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*Research Article*

# Weighted approximations by sampling type operators: recent and new results

OSMAN ALAGÖZ*

ABSTRACT. In this paper, we collect some recent results on the approximation properties of generalized sampling operators and Kantorovich operators, focusing on pointwise and uniform convergence, rate of convergence, and Voronovskaya-type theorems in weighted spaces of functions. In the second part of the paper, we introduce a new generalization of sampling Durrmeyer operators including a special function $\rho$ which satisfies certain assumptions. For the family of newly constructed operators, we obtain pointwise convergence, uniform convergence and rate of convergence for functions belonging to weighted spaces of functions.

**Keywords:** Sampling series, generalized sampling operator, Kantorovich operator, Durrmeyer type sampling operator, weighted approximation.

**2020 Mathematics Subject Classification:** 41A25, 41A35.

## 1. INTRODUCTION

The reconstruction of a function from its sample values is an extensively studied problem in approximation theory. Butzer and his school extended the approximation to the entire real axis (see [9, 10, 11, 12]) by defining the family of generalized sampling operators:

$$(S_w^\phi f)(t) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\phi(wt-k), \quad x\in\mathbb{R}, w>0, \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a kernel that meets specific approximate identities, and $f : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function on $\mathbb{R}$.

The series given by (1.1) is meaningful for functions that make the series converge and provides an approximation method in the case, where the function $f$ is continuous. However, the reconstruction problem of functions that do not need to be continuous was solved in [7], by replacing the data (sample) points $\frac{k}{w}$, for $k \in \mathbb{Z}$ and $w > 0$, with the integral mean value $w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)\,du$ and defining the generalized sampling Kantorovich operators, which are the $L^1$ version of the generalized sampling operators

$$(K_w^\chi f)(x) = \sum_{k\in\mathbb{Z}} \left\{ w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)\,du \right\} \chi(wx-k), \quad x\in\mathbb{R}. \tag{1.2}$$

Here, $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function and $\chi : \mathbb{R} \to \mathbb{R}$ is a kernel satisfying certain suitable conditions.

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*Corresponding author: Osman Alagöz; osman.alagoz@bilecik.edu.tr
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The generalized sampling Kantorovich operators, represented by (1.2), have been effectively utilized in the engineering fields. Significant numerical results have been obtained, particularly in the study of thermal bridges and the behavior of buildings under seismic actions using thermographic images (see, [6, 13, 18]).

While the sampling Kantorovich series offers an approximation for functions belonging to the $L^1$ space, it does not provide an approximation for functions in $L^p$ spaces. To solve this problem, C. Bardaro and I. Mantellini [14] introduced the sampling Durrmeyer series, meaningful for $L^p$, $1 \leq p < \infty$, by taking the convolution of function $f$ with a kernel function instead of the mean values of $f$. This is given by

$$(S_w^{\varphi,\psi} f)(x) := \sum_{k\in\mathbb{Z}} \varphi(wx-k) w \int_{\mathbb{R}} \psi(wu-k) f(u)\, du, \quad x \in \mathbb{R},\, w > 0. \tag{1.3}$$

For more recent papers about sampling type series see [16, 17].

## 2. Preliminaries

Throughout this paper, we denote the sets of all positive integers, integers, and real numbers by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, respectively. The space of all continuous functions on $\mathbb{R}$ (not necessarily bounded) is represented by $C(\mathbb{R})$. The space of all bounded continuous functions on $\mathbb{R}$, denoted by $C_B(\mathbb{R})$, is equipped with the norm $\|f\| := \sup_{x\in\mathbb{R}} |f(x)|$. Additionally, $UC(\mathbb{R})$ refers to the subspace of $C_B(\mathbb{R})$ that includes all uniformly continuous functions and for $r \in \mathbb{R}$, we denote the space of $C^r(\mathbb{R})$ which consists of all $r$-times continuously differentiable functions on $\mathbb{R}$.

A function $\chi : \mathbb{R} \to \mathbb{R}$ is called a kernel function if it satisfies the following assumptions:

($\chi$1) $\chi$ is continuous on $\mathbb{R}$.

($\chi$2) The discrete algebraic moment of order 0

$$m_0(\chi, u) = \sum_{k\in\mathbb{Z}} \chi(u-k) = 1 \quad \text{for every } u \in \mathbb{R}.$$

($\chi$3) There exists $\beta > 0$ such that the discrete absolute moment of order $\beta$ is finite, i.e.,

$$M_\beta(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)||u-k|^\beta < +\infty.$$

**Lemma 2.1** ([7]). *Let $\chi$ be a kernel satisfying ($\chi$1) and ($\chi$3). For every $\delta > 0$ there holds:*

$$\lim_{w\to+\infty} \sum_{|k-wx|>w\delta} |\chi(wx-k)| = 0$$

*uniformly with respect to $x \in \mathbb{R}$.*

From [15, Lemma 2.1 (i)], if $\chi$ satisfies the assumptions ($\chi$1) and ($\chi$3), it follows that

$$M_\gamma(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)||u-k|^\gamma < +\infty$$

for every $0 \leq \gamma \leq \beta$.

