Celal Bayar University Journal of Science
Volume 21, Issue 2, 2025, p 147-151
Doi: 10.18466/cbayarfbe.1528991


**Celal Bayar University Journal of Science**

# Tribonacci numbers as sum or difference of powers of 2

Fatih Erduvan[1]*

[1] MEB, İzmit Namık Kemal Anatolia High School, Kocaeli, Türkiye
* erduvanmat@hotmail.com
* Orcid No: 0000-0001-7254-2296

Received: 6 August 2024
Accepted: 25 February 2025
DOI: 10.18466/cbayarfbe.1528991

**Abstract**

This paper investigates Tribonacci numbers can be expressed as either the sum or difference of two distinct powers of 2. Namely, we address the problem of expressing Tribonacci numbers in the form

$$T_n = 2^x \pm 2^y$$

in positive integers with $1 \leq y \leq x$. Our findings reveal specific instances where such representations are possible, including examples like the seventh Tribonacci number expressed both as the sum and the difference of powers of 2. Additionally, we identify Tribonacci numbers that can be represented as the differences of Mersenne numbers, specifically, the numbers 2, 4, 24, and 504. These results enhance the understanding of the structural properties of Tribonacci sequences and their relationships with exponential and Mersenne-based number systems.

**Keywords:** Diophantine equations, Tribonacci numbers, Baker's Theory.

## 1. Introduction

The exploration of the properties of linear recurrence sequences has a rich history and has produced an extensive body of literature. The quest to find all integer solutions to Diophantine equations with Fibonacci numbers has drawn considerable interest from mathematicians, leading to a diverse array of literature on the subject. Tribonacci numbers are a generalization of Fibonacci numbers and therefore, it is inevitable that equations containing Tribonacci numbers will also be interesting. Let $(T_n)$ be the sequence of Tribonacci numbers defined by $T_0 = 0, T_1 = T_2 = 1$ and

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{for } n \geq 0.$$

In recent years, many studies have been seen involving integer sequences and powers integers. Readers can see these works in [1-9]. Especially, the equation

$$T_n = 2^a + 3^b + 5^c + \delta$$

related to Tribonacci numbers has been solved by Irmak and Szalay in [10]. Here, variables are non-negative integers with $0 \leq \delta \leq 10$ and $0 \leq a, b \leq c$.

In this paper, using the linear forms in logarithms, we obtain large upper bounds thanks to Lemma 2.1 and then, we reduce these bounds using Lemma 2.2. Now, we give our main theorem.

**Theorem 1.1:** Let $T_n$ be n-th Tribonacci number. The only solutions to the Diophantine equations

$$T_n = 2^x + \epsilon 2^y \qquad (1)$$

in non-negative integers with $1 \leq y \leq x$ and $\epsilon = \pm 1$ are given by

$\epsilon = 1$: $(n, x, y) = (4,1,1), (7,4,3)$;

$\epsilon = -1$: $(n, x, y) = \{(0, x, x), (3,2,1), (4,3,2), (7,5,3), (12,9,3)\}$.

## 2. The Tools

In this section, we will remind you about Tribonacci numbers and linear forms in logarithms. We will also give some lemmas that are necessary to prove the main theorem. The characteristic equation

$$\psi(x) := x^3 - x^2 - x - 1 = 0$$

has roots $\alpha, \beta, \gamma = \bar{\beta}$ where

$$\alpha = \left(\frac{1+(s_1+s_2)}{3}\right), \bar{\gamma} = \beta = \left(\frac{2-(s_1+s_2)+\sqrt{-3}(s_1-s_2)}{6}\right)$$

and

$$s_1 = \sqrt[3]{19+3\sqrt{33}},\ s_2 = \sqrt[3]{19-3\sqrt{33}}.$$

Binet formula for this number is

$$T_n = a\alpha^n + b\beta^n + c\gamma^n \text{ for all } n \geq 0, \qquad (2)$$

where

$$a = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}, b = \frac{1}{(\beta-\alpha)(\beta-\gamma)}, \text{and } c = \frac{1}{(\gamma-\alpha)(\gamma-\beta)}.$$

