POLİTEKNİK DERGİSİ

JOURNAL of POLYTECHNIC

ISSN: 1302-0900 (PRINT), ISSN: 2147-9429 (ONLINE)
URL: http://dergipark.gov.tr/politeknik

Flexure analysis of laminated composite and sandwich beams using timoshenko beam theory

Timoshenko kiriş teorisi kullanılarak lamine kompozit ve sandviç kirişlerin eğilme analizleri

Yazar(lar)(Author((s)): *Armağan KARAMANLI*

ORCID: 0000-0003-3990-6515

Bu makaleye şu şekilde atıfta bulunabilirsiniz(To cite to this article): Karamanlı A., "Flexure analysis of laminated composite and sandwich beams using timoshenko beam theory ", ***Politeknik Dergisi***, 21(3): 633-643, (2018).

Erişim linki (To link to this article): http://dergipark.gov.tr/politeknik/archive

DOI: **10.2339/politeknik.386958**

Flexure Analysis of Laminated Composite and Sandwich Beams Using Timoshenko Beam Theory

Araştırma Makalesi / Research Article

Armağan KARAMANLI*

Mühendislik ve Mimarlık Fakültesi, Mekatronik Mühendisliği Bölümü, İstanbul Gelişim Üniversitesi, Türkiye

(Received : 12.07.2017 ; Accepted : 23.08.2017)

ABSTRACT

The static behaviour of laminated composite and sandwich beams subjected to various sets of boundary conditions is investigated by using the Timoshenko beam theory and the Symmetric Smoothed Particle Hydrodynamics (SSPH) method. In order to solve the problem, a SSPH code which consists of up to sixth order derivative terms in Taylor series expansion is developed. The validation and convergence studies are performed by solving symmetric and anti-symmetric cross-ply composite beam problems with various boundary conditions and aspect ratios. The results in terms of mid-span deflections, axial and shear stresses are compared with those from previous studies to validate the accuracy of the present method. The effects of fiber angle, lay-up and aspect ratio on mid-span displacements and stresses are studied. At the same time, the problems not only for the convergence analysis but also for the extensive analysis are also solved by using the Euler-Bernoulli beam theory for comparison purposes.

Keywords: Meshless method, composite beam, SSPH method, Timoshenko beam theory.

Timoshenko Kiriş Teorisi Kullanılarak Lamine Kompozit ve Sandviç Kirişlerin Eğilme Analizleri

ÖZ

Timoshenko kiriş teorisi ve Simetrik Düzgünleştirilmiş Parçacık Hidrodinamiği (SDPH) yöntemi kullanılarak çeşitli sınır koşullarına sahip lamine kompozit ve sandviç kirişlerin static davranışları incelenmiştir. Problemin çözümü için Taylor serisi açılımında 6.mertebeye kadar türev ifadelerini içeren SDPH algoritması geliştirilmiştir. Çeşitli sınır koşullarına ve en boy oranlarına sahip simetrik ve simetik olmayan çapraz destekli kompozit kiriş problemleri çözülerek doğrulama ve yakınsaklık çalışmaları gerçekleştirilmiştir. Sunulan yöntemin doğruluğunu sağlamak üzere boyutsuz formda elde edilen orta nokta çökmesi, eksenel ve kayma gerilme değerleri daha önceki çalışmalardan elde edilen sonuçlarla karşılaştırılmıştır. Fiber açısının, lamine yerleşimlerinin, ve en boy oranlarının orta nokta çökmesi ve gerilmeler üzerindeki etkileri çalışılmıştır. Aynı zamanda, karşılaştırma amacıyla hem yakınsaklık hem de detaylı analiz çalışmaları için çözülen problemler Euler-Bernoulli kiriş teorisi kullanılarak da çözülmüştür.

Anahtar Kelimeler: Ağsız yöntem, kompozit kiriş, SSPH yöntemi, Timoshenko kiriş teorisi

1. INTRODUCTION

In recent years, the use of the structures which are made of composite materials have been increasing in many modern engineering applications such as aerospace, marine, automotive, and civil engineering due to attractive properties in strength, stiffness and lightness.