Now, we recall the weighted spaces of continuous functions. A function $\omega$ is said to be a weight function if it is a positive continuous function on the whole real axis $\mathbb{R}$. Here, we consider the weight function

$$\omega(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

By $B_\omega(\mathbb{R})$, we denote the space

$$B_\omega(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} \omega(x)|f(x)| \in \mathbb{R} \right\}.$$

The following natural subspaces of $B_w(\mathbb{R})$ will be used in the rest of the paper

$$C_\omega(\mathbb{R}) := C(\mathbb{R}) \cap B_\omega(\mathbb{R}),$$
$$C^*_\omega(\mathbb{R}) := \left\{ f \in C_\omega(\mathbb{R}) : \lim_{x\to\pm\infty} \omega(x)f(x) \in \mathbb{R} \right\},$$
$$U_\omega(\mathbb{R}) := \{f \in C_\omega(\mathbb{R}) : \omega f \text{ is uniformly continuous}\}.$$

The linear space of functions $B_\omega(\mathbb{R})$, and its above subspaces are normed spaces with the norm

$$\|f\|_\omega := \sup_{x\in\mathbb{R}} \omega(x)|f(x)|$$

(see [3, 4, 5, 8, 19, 20]).

The weighted modulus of continuity, considered in [22] and denoted by $\Omega(f; \cdot)$ is defined for $f \in C_\omega(\mathbb{R})$ by

$$\Omega(f;\delta) = \sup_{|h|<\delta, x\in\mathbb{R}} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)} \quad \text{for } \delta > 0. \tag{2.4}$$

Some elementary properties of $\Omega(f;\delta)$ are collected in the following lemma.

**Lemma 2.2** ([22])**.** *Let $\delta > 0$, $x \in \mathbb{R}$. Then,*

(i) *$\Omega(f;\delta)$ is an increasing function of $\delta$,*
(ii) *$\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$ when $f \in C^*_\omega(\mathbb{R})$,*
(iii) *For each $\lambda > 0$ and $f \in C_\omega(\mathbb{R})$,*

$$\Omega(f;\lambda\delta) \le 2(1+\lambda)\left(1+\delta^2\right)\Omega(f;\delta). \tag{2.5}$$

**Remark 2.1** ([1])**.** *Using the inequality (2.5) with $\lambda = \frac{|y-x|}{\delta}$, $x, y \in \mathbb{R}$, $\delta > 0$ and choosing $0 < \delta \le 1$, we get*

$$|f(y) - f(x)| \le 16(1+x^2)\Omega(f;\delta)\left(1 + \frac{|y-x|^3}{\delta^3}\right)$$

*for every $f \in C_\omega(\mathbb{R})$, $x, y \in \mathbb{R}$.*

In a very recent paper [23], Turgay and Acar introduced a new generalization of generalized sampling operators (1.1) by considering a special function $\rho$.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function that satisfies the following conditions:

$(\rho_1)$ $\rho \in C(\mathbb{R})$;
$(\rho_2)$ $\rho(0) = 0$, $\lim_{x\to\pm\infty} \rho(x) = \pm\infty$.

Let $\tau \in C(\mathbb{R})$ and $\varphi \in L^1(\mathbb{R})$ be functions such that for every $u, x \in \mathbb{R}$,

$$m_0^\rho(\tau, x) = \sum_{k\in\mathbb{Z}} \tau(\rho(x) - k) = 1, \quad \mathsf{m}_0(\varphi, u) = \int_{\mathbb{R}} \varphi(u)du = 1. \tag{2.6}$$

For any $\beta \in \mathbb{N}_0$, let us define the $\rho$-algebraic moment of order $\beta$ of $\tau$ and algebric moment of order $\beta$ of $\varphi$, respectively, by

$$m_\beta^\rho(\tau, x) = \sum_{k\in\mathbb{Z}} \tau(\rho(x) - k)(k - \rho(x))^\beta$$

$$\mathsf{m}_\beta(\varphi, u) = \int_{\mathbb{R}} \varphi(u)u^\beta du$$

and for $\alpha \geq 0$ the $\rho$-absolute moment of order $\alpha$ of $\tau$ and absolute moment of order $\alpha$ of $\varphi$, respectively, by

$$M_\alpha^\rho(\tau) = \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\tau(\rho(x)-k)||(k-\rho(x))|^\alpha$$

$$\mathcal{M}_\alpha(\varphi) = \int_{\mathbb{R}} |\varphi(u)||u|^\alpha du.$$

From now on, $\tau$ will be called a $\rho$-kernel and $\varphi$ will be called a kernel, if they satisfy the condition (2.6) such that there exists $\eta, \nu > 0$ with $M_\eta^\tau(\tau) < +\infty$ and $\mathcal{M}_\nu(\varphi) < +\infty$.

**Lemma 2.3** ([1]). *Let $\tau$ be a kernel satisfying the conditions*

*(1) $\tau$ is continuous on $\mathbb{R}$,*
*(2) there exists $\alpha \geq 0$, such that*

$$M_\alpha^\rho(\tau) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\tau(\rho(u)-k)||k-\rho(u)|^\beta$$

*is finite.*
*For every $\delta > 0$ there holds:*

$$\lim_{w\to\infty} \sum_{|k-w\rho(x)|>w\delta} |\tau(w\rho(x)-k)| = 0$$

*uniformly with respect to $x \in \mathbb{R}$.*

Now, we introduce the modified Durrmeyer type sampling operators as follows

$$(S_w^{\tau,\varphi} f)(x) = \sum_{k\in\mathbb{Z}} \tau(w\rho(x)-k) w \int_{\mathbb{R}} \varphi(wu-k)(fo\rho^{-1})(u)du. \tag{2.7}$$