Moreover, $a = \frac{\alpha}{\alpha^2+2\alpha+3}$ and the minimal polynomial of $a$ over $\mathbb{Z}$ is given by $44x^3 + 44x - 1$. Zeros of this equation are $a, b$ and $c$. With simple calculations, it can be shown the following estimates hold:

$$1.83 < \alpha < 1.84, 0.73 < |\beta| = |\gamma| < \alpha^{1/2} < 0.74$$

and

$$0.18 < a < 0.19, 0.35 < |b| = |c| < 036.$$

Let

$$e(n) := T_n - a\alpha^n = b\beta^n + c\gamma^n.$$

Then, from the above inequalities we conclude that

$$|e(n)| := \frac{1}{\alpha^{n/2}} \qquad (3)$$

for $k \geq 1$. The relation between $T_n$ with $\alpha$ is given by

$$\alpha^{n-2} \leq T_n \leq \alpha^{n-1} \text{ for all } n \geq 1. \qquad (4)$$

Baker's Theory, developed by Alan Baker, is an important theory on number theory and Diophantine equations. This theory specifically includes the concepts of linear form and logarithmic height and plays a large role in the context of solving Diophantine equations. Now, we present materials related to this theory. Let $\gamma$ be an algebraic number of degree $d$ over $\mathbb{Q}$ with minimal primitive polynomial

$$c_0x^d + c_1x^{d-1} + \cdots + c_d = c_0\sum_{i=1}^{d}(x - \gamma^{(i)}) \in \mathbb{Z}[x],$$

with $\gamma^{(i)}$'s are conjugates of $\gamma$ and $c_0 > 0$. Then logarithmic height of $\gamma$ is given

$$h(\gamma) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d}\log\left(\max\{|\gamma^{(i)}|, 1\}\right)\right).$$

The following properties are given in [11].

$$h(\gamma_1 \mp \gamma_2) \leq \log 2 + h(\gamma_1) + h(\gamma_2)$$

$$h(\gamma_1\gamma_2^{\pm 1}) \leq h(\gamma_1) + h(\gamma_2)$$

$$h(\gamma_1^{\,r}) = |r|h(\gamma_1).$$

The following two lemmas can be found in [12,13].

**Lemma 2.1.** Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be positive real algebraic numbers and let $b_1, b_2, \ldots, b_n$ be nonzero integers. Let $D$ be the degree of the number field $\mathbb{Q}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ over $\mathbb{Q}$. Let

$$B \geq \max\{|b_1|, |b_2|, \ldots, |b_n|\},$$

$$A_i \geq \max\{D \cdot h(\gamma_i), |\log\gamma_i|, (0.16)\}$$

for all $i = 1,2,\ldots,n$. If

$$\Gamma := \gamma_1^{b_1} \cdot \gamma_2^{b_2} \cdots \gamma_n^{b_n} - 1 \neq 0$$

then

$$|\Gamma| > \exp(-1.4 \cdot 30^{n+3} \cdot n^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdot A_2 \cdots A_n).$$

**Lemma 2.2.** Let $\eta$ be irrational number, $M$ be a positive integer and $\frac{p}{q}$ be a convergent of the continued fraction of $\eta$ such that $q > 6M$, and let $A, B, \mu$ be some real numbers with $A > 0$ and $B > 1$. Put

$$\varepsilon := \|\mu q\| - M\|\eta q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no positive integer solution $(r, s, t)$ to the inequality

$$0 < |r\eta - s + \mu| < A \cdot B^{-t}$$

subject to the restrictions that

$$r \leq M \text{ and } t \geq \frac{\log\left(Aq/\varepsilon\right)}{\log B}.$$

We give the following lemma without proof since its proof can be easily seen.

**Lemma 2.3.** If the real numbers $x$ and $K$ satisfy

$$|e^x - 1| < K < 3/4,$$

then $|x| < 2K$.