Researchers have been developed various beam theories for analysis of the structural behaviour of the composite beams during the last decade, the review of these theories is given in [1]. The Euler-Bernoulli beam theory (EBT) is widely used to solve the bending behaviour of the thin beams. When the beam is thick or short, the effect of the transverse shear deformation cannot be neglected and refined shear deformation theories are needed. One of the theories which have been developed to eleminate the assumption which is that the cross sections which are normal to the mid-plane before deformation remain plane/straight and normal to the mid-plane after deformation in the EBT is the first order shear deformation theory called as Timoshenko beam theory (TBT). In the TBT, the normality assumption of the EBT is relaxed and the cross sections do not need to normal to the mid-plane but still remain plane. The TBT requires the shear correction factor (SCF) to compensate the error due to the assumption of the constant transverse shear strain and shear stress through the beam thickness. The SCF depends on the geometric and material parameters of the beam but the loading and boundary conditions are also important to determine the SCF [2-3].

Many higher order beam theories (HBT) including quasi-3D ones have been used to study the bending behaviour of composite beams and only some of them [4-14] are referenced here. In [4], a set of theoretical models which include all the secondary effects such as the transverse shear stress, shear strains and their variation across the cross section is developed. Exact solutions have been developed for symmetric and antisymmetric cross-ply beams with arbitrary boundary conditions subjected to arbitrary loadings based on EBT, TBT and RBT by using

** Corresponding Author*
e-mail : armagan_k@yahoo.com

the state space concept in [5]. A general four-degrees-of-freedom beam theory (G4DOFBT) which takes into consideration the effects of both transverse shear and normal deformation is presented in [6]. A third-order composite beam element which possesses a linear bending strain is presented for the analysis of composite beams and plates based on the RBT [7]. In [8], a shear deformation theory which includes two variables considering the effect of the normal strain and satisfies the zero tangential traction boundary conditions on the surfaces of the beam is developed. A multi-layered laminated composite structure model which satisfies the continuity condition of displacements and transverse shear stresses at interfaces, as well as the boundary conditions for a laminated composite with the help of the Heaviside step function is presented in [9]. A refined 2-node, 4 DOF/node beam element is derived based on the RBT theory for axial–flexural-shear coupled deformation in asymmetrically stacked laminated composite beams is developed [10]. In [11], the kinematics of the laminated composite beam is presented by using a sinus function for the transverse shear strain distribution. The numerical assessment of different finite element models (FEM) for the static analysis of laminated composite beams of various cross-sections, considering equivalent single layer theories is presented in [12]. A refined formulation ZigZag theory is presented for the analysis of laminated composite beams (RZT) [13]. A four unknown shear and normal deformation theory is used to flexural analysis of laminated composite and sandwich beams in [14].

Analytical, experimental and numerical methods have been used to explore the static behaviours of composite and sandwich beams. However, in some cases it is impossible to obtain the analytical solution and the cost of experimental studies are being expensive. By the advancement in the computer technology, the solution of these complex problems becomes possible via different numerical approaches such as the finite element methods (FEM), meshless methods, generalized differential quadrature method (GDQM), etc. The finite element method (FEM) is one of the most commonly used numerical methods for engineering problems. However, they have some drawbacks which can be eleminated by using meshless methods, for instance avoiding the re-mesh at every step during the evolution of the analysis.

Meshless methods are the most promising and have attracted considerable attention for the analysis of engineering problems with intrinsic complexity. Meshless methods are widely used in static and dynamic analyses of the isotropic, laminated composite and functionally graded beam problems [15-23]. However, the studies are very limited regarding to the flexure analysis of laminated composite and sandwich beams by employing a meshless method [24-27].

As it is seen form above literature survey, the studies related to flexure analysis of the laminated composite and sandwich beams by employing a meshless method are very limited in the literature. The main scope of this work is to investigate the flexure behaviour of the laminated composite and sandwich beams based on various beam theories such as Euler Bernoulli Beam Theory (EBT) and Timoshenko Beam Theory (TBT) by using the Symmetric Smoothed Particle Hydrodynamics (SSPH) method. In this paper, the elastostatic analysis of the laminated composite and sandwich beams are presented by considering fibre angles, lay-ups, aspect ratios and sets of boundary conditions.

In section 2, the formulation of the basis function of the SSPH method is given. In section 3, the constitutive equations of the composite and sandwich beams are presented. The formulation of the EBT and TBT based on the studied beam problems are given in Section 4. In Section 5, numerical results are given for the problems with four different boundary conditions which are simply supported (SS), clamped-simply supported (CS), clamped-clamped (CC) and clamped-free (CF).