**Remark 2.2.** *The operator (2.7) is well-defined if, for example, $f$ is bounded. Indeed, if $|f(x)| \leq L$ for every $x \in \mathbb{R}$, then $fo\rho^{-1}$ is also bounded function. Then*

$$\begin{aligned}
|(S_w^{\tau,\varphi} f)(x)| =& \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)||(fo\rho^{-1})(u)|du \\
\leq& L \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|du \\
\leq& L M_0^\rho(\tau)\mathcal{M}_0(\varphi) < \infty.
\end{aligned}$$

**Remark 2.3.** *In the special case of $\rho(x) = x$ (it is clear that $(\rho_1)$ and $(\rho_2)$ are satisfied), the operators (2.7) reduce to the classical sampling Durrmeyer operators*

$$(S_w^{\tau,\varphi} f)(x) = \sum_{k\in\mathbb{Z}} \tau(wx-k) w \int_{\mathbb{R}} \varphi(wu-k) f(u)du$$

*which was introduced in* [14].

## 3. Recent Results

In this section, we present some recent results on the convergence theorems of generalized sampling operators and Kantorovich forms in weighted spaces of continuous functions. The proofs of these theorems are omitted here. For further details, readers are referred to the original sources (see [1, 2]).

### 3.1. Pointwise and uniform convergences of $G_w^\chi$ and $K_w^\chi$ in weighted spaces.

**Theorem 3.1** ([1]). *Let $\chi$ be a kernel satisfying the assumptions $(\chi_1)$, $(\chi_2)$, and $(\chi_3)$ for $\beta = 2$. Moreover, let $f \in C_\omega(\mathbb{R})$ be fixed. Then,*

$$\lim_{w\to\infty} (G_w^\chi f)(x) = f(x)$$

*holds for every $x \in \mathbb{R}$. Moreover, if $f \in U_\omega(\mathbb{R})$, then*

$$\lim_{w\to\infty} \|G_w^\chi f - f\|_\omega = 0.$$

**Theorem 3.2** ([2]). *Let $\chi$ be a kernel satisfying $(\chi_1)$, $(\chi_2)$, and $(\chi 3)$ for $\beta = 2$ and $f \in C_\omega(\mathbb{R})$ be fixed. Then,*

$$\lim_{w\to+\infty} (K_w^\chi f)(x) = f(x)$$

*holds for every $x \in \mathbb{R}$. Moreover, if $f \in U_\omega(\mathbb{R})$, then*

$$\lim_{w\to+\infty} \|K_w^\chi f - f\|_\omega = 0.$$

### 3.2. Rate of convergences of $G_w^\chi$ and $K_w^\chi$ in weighted spaces.

**Theorem 3.3** ([1]). *Let $\chi$ be a kernel satisfying the assumptions $(\chi_1)$, $(\chi_2)$, and $(\chi_3)$ for $\beta = 3$. Then, for $f \in C_\omega^*(\mathbb{R})$, we have*

$$\|G_w^\chi f - f\|_\omega \le 16\Omega(f; w^{-1})(M_0(\chi) + M_3(\chi)), \quad \text{for } w \ge 1.$$

**Theorem 3.4** ([2]). *Let $\chi$ be a kernel satisfying the assumptions $(\chi_1)$, $(\chi_2)$ and $(\chi 3)$ with $\beta = 3$. For $f \in C_\omega^*(\mathbb{R})$, we have*

$$\|K_w^\chi f - f\|_\omega \le 32\Omega(f; w^{-1})[M_0(\chi) + 2M_3(\chi)]$$

*for every $w \ge 1$.*

### 3.3. Voronovskaja type formulae for $G_w^\chi$ and $K_w^\chi$.

A quantitative form of Voronovskaja theorem for the operators (1.1) was obtained as following.

**Theorem 3.5** ([1]). *Let $\chi$ be a kernel satisfying the assumptions $(\chi_1)$, $(\chi_2)$ and $(\chi_3)$ for $\beta = 4$. Furthermore, we assume in addition that the first-order algebraic moment of $\chi$ is constant, i.e.:*

$$m_1(\chi, x) = m_1(\chi) \in \mathbb{R} \setminus \{0\} \text{ for every } x \in \mathbb{R}.$$

*If $f \in C_\omega^*(\mathbb{R})$, then we have for $x \in \mathbb{R}$ that*

$$|w(G_w^\chi f)(x) - f(x) - f'(x)m_1(\chi)| \le 16(1 + x^2)\Omega(f'; w^{-1})\left(M_1(\chi) + M_4(\chi)\right).$$

*If we suppose in addition $m_j(\chi, x) = 0$, for every $x \in \mathbb{R}$, for $j = 1, \ldots, r-1$, $r \in \mathbb{N}$, that $(\chi_3)$ is satisfied for $\beta = r + 3$, and $m_r(\chi, x) = m_r(\chi) \in \mathbb{R} \setminus \{0\}$, for every $x \in \mathbb{R}$, then we have for $f^{(r)} \in C_\omega^*(\mathbb{R})$ that*

$$\left| w^r (G_w^\chi f)(x) - f(x) - f^{(r)}(x)\frac{m_r(\chi)}{r!} \right| \le \frac{16}{r!}(1 + x^2)\Omega(f^{(r)}; w^{-1})\left(M_r(\chi) + M_{r+3}(\chi)\right).$$