## 3. Proof of Main Theorem

In the case $x = y$, we have the trivial solution

$(n,x,y)=(0,x,x)$ for $\epsilon=-1$ and the only solution $(n,x,y)=(4,1,1)$ for $\epsilon=1$, from [14]. From now on, we will take $1\le y<x$. As $y\le x-1$ and $x\ge 2$, we can say

$$2<2^{x-1}=2^x-2^{x-1}\le 2^x-2^y<T_n$$

by the equation (1). So, we get $n\ge 3$. Now let's find a relationship between $n$ and $x$. Considering (1), we obtain

$$T_n\le 2^x+2^y\le 2^x+2^{x-1}=3.2^{x-1}<\alpha^{1.9+1.2(x-1)}.$$

If we combine the above inequality and (4), we get

$$n<1.7+1.2x \qquad (5)$$

Using (2) and (3), we arrange the equation (1) as

$$|\Gamma|=|a\cdot\alpha^n\cdot 2^{-x}-1|\le\frac{1}{\alpha^{\frac{n}{2}}\cdot 2^x}+\frac{1}{2^{x-y}}<\frac{1.21}{2^{x-y}} \qquad (6)$$

and

$$|\Gamma'|=|(a^{-1}(1\pm 2^{y-x}))^{-1}\cdot\alpha^n\cdot 2^{-x}-1|\le\frac{1}{\alpha^{\frac{n}{2}}\cdot 2^x\cdot|1\pm 2^{y-x}|}<\frac{0.81}{2^x}. \qquad (7)$$

Here, we have used that $\alpha^{-\frac{n}{2}}<0.401$ for $n\ge 3$ and

$$|1\pm 2^{y-x}|^{-1}<2$$

for $1\le y<x$. To apply Lemma 2.1, we take

$$(\gamma_1,b_1):=(a,1)$$

$$(\gamma'_1,b'_1):=((a^{-1}(1\pm 2^{y-x}))^{-1},-1)$$

$$(\gamma_2,b_2)=(\gamma'_2,b'_2):=(\alpha,n)$$

$$(\gamma_3,b_3)=(\gamma'_3,b'_3):=(2,-x)$$

in the inequalities (6) and (7). Moreover, $D=D'=3$. Now, we show that

$$\Gamma:=a\cdot\alpha^n\cdot 2^{-x}-1\ne 0.$$

If $\Gamma=0$, then we can write $a\cdot\alpha^n=2^x\in\mathbb{Z}$, which is not possible. Similarly,

$$\Gamma':=(a^{-1}(1\pm 2^{y-x}))^{-1}\cdot\alpha^n\cdot 2^{-x}-1\ne 0.$$

Using the definition and properties of logarithmic height, we get

$$h(\gamma_1)=h(a)=\frac{\log 44}{3},$$

$$h(\gamma'_1)=h((a^{-1}(1\pm 2^{y-x}))^{-1}),\quad <\frac{\log 44}{3}+(x-y)\log 2+\log 2,\quad <3.9(x-y)\log 2,$$