2. FORMULATION OF SYMMETRIC SMOOTHED PARTICLE HYDRODYNAMICS METHOD

A scalar function for 1D case can be presented by using Taylor Series Expansion (TSE) as follows

$$f(\xi)=f(x)+(\xi-x)f'(x)+\frac{1}{2!}(\xi-x)^2f''(x)+ +\frac{1}{3!}(\xi-x)^3f'''(x)+\frac{1}{4!}(\xi-x)^4f^{(IV)}(x)+ \frac{1}{5!}(\xi-x)^5f^{(V)}(x)+\frac{1}{6!}(\xi-x)^6f^{(VI)}(x)+\ldots \quad (1)$$

where $f(\xi)$ is the value of the function at ξ located in near of x. The Eq. (1) can be given by employing the zeroth to sixht order terms and neglecting the higher order terms

$$f(\xi)=\boldsymbol{P}(\xi,x)\boldsymbol{Q}(x) \quad (2)$$

where

$$\boldsymbol{Q}(x)=\left[f(x),\frac{df(x)}{dx},\frac{1}{2!}\frac{d^2f(x)}{dx^2},\ldots,\frac{1}{6!}\frac{d^6f(x)}{dx^6}\right]^T \quad (3)$$

$$\boldsymbol{P}(\xi,x)=[1,(\xi-x),(\xi-x)^2,\ldots,(\xi-x)^6] \quad (4)$$

The number of terms employed in the TSE can be increased to improve the accuracy depending on the order of the governing equations. However, increasing the number of terms to be employed definitely increases the CPU time and may decrease the effectiveness of the method. Determination of the number of terms mainly depends on the experience of the researcher. To determine the unknown variables given in the $\mathbf{Q}(x)$, both sides of Eq. (2) are multiplied with $W(\xi,x)\mathbf{P}(\xi,x)^T$ and evaluated for every node in the CSD. In the global numbering system, let the particle number of the j^{th} particle in the compact support of $W(\xi,x)$ be r (j). The following equation is obtained

$$\sum_{j=1}^{N(x)} f\left(\xi^{r(j)}\right)W\left(\xi^{r(j)},x\right)\boldsymbol{P}\left(\xi^{r(j)},x\right)^T = \sum_{j=1}^{N(x)}\left[\boldsymbol{P}\left(\xi^{r(j)},x\right)^TW\left(\xi^{r(j)},x\right)\boldsymbol{P}\left(\xi^{r(j)},x\right)\right]\boldsymbol{Q}(x) \quad (5)$$

where N(x) is the number nodes in the compact support domain (CSD) of the W(ξ, x) as shown in Figure 1.

Then, Eq. (5) can be given by

$$\boldsymbol{C}(\xi,x)\boldsymbol{Q}(x) = \boldsymbol{D}(\xi,x)\boldsymbol{F}^{(x)}(\xi,x) \tag{6}$$

Where $\mathbf{C}(\xi,x) = \mathbf{P}(\xi,x)^{T}\mathbf{W}(\xi,x)\mathbf{P}(\xi,x)$ and $\mathbf{D}(\xi,x) = \mathbf{P}(\xi,x)^{T}\,\mathbf{W}(\xi,x)$.

The solution of Eq. (6) is given by

$$\boldsymbol{Q}(x) = \boldsymbol{K}(\xi,x)\boldsymbol{F}(\xi) \tag{7}$$

where $\mathbf{K}^{(x)}(\xi,x) = \mathbf{C}(\xi,x)^{-1}\mathbf{D}(\xi,x)$.

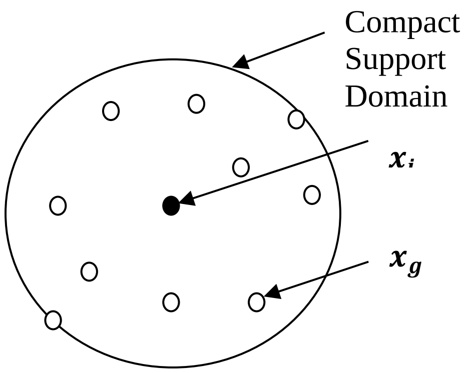


Figure 1. Compact support of the weight function W(ξ, x) for the node located at $x = (x_i, y_i)$.