**Theorem 3.6** ([2]). *Let $\chi$ be a kernel satisfying the assumptions $(\chi_1)$, $(\chi_2)$, and $(\chi_3)$ for $\beta = r + 3$, $r \in \mathbb{N}$. Then, for $f \in C^r(\mathbb{R})$ such that $f^{(r)} \in C_\omega^*(\mathbb{R})$, there holds*

$$\begin{aligned} &\left| w\left[(K_w^\chi f)(x) - f(x)\right] - \sum_{j=1}^{r} \frac{f^{(j)}(x)}{j! w^{j-1}} \sum_{l=0}^{j} \binom{j}{l} \frac{m_l(\chi)}{(j-l+1)} \right| \\ &\le \frac{2^{r+3}}{w^r r!}(1 + x^2)\Omega\left(f^{(r)}, w^{-1}\right)\left[M_r(\chi) + \frac{M_0(\chi)}{r+1} + 8M_{r+3}(\chi) + \frac{8M_0(\chi)}{r+4}\right]. \end{aligned}$$

## 4. New Results

In [23], Turgay and Acar studied the approximation properties of the modified generalized sampling operators in weighted spaces of continuous functions. In this section, we present the approximation properties of the modified Durrmeyer type sampling operators in weighted spaces of continuous functions. For the weight function $\psi : \mathbb{R} \to \mathbb{R}$, defined by $\psi(x) = 1+\rho^2(x)$, we consider the following classes of functions:

$$B_\psi(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \text{for every } x \in \mathbb{R},\ \frac{|f(x)|}{\psi(x)} \le M_f \right\},$$
$$C_\psi(\mathbb{R}) = C(\mathbb{R}) \cap B_\psi(\mathbb{R}),$$
$$U_\psi(\mathbb{R}) = \left\{ f \in C_\psi(\mathbb{R}) \mid \frac{|f(x)|}{\psi(x)} \text{ is uniformly continuous on } \mathbb{R} \right\},$$

where $M_f$ is a constant depending only on $f$. These spaces are normed linear spaces with the norm

$$\|f\|_\psi = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\psi(x)}.$$

The weighted modulus of continuity, defined in [21], is given by

$$\omega_\psi(f;\delta) = \sup_{x,t\in\mathbb{R},|\rho(t)-\rho(x)|\le\delta} \frac{|f(t)-f(x)|}{\psi(t)+\psi(x)}$$

for each $f \in C_\psi(\mathbb{R})$ and for every $\delta > 0$. We observe that

$$\omega_\psi(f;0) = 0$$

for every $f \in C_\psi(\mathbb{R})$, and the function $\omega_\psi(f;\delta)$ is nonnegative and nondecreasing with respect to $\delta$ for $f \in C_\varphi(\mathbb{R})$. Additionally,

$$\lim_{\delta\to 0} \omega_\psi(f;\delta) = 0$$

for every $f \in U_\psi(\mathbb{R})$ (for more details, see [21]).

We recall the following auxiliary lemma to obtain an estimate for $|f(u) - f(x)|$.

**Lemma 4.4** ([21]). *For every $f \in C_\psi(\mathbb{R})$ and $\delta > 0$*

$$|f(u) - f(x)| \le (\psi(u) + \psi(x)) \left( 2 + \frac{|\rho(u) - \rho(x)|}{\delta} \right) \omega_\psi(f,\delta) \tag{4.8}$$

*holds for all $x, u \in \mathbb{R}$.*

**Remark 4.4** ([23]). *If we consider inequality (4.8), since*

$$\psi(u) + \psi(x) \le \delta^2 + 2\rho^2(x) + 2|\rho(x)|\delta \quad \textit{whenever} \quad |\rho(u) - \rho(x)| \le \delta,$$

*and*

$$\psi(u) + \psi(x) \le \left(\delta^2 + 2\rho^2(x) + 2|\rho(x)|\delta\right) \left( \frac{|\rho(u) - \rho(x)|}{\delta} \right)^2 \quad \textit{whenever} \quad |\rho(u) - \rho(x)| > \delta,$$

*by choosing $\delta \le 1$, it turns out that*

$$|f(u) - f(x)| \le 9(1 + |\rho(x)|)^2 \omega_\varphi(f;\delta) \left( 1 + \frac{|\rho(u) - \rho(x)|^3}{\delta^3} \right). \tag{4.9}$$

As a first main result of this section, we present the well-definiteness of the family of operators $(S_w^{\tau;\varphi})$ in weighted spaces of continuous functions. To prove this, we need the following proposition.

**Lemma 4.5.** *Let $\tau$ be a $\rho$- kernel and $\varphi$ be a kernel with $\beta = 2$. Further we denote by $v(x) := 1+\rho^2(x)$, $x \in \mathbb{R}$ and for any fixed $w > 0$. Then the following inequality holds:*

$$|(S_w^{\tau,\varphi}v)(x)| \leq M_0^\rho(\tau)\Big(\frac{1}{w}\mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\rho^2(x)\mathcal{M}_0(\varphi)\Big) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi). \tag{4.10}$$