$$h(\gamma_2)=h(\gamma'_2)=h(\alpha)=\frac{\log\alpha}{3},$$

and

$$h(\gamma_3)=h(\gamma'_3)=h(2)=\log 2.$$

Thus, we can write

$$A=A_1\cdot A_2\cdot A_3=\log 44\cdot\log\alpha\cdot\log 8,$$

$$A'=A'_1\cdot A'_2\cdot A'_3:=11.7(x-y)\log 2\cdot\log\alpha\cdot\log 8.$$

The inequality (5) lead to $B=B':=1.7+1.2x$. Let

$$T=(-1.4)\cdot 30^6\cdot 3^{4.5}\cdot 3^2\cdot\log 9.$$

From Lemma 2.1, the inequalities (6) and (7) give us

$$1.21\cdot 2^{-(x-y)}>|\Gamma|>\exp((T\cdot(3.5\log x)\cdot A)$$

i.e.,

$$x-y<6.55\cdot 10^{13}\cdot\log x \qquad (8)$$

and

$$0.81\cdot 2^{-x}>|\Gamma'|>\exp((T\cdot(3.5\log x)\cdot A')$$

i.e.,

$$x<1.41\cdot 10^{14}\cdot(x-y)\cdot\log x, \qquad (9)$$

where we have used

$$1+\log(1.7+1.2x)<3.5\log x$$

for $x\ge 2$. The inequalities (8) and (9) tell us

$$x<9.24\cdot 10^{27}\cdot(\log x)^2$$

or $x<4.93\cdot 10^{31}$. From (5), we obtain

$$n<5.92\cdot 10^{31}. \qquad (10)$$

Let

$$z:=n\log\alpha-x\log 2+\log a$$

and

$$z':=n\log\alpha-x\log 2-\log\left((a^{-1}(1\pm 2^{y-x})\right).$$

From (6) and (7), we can write

$$|\Gamma| \coloneqq |e^{z} - 1| < \frac{1.21}{2^{x-y}} < 0.61$$

for $x - y \geq 1$ and

$$|\Gamma'| \coloneqq \left|e^{z'} - 1\right| < \frac{0.81}{2^{x}} < 0.21.$$

for $x \geq 2$. According to Lemma 2.3, we can say

$$|z| = |n\log\alpha - x\log 2 + \log a| < \frac{2.42}{2^{x-y}}$$

and

$$|z'| = \left|n\log\alpha - x\log 2 - \log\left((a^{-1}(1 \pm 2^{y-x})\right)\right| < \frac{1.62}{2^{x}}.$$

With necessary arrangement, these inequalities convert to

$$0 < \left|n\left(\frac{\log\alpha}{\log 2}\right) - x + \frac{\log a}{\log 2}\right| < \frac{3.5}{2^{x-y}} \qquad (11)$$

and

$$0 < \left|n\left(\frac{\log\alpha}{\log 2}\right) - x - \frac{\log\left((a^{-1}(1\pm 2^{y-x})\right)}{\log 2}\right| < \frac{2.34}{2^{x}}. \qquad (12)$$

To apply Lemma 2.2, we choose

$$\eta = \eta' \coloneqq \frac{\log\alpha}{\log 2},\ \mu \coloneqq \frac{\log a}{\log 2},\ A \coloneqq 3.5, B \coloneqq 2,\ t \coloneqq x - y$$

and

$$\mu' \coloneqq \frac{\log\left((a^{-1}(1\pm 2^{y-x})\right)}{\log 2},\ A' \coloneqq 2.34,\ B' \coloneqq 2,\ t' \coloneqq x$$

by considering (11) and (12). To reduce $x - y$, we take

$$n < 5.92 \cdot 10^{31} = M$$

from (10) in the equation (11). We find that $q_{62} > 6M$ for $\eta$. Moreover, we compute

$$\varepsilon \coloneqq \|\mu q_{62}\| - M\|\eta q_{62}\| > 0.37.$$

Thanks to Lemma 2.2, we have

$$x - y \leq \frac{\log\left(\frac{Aq_{62}}{\varepsilon}\right)}{\log B} \leq 112.61,$$

and so $x - y \leq 112$. This upper bound and (9) give us

$$x < 6.48 \cdot 10^{18}.$$

The above inequality and (5) imply that

$$n < 7.78 \cdot 10^{18}. \qquad (13)$$

Considering (12) and (13), we choose

$$n < 7.78 \cdot 10^{18} = M'.$$

It can be seen that $q_{33} > 6M'$ for $\eta'$. Furthermore, we compute

$$\varepsilon \coloneqq \|\mu q_{33}\| - M\|\eta q_{33}\| > 0.001.$$

From Lemma 2.2, we conclude that

$$x \leq \frac{\log\left(\frac{Aq_{33}}{\varepsilon}\right)}{\log B} \leq 78.86.$$

From this, we get $x \leq 78$ and so $n \leq 95$. With help a computer program, we get the only solutions displayed as

$$24 = T_7 = 2^4 + 2^3, \quad 2 = T_3 = 2^2 - 2^1,$$

$$4 = T_4 = 2^3 - 2^2, \quad 24 = T_7 = 2^5 - 2^3,$$

$$504 = T_{12} = 2^9 - 2^3$$

for $1 \leq y < x \leq 78$ and $3 \leq n \leq 95$. Therefore, the proof ends.

