofreeness and cointeraction*, arXiv:1905.10199 [math.RA] (2019).

[9] B. Ho and S. Nelson, *Matrices and finite quandles*, Homology Homotopy Appl., 7(1) (2005), 197-208.

[10] A. Joyal, *Une theorie combinatoire des series formelles*, Adv. in Math, 42(1) (1981), 1-82.

[11] A. Joyal, *Foncteurs analytiques et espèces de structures*, Combinatoire enumerative (Montreal, Que., 1985/Quebec, Que., 1985), Lecture Notes in Math., 1234 (1986), 126-159.

[12] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, 23(1) (1982), 37-65.

[13] P. Lopes and D. Roseman, *On finite racks and quandles*, Comm. Algebra, 34(1) (2006), 371-406.

[14] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.), 119(161) (1982), 78-88,

[15] R. L. Rubinsztein, *Topological quandles and invariants of links*, J. Knot Theory Ramifications, 16(6) (2007), 789-808.

[16] A. K. Steiner, *The lattice of topologies: Structure and complementation*, Trans. Amer. Math. Soc., 122 (1966), 379-398.

[17] R. E. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc., 123 (1966), 325-340.

[18] R. Vaidyanathaswamy, Set Topology, 2nd ed. Chelsea Publishing Co., New York, 1960.

[19] D. N. Yetter, *Quandles and monodromy*, J. Knot Theory Ramifications, 12(4) (2003), 523-541.

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