*Proof.* By using the definition and linearity of the operators, we get

$$\begin{aligned}
|(S_w^{\tau,\varphi}v)(x)| \leq &\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|(1+\rho^2(\rho^{-1}(u))du \\
\leq &\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|du + \sum_{k\in\mathbb{Z}} |\tau(w\rho-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|u^2 du \\
\leq &\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)|du \\
&+\frac{1}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|(wu-k+k)^2 du \\
\leq &M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|([wu-k]^2+k^2)du \\
\leq &M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|(wu-k)^2 du \\
&+\frac{2}{w}\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|(k-w\rho(x)+w\rho(x))^2 du \\
\leq &M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w^2}M_0^\rho(\tau)\mathcal{M}_2(\varphi) \\
&+\frac{4}{w}\Big(\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)|(k-w\rho(x))^2 \int_{\mathbb{R}} |\varphi(wu-k)|du \\
&+w^2\rho^2(x)\sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)|du\Big) \\
\leq &M_0^\rho(\tau)\mathcal{M}_0(\varphi) + \frac{2}{w^2}M_0^\rho(\tau)\mathcal{M}_2(\varphi) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi) + 4\rho^2(x)M_0^\rho(\tau)\mathcal{M}_0(\varphi) \\
\leq &M_0^\rho(\tau)\Big(\mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\rho^2(x)\mathcal{M}_0(\varphi)\Big) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi).
\end{aligned}$$

This completes the proof. □

*Now, we give the well definiteness of modified sampling Durrmeyer type operator and some convergence results.*

**Theorem 4.7.** *Let $\tau$ be a $\rho$- kernel and $\varphi$ be a kernel with $\beta = 2$. For any fixed $w > 0$ the operator $S_w^{\tau,\varphi}$ is a linear operator from $B_\psi(\mathbb{R})$ to $B_\psi(\mathbb{R})$ and the inequality*

$$\|S_w^{\tau,\varphi}f\|_{B_\psi(\mathbb{R})\to B_\psi(\mathbb{R})} \leq M_0^\rho(\tau)\Big(\mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\mathcal{M}_0(\varphi)\Big) + \frac{4}{w^2}M_2^\rho(\tau)\mathcal{M}_0(\varphi)$$

*holds.*

*Proof.* By using the Lemma (4.5), we can easily obtain the inequality

$$\begin{aligned}
|(S_w^{\tau,\varphi} f)(x)| \leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| f(\rho^{-1}(u)) du \\
= & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| \frac{f(\rho^{-1}(u))}{1+\rho^2(\rho^{-1}(u))} \big(1+\rho^2(\rho^{-1}(u)\big) du \\
\leq & \|f\|_\psi \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| (1+\rho^2(\rho^{-1}(u)) du \\
\leq & \|f\|_\psi \Big[ M_0^\rho(\tau) \Big( \mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\rho^2(x)\mathcal{M}_0(\varphi) \Big) + \frac{4}{w^2} M_2^\rho(\tau)\mathcal{M}_0(\varphi) \Big].
\end{aligned}$$

Now if we multiply both sides with $\dfrac{1}{1+\rho^2(x)}$, we get

$$\frac{|(S_w^{\tau,\varphi} f)(x)|}{1+\rho^2(x)} \leq \|f\|_\psi \Big[ M_0^\rho(\tau) \Big( \mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\mathcal{M}_0(\varphi) \Big) + \frac{4}{w^2} M_2^\rho(\tau)\mathcal{M}_0(\varphi) \Big]$$

for every $x \in \mathbb{R}$. By assumptions, we conclude that $\|S_w^{\tau,\varphi} f\|_\psi < +\infty$ that is $S_w^{\tau,\varphi} \in B_\psi(\mathbb{R})$. Now taking supremum over $x \in \mathbb{R}$ and the supremum with respect to $f \in B_\psi(\mathbb{R})$ with $\|f\| \leq 1$, it turns out

$$\|S_w^{\tau,\varphi}\|_{B_\psi(\mathbb{R})\to B_\psi(\mathbb{R})} \leq M_0^\rho(\tau) \Big( \mathcal{M}_0(\varphi) + \frac{2}{w^2}\mathcal{M}_2(\varphi) + 4\mathcal{M}_0(\varphi) \Big) + \frac{4}{w^2} M_2^\rho(\tau)\mathcal{M}_0(\varphi).$$

□

**Theorem 4.8.** *Let $\tau$ be a $\rho$- kernel and $\varphi$ be a kernel with $\beta = 2.$ and $f \in C_\psi(\mathbb{R})$. Then, we have*

$$\lim_{w\to\infty} (S_w^{\tau,\varphi} f)(x) = f(x). \tag{4.11}$$

*Proof.* By straightforward calculations, we have

$$\begin{aligned}
|(S_w^{\tau,\varphi} f)(x) - f(x)| \leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| \Big[ (fo\rho^{-1})(u) - f(x) \Big] du \\
\leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| \Big\{ \frac{|(fo\rho^{-1})(u)|}{(\psi o\rho^{-1})(u)} |(\psi o\rho^{-1})(u) - \psi(x)| \\
& + \psi(x) \Big| \frac{|(fo\rho^{-1})(u)|}{(\psi o\rho^{-1})(u)} - \frac{f(x)}{\psi(x)} \Big| \Big\} du \\
:= & I_1 + I_2. \qquad (4.12)
\end{aligned}$$