Mersenne numbers are given by $2^a - 1$ for $a \geq 1$. Difference of two Mersenne numbers is expressed as $2^a - 2^b$ for $a, b \geq 1$.

**Corollary 3.1:** Tribonacci numbers that can be written as the difference of two Mersenne numbers are 2, 4, 24 and 504.

## 4. Conclusion and Suggestion

Recently, investigators have used linear forms in logarithms and reduction method in many of their studies. It turned out that the Diophantine equation produced the interesting result, which is Tribonacci numbers in the form of the difference of two Mersenne numbers. This result not only show cases the power of logarithmic forms in solving Diophantine equations but also provides a deeper understanding of the interplay between different number-theoretic structures.

As a more general version of this study, solutions to these equations can be investigated by substituting various integer sequences for 2 in these equations. By doing so, one can explore a broader class of Diophantine equations that involve powers of integers, such as powers of primes or other specially constructed sequences. This approach could yield new insights into the nature of integer solutions and their distribution, potentially revealing deeper symmetries in number theory.

**Acknowledgement**

The authors thank the anonymous reviewers for their contribution to this paper.

**Author's Contributions**

**Fatih Erduvan:** Supervision, Methodology, Validation, Writing-original draft, Investigation

**Ethics**

There are no ethical issues after the publication of this manuscript.

**References**

**[1].** Bravo, JJ, Luca, F. 2014. Powers of two as sums of two Lucas numbers. *Journal of Integer Sequences*; 17(1): article no 14.8.3.

**[2].** Bravo, EF, Bravo, JJ. 2015. Powers of two as sums of three Fibonacci numbers. *Lithuanian Mathematical Journal*; 55(3). 301-311.

**[3].** Bravo, JJ, Gómez, CA, Luca, F. 2016. Powers of two as sums of two k-Fibonacci numbers. *Miskolc Mathematical Notes*; 17(1): 85-100.

**[4].** Bravo, JJ, Faye, B, Luca, F. 2017. Powers of two as sums of three Pell numbers. *Taiwanese Journal of Mathematics;* 21(4): 739-751.

**[5].** Erduvan, F, Keskin, R. 2019. Non-negative integer solutions of the equation $F_n - F_m = 5^a$. *Turkish Journal of Mathematics;* 43(3): 1115-1123.

**[6].** Demirtürk, B. 2019. On the Diophantine equation $L_n - L_m = 2 \cdot 3^a$. *Periodica Mathematica Hungaricas;* 79(2): 210-217.

**[7].** Tiebekabe, P, Diouf, I. 2021. Powers of three as difference of two Fibonacci numbers. *JP Journal of Algebra, Number Theory and Applications;* 49(2): 185-196.

**[8].** Aboudja, H., Hernane, M, Rihane, SE, Togbé, A. 2021. On perfect powers that are sums of two Pell numbers. *Periodica Mathematica Hungarica; 82*: 11-15.

**[9].** Şiar, Z, Keskin, R. 2020. On the Diophantine equation $F_n - F_m = 2^a$. *Colloquium Mathematicum;* 159: 119-126.

**[10].** Irmak, N, Szalay, L. 2016. Tribonacci Numbers Close to the Sum $2^a+3^b + 5^c$. *Mathematica Scandinavica;* 118: 27-32.

**[11].** Y. Bugeaud, 2018. ''Linear Forms in Logarithms and Applications'' IRMA Lectures in Mathematics and Theoretical Physics 28, Zurich, *European Mathematical Society*, 1-176.

**[12].** Y. Bugeaud, M. Mignotte S. Siksek, 2006. ''Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers'' *Annals of Mathematics*, 163(3), 969-1018.

**[13].** J.J. Bravo, C.A. Gomez, F. Luca, 2016. ''Powers of two as sums of two k-Fibonacci numbers'' *Miskolc Mathematical Notes*, 17(1), 85-100.

**[14].** Bravo, J.J., Luca, F. (2012). Powers of two in generalized Fibonacci sequences. *Revista Colombiana de Matemáticas*, *46*(1), 67-79.