Let's first estimate $I_1$. Since $f \in C_\psi(\mathbb{R})$, we have

$$
\begin{aligned}
I_1 \leq & \|f\|_\psi \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)||u^2-\rho^2(x)| du \\
\leq & \|f\|_\psi \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| \Big[ |u-\rho(x)||u+\rho(x)| \Big] du \\
\leq & \frac{\|f\|_\psi}{w} \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-w\rho(x)||wu+w\rho(x)| du \\
\leq & \frac{\|f\|_\psi}{w} \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)| \Big\{ (|wu-k|+|k-w\rho(x)|)(|wu-k|+|k+w\rho(x)|) \Big\} du \\
\leq & \frac{\|f\|_\psi}{w} \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)| \Big\{ |wu-k|^2 + |wu-k||k+w\rho(x)| \\
& + |k-w\rho(x)||wu-k| + |k-w\rho(x)||k+w\rho(x)| \Big\} du \\
\leq & \frac{\|f\|_\psi}{w} \Big[ \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-k|^2 du \\
& + \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)||k+w\rho(x)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-k| du \\
& + \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)||k-w\rho(x)| \int_{\mathbb{R}} |\varphi(wu-k)||wu-k| du \\
& + \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)||k-w\rho(x)||k+w\rho(x)| \int_{\mathbb{R}} |\varphi(wu-k)| du
\end{aligned}
$$

(4.13)

$$:= I_{1.1} + I_{1.2} + I_{1.3} + I_{1.4}.$$

Since $|k + w\rho(x)| = |k - w\rho(x) + 2w\rho(x)| \leq |k - w\rho(x)| + |2\rho(x)|$, it is clear to see that the following statements hold

$$
\begin{aligned}
I_{1.1} =& \frac{\|f\|_\psi}{w^2} M_0^\rho(\tau)\mathcal{M}_2(\varphi), \\
I_{1.2} =& \frac{\|f\|_\psi}{w^2} \Big[ M_1^\rho(\tau)\mathcal{M}_1(\varphi) + 2|w\rho(x)| M_0^\rho(\tau)\mathcal{M}_1(\varphi) \Big], \\
I_{1.3} =& \frac{\|f\|_\psi}{w^2} M_1^\rho(\tau)\mathcal{M}_1(\varphi), \\
I_{1.4} =& \frac{\|f\|_\psi}{w^2} \Big[ M_2^\rho(\tau)\mathcal{M}_0(\varphi) + 2|w\rho(x)| M_1^\rho(\tau)\mathcal{M}_0(\varphi) \Big].
\end{aligned}
$$

If we substitute $I_{1.1}, I_{1.2}, I_{1.3}$ and $I_{1.4}$ in (4.13), we can get

$$
\begin{aligned}
I_1 \leq & \frac{\|f\|_\psi}{w^2} \Big[ M_0^\rho(\tau)\mathcal{M}_2(\varphi) + M_1^\rho(\tau)\mathcal{M}_1(\varphi) + 2w|\rho(x)| M_0^\rho(\tau)\mathcal{M}_1(\varphi) + M_1^\rho(\tau)\mathcal{M}_1(\varphi) + M_2^\rho(\tau)\mathcal{M}_0(\varphi) \\
& + 2w|\rho(x)| M_1^\rho(\tau)\mathcal{M}_0(\varphi) \Big] \\
= & \frac{\|f\|_\psi}{w^2} \Big[ M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 2M_1^\rho(\tau)\mathcal{M}_1(\varphi) + 2w|\rho(x)| \Big( M_0^\rho(\tau)\mathcal{M}_1(\varphi) + M_1^\rho(\tau)\mathcal{M}_0(\varphi) \Big) \\
& + M_2^\rho(\tau)\mathcal{M}_0(\varphi) \Big).
\end{aligned}
$$

Now lets consider $I_2$. Let $x \in \mathbb{R}$ and $\epsilon > 0$ be fixed. Then there exists $\delta > 0$ such that

$$\Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| < \epsilon$$

when $|\rho^{-1}(u) - x| < \delta$. Hence we can write

$$\begin{aligned}
I_2 =& \psi(x) \sum_{k\in\mathbb{Z}} |\tau(w\rho(x)-k)| w \int_{\mathbb{R}} |\varphi(wu-k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \\
=& w\psi(x)\Big[ \sum_{|k-w\rho(x)|<\frac{w\delta}{2}} |\tau(w\rho(x)-k)| \int_{|wu-k|\le\frac{w\delta}{2}} |\varphi(wu-k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \\
&+ \sum_{|k-w\rho(x)|\le\frac{w\delta}{2}} |\tau(w\rho(x)-k)| \int_{|wu-k|>\frac{w\delta}{2}} |\varphi(wu-k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \\
&+ \sum_{|k-w\rho(x)|>\frac{w\delta}{2}} |\tau(w\rho(x)-k)| \int_{\mathbb{R}} |\varphi(wu-k)| \Big|\frac{(fo\rho^{-1}(u)}{(\psi o\rho^{-1}(u)} - \frac{f(x)}{\psi(x)}\Big| du \Big] \\
:=& w\psi(x)\Big[I_{2.1} + I_{2.2} + I_{2.3}\Big].
\end{aligned}$$

For $|k - w\rho(x)| \le \dfrac{w\delta}{2}$, if $|wu - k| \le \dfrac{w\delta}{2}$, we have

$$|u - \rho(x)| \le |u - \frac{k}{w}| + |\frac{k}{w} - \rho(x)| < \delta.$$

Since $f \in C_\psi(\mathbb{R})$, we get

$$I_{2.1} \le \epsilon M_0^\rho(\tau)\mathcal{M}_0(\varphi).$$

Taking supremum for $u \in \mathbb{R}$, we can write

$$I_{2.2} \le 2\|f\|_\psi \sum_{|k-w\rho(x)|\le\frac{w\delta}{2}} |\tau(w\rho(x)-k| \int_{|wu-k|>\frac{w\delta}{2}} |\varphi(wu-k)| du$$

and $\int_{|wu-k|>\frac{w\delta}{2}} |\varphi(wu-k)|du = \int_{|t|>\frac{w\delta}{2}} |\varphi(t)|dt \to 0$ as $w \to \infty$ for sufficiently large $w$. Hence, we get

$$I_{2.2} \le \frac{2}{w}\|f\|_\psi M_0^\rho(\tau)\epsilon.$$

Finally, by Lemma (2.3), since

$$\lim_{w\to+\infty} \sum_{|k-w\rho(x)|>\frac{w\delta}{2}} |\tau(w\rho(x)-k)| = 0,$$

then we get

$$I_{2.3} \le \frac{2}{w}\|f\|_\psi \mathcal{M}_0(\varphi)\epsilon$$

for sufficiently large $w$. Combining the above estimates, we have

$$\begin{aligned}
|(S_w^\tau, \varphi f)(x) - f(x)| \le& I_1 + I_{2.1} + I_{2.2} + I_{2.3} \\
\le& \frac{\|f\|_\psi}{w^2}\Big[M_0^\rho(\tau)\mathcal{M}_2(\varphi) + 3M_1^\rho(\tau)\mathcal{M}_1(\varphi) + |4\rho(x)|M_0^\rho(\tau)\mathcal{M}_1(\varphi)\Big] \\
&+ \psi(x)\Big[\epsilon\Big(M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 2\|f\|_\psi M_0^\rho(\tau) + 2\|f\|_\psi\mathcal{M}_0(\varphi)\Big)\Big]. \qquad (4.14)
\end{aligned}$$

By taking limit as $w \to \infty$, we get the desired result. □

**Theorem 4.9.** *Let $\tau$ be a $\rho$- kernel and $\varphi$ be a kernel with $\beta = 2$ and $\dfrac{fo\rho^{-1}}{\varphi o\rho^{-1}} \in U_\psi(\mathbb{R})$. Then*

$$\lim_{w\to\infty} \|S_w^{\tau,\varphi} f - f\|_\psi = 0$$

*holds.*

*Proof.* For functions $f \in U_\psi(\mathbb{R})$, let us follow the same steps with the proof of Theorem (4.11) and replace $\delta$ with corresponding parameter of the uniform continuity of $\dfrac{fo\rho^{-1}}{\psi o\rho^{-1}} \in U_\psi(\mathbb{R})$ also considering the inequality (4.14), we have

$$\begin{aligned}\frac{|(S_w^{\tau,\varphi} f(x) - f(x)|}{\psi(x)} \leq & \frac{\|f\|_\varphi}{w^2\psi(x)}\Big[M_0^\rho(\tau)\mathcal{M}_2(\varphi) + 3M_1^\rho(\tau)\mathcal{M}_1(\varphi) + |4\rho(x)|M_0^\rho(\tau)\mathcal{M}_1(\varphi)\Big] \\ & + \epsilon\Big(M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 2\|f\|_\psi M_0^\rho(\tau) + 2\|f\|_\psi \mathcal{M}_0(\varphi)\Big)\end{aligned}$$

and taking supremum over $x \in \mathbb{R}$ we obtain the desired result. □

**Theorem 4.10.** *Let $\tau$ be a $\rho$- kernel and $\varphi$ be a kernel with $\beta = 3$. Then for $f \in C_\psi(\mathbb{R})$, we get*

$$|(S_w^{\tau,\varphi} f)(x) - f(x)| \leq 9(1+|\rho(x)|)^2\omega_\varphi(fo\rho^{-1}; w^{-1})\Big(M_0^\rho(\tau)\mathcal{M}_0(\varphi) + 4(M_0^\rho(\tau)\mathcal{M}_3(\varphi) + M_3^\rho(\tau)\mathcal{M}_0(\varphi)\Big).$$

*Proof.* Using the definition of the operators $S_w^{\tau,\varphi}$ and (4.9), we have

$$\begin{aligned}|(S_w^{\tau,\varphi} f)(x) - f(x)| \leq & \sum_{k\in\mathbb{Z}} |\tau(w\rho(x) - k)| w \int_{\mathbb{R}} |\varphi(wu - k)||(fo\rho^{-1}(u) - f(x)|du \\ \leq & 9w(1 + |\rho(x)|)^2\omega_\varphi(f;\delta) \sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu - k)|\Big(1 + \frac{|u - \rho(x)|^3}{\delta^3}\Big)du \\ = & 9w(1 + |\rho(x)|)^2\omega_\varphi(f;\delta)\Big[\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu - k)|du \\ & + \frac{1}{\delta^3}\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu - k)||u - \rho(x)|^3 du \\ = & 9w(1 + |\rho(x)|)^2\omega_\varphi(fo\rho^{-1};\delta)(I_1 + I_2).\end{aligned}$$

It can be easily seen that

$$I_1 \leq \frac{1}{w} M_0^\rho(\tau)\mathcal{M}_0(\varphi).$$

Now, we need to estimate $I_2$. By elementary calculations, we have

$$\begin{aligned}I_2 \leq & \frac{1}{\delta^3}\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu - k)\Big(|u - \frac{k}{w}| + |\frac{k}{w} - \rho(x)|\Big)^3 du \\ \leq & \frac{4}{\delta^3 w^3}\sum_{k\in\mathbb{Z}} |w\rho(x) - k| \int_{\mathbb{R}} |\varphi(wu - k)\Big(|wu - k|^3 + |k - w\rho(x)|^3\Big)du \\ = & \frac{4}{\delta^3 w^4}\Big(M_0^\rho(\tau)\mathcal{M}_3(\varphi) + M_3^\rho(\tau)\mathcal{M}_0(\varphi)\Big).\end{aligned}$$

Substituting $I_1$ and $I_2$ and choosing $\delta = w^{-1}$, we immediately obtain the result.

□

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## References

[1] T. Acar, O. Alagoz, A. Aral, D. Costarelli, M. Turgay and G. Vinti: *Convergence of generalized sampling series in weighted spaces*, Demonstr. Math., **55** (2022), 153–162.
[2] T. Acar, O. Alagoz, A. Aral, D. Costarelli, M. Turgay and G. Vinti: *Approximation by sampling Kantorovich series in weighted spaces of functions*, Turkish J. Math., **46** (7) (2022), 2663–2676.
[3] T. Acar, M. C. Montano, P. Garrancho and V. Leonessa: *Voronovskaya type results for Bernstein-Chlodovsky operators preserving* $e^{-2x}$, J. Math. Anal. Appl., **491** (1) (2020), 124307.
[4] T. Acar, M. C. Montano, P. Garrancho and V. Leonessa: *On Bernstein-Chlodovsky operators preserving* $e^{-2x}$, Bull. Belg. Math. Soc. Simon Stevin, **26** (5) (2019), 681–698.
[5] A. Aral: *Weighted approximation: Korovkin and quantitative type theorems*, Modern Math. Methods, **1** (1) (2023), 1–21.
[6] F. Asdrubali, G. Baldinelli, F. Bianchi, D. Costarelli, A. Rotili, M. Seracini and G. Vinti: *Detection of thermal bridges from thermographic images by means of image processing approximation algorithms*, Appl. Math. Comput., **317** (2018), 160–171.
[7] C. Bardaro, P. L. Butzer, R. L. Stens and G. Vinti: Kantorovich-type generalized sampling series in the setting of Orlicz spaces, Sampl. Theory Signal Image Process., **6** (1) (2007), 29–52.
[8] B. R. Dragonov: *A fast converging sampling operator*, Constr. Math. Anal., **5** (4) (2022), 190–201.
[9] P. L. Butzer, R. L. Stens: *The sampling theorem and linear prediction in signal analysis*, Jahresber. Dtsch. Math. Ver., **90** (1) (1998), 1–70.
[10] P. L. Butzer, R. L. Stens: *Linear prediction by samples from the past*, Advanced topics in Shannon sampling and interpolation theory, Springer, New York, 1993, 157–183.
[11] P. L. Butzer, W. Engels, S. Ries and R. L. Stens: *The Shannon sampling series and the reconstruction of signals in terms of linear, quadratic and cubic splines*, SIAM J. Appl. Math., **46** (2) (1986), 299–323.
[12] P. L. Butzer, W. Splettstosser, *A sampling theorem for duration-limited functions with error estimates*, Inf. Control, **34** (1) (1977), 55–65.
[13] G. Baldinelli, F. Bianchi, A. Rotili, D. Costarelli, M. Seracini, G. Vinti and L. Evangelisti: *A model for the improvement of thermal bridges quantitative assessment by infrared thermography*, Appl. Energy, **211** (2018), 854–864.
[14] C. Bardaro, I. Mantellini: *Asymptotic expansion of generalized Durrmeyer sampling type series*, Jaen J. Approx., **6** (2) (2014), 143–165.
[15] D. Costarelli, G. Vinti: *Rate of approximation for multivariate sampling Kantorovich operators on some functions spaces*, J. Integral Equ. Appl., **26** (4) (2014), 455–481.
[16] D. Costarelli, A. R. Sambucini: *A comparison among a fuzzy algorithm for image rescaling with other methods of digital image processing*, Constr. Math. Anal., **7** (2) (2024), 45–68.
[17] L. Boccali, D. Costarelli, G. Vinti: *A Jackson-type estimate in terms of the* $\tau$*-modulus for neural network operators in* $L^p$*-spaces*, Modern Math. Methods, **2** (2) (2024), 90–102.
[18] F. Cluni, D. Costarelli, A. M. Minotti and G. Vinti: *Enhancement of thermographic images as tool for structural analysis in earthquake engineering*, NDT E Int., **70** (2015), 60–72.
[19] A. D. Gadjiev: *The convergence problem for a sequence of positive linear operators on unbounded sets, and Theorems analogous to that of P. P. Korovkin*, Dokl. Akad. Nauk SSSR, **218** (5) (1974), 1001–1004.
[20] A. D. Gadjiev: *Theorems of Korovkin type*, Math. Notes Acad. Sci. USSR, **20** (1976), 995–998.
[21] A. Holhos: *Quantitative estimates for positive linear operators in weighted space*, Gen. Math., **16** (4) (2008), 99–110.
[22] N. Ispir: *On modified Baskakov operators on weighted spaces*, Turk. J. Math., **25** (3) (2001), 355–365.
[23] M. Turgay, T. Acar: *Approximation by Modified Generalized Sampling Series*, Mediterr. J. Math., **21** (2024), 107.

Osman Alagöz
Bilecik Şeyh Edebali University
Department of Mathematics
Bilecik, Türkiye
ORCID: 0000-0002-0587-460X
*Email address*: osman.alagoz@bilecik.edu.tr