Eq. (7) can be also written as follows

$$Q_I(x) = \sum_{J=1}^{M} K_{IJ}F_J\ , \qquad I = 1,2,\dots,7 \tag{8}$$

where M is the number of nodes and $F_J = f(\xi^J)$. Seven components of Eq. (8) for 1D case are written as

$$f(x) = Q_1(x) = \sum_{J=1}^{M} K_{1J}F_J$$

$$\frac{df(x)}{dx} = Q_2(x) = \sum_{J=1}^{M} K_{2J}F_J$$

$$\frac{d^2f(x)}{dx^2} = 2!\,Q_3(x) = 2!\sum_{J=1}^{M} K_{3J}F_J$$

$$\frac{d^3f(x)}{dx^3} = 3!\,Q_4(x) = 3!\sum_{J=1}^{M} K_{4J}F_J$$

$$\frac{d^4f(x)}{dx^4} = 4!\,Q_5(x) = 4!\sum_{J=1}^{M} K_{5J}F_J$$

$$\frac{d^5f(x)}{dx^5} = 5!\,Q_6(x) = 5!\sum_{J=1}^{M} K_{6J}F_J$$

$$\frac{d^6f(x)}{dx^6} = 6!\,Q_7(x) = 6!\sum_{J=1}^{M} K_{7J}F_J \tag{9}$$

3. CONSTITUTIVE EQUATIONS

In Figure 2., a laminated composite beam which is made of many plies of orthotropic materials in different orientations with respect to x-axis is presented. The formulation of the constitutive equations following assumptions are done;

1. A lamina is continuum; i.e. , no gaps or empty spaces exist,

2.A lamina behaves as a linear elastic material

3.Each lamina is bounded perfectly to each other.

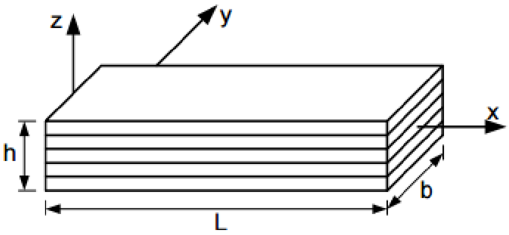


Figure 2. Geometry of a laminated composite beam.

The stress-strain relationship of a k^{th} orthotropic lamina in the material coordinate axes is given by:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{xz}\end{Bmatrix}^k = \begin{bmatrix}Q_{11} & 0\\ 0 & Q_{55}\end{bmatrix}^k \begin{Bmatrix}\varepsilon_{xx}\\ \gamma_{xz}\end{Bmatrix} \tag{10}$$

where (σ_{xx}, σ_{xz}) are the stresses and (ε_{xx}, γ_{xz}) are the normal strain and shear strain, respectively, with respect to the laminate axes. Q_{ij}'s are the transformed elastic constants or stiffness matrix with respect to laminate axis x. The transformed elastic constant can be given by:

$$Q_{11} = C_{11}cos^4\theta + 2(C_{12} + 2C_{66})cos^2\theta sin^2\theta + C_{22}sin^4\theta$$

$$Q_{55} = C_{44}sin^2\theta + C_{55}cos^2\theta \tag{11}$$

where

$$C_{11} = \frac{E_1}{1 - \upsilon_{12}\upsilon_{21}};\ C_{12} = \frac{E_1\upsilon_{21}}{1 - \upsilon_{12}\upsilon_{21}};$$

$$C_{22} = \frac{E_2}{1 - \upsilon_{12}\upsilon_{21}}; C_{66} = G_{12}; C_{55} = G_{13}; C_{44} = G_{23};$$

$E_1, E_2, G_{12}, G_{13}, G_{23}, \upsilon_{12}$ and υ_{21} are the six independent engineering constants. E is the Young's Modulus, G is the Shear Modulus and υ is the Poisson's ratio.

4. THEORETICAL FORMULATION OF BEAM THEORIES

The kinematics of deformation of a beam can be represented by using various beam theories. Among them, the Euler Bernoulli Beam Theory (EBT) and the Timoshenko Beam Theory (TBT) are commonly used. To describe the EBT and TBT the following coordinate system is introduced. The x-coordinate is taken along the axis of the beam and the z-coordinate is taken through the height (thickness) of the beam. In the general beam theory, all the loads and the displacements (u,w,ϕ) along the coordinates (x,z) are only the functions of the x and z coordinates. The formulations of the beam theories based on the laminated composite beams are given below.

4.1 Euler Bernoulli Beam Theory

According to EBT, the displacement field is given by,

$$U(x,z) = u(x) - z\frac{dw(x)}{dx}$$
$$W(x,z) = w(x) \quad (12)$$

where u and w are two variables to be determined. The only the axial strain which is nonzero is given by,

$$\varepsilon_{xx} = \frac{dU}{dx} = \frac{du}{dx} - z\frac{d^2w}{dx^2} \quad (13)$$

The virtual strain energy of the beam can be presented by using the axial stress and the axial strain as follows

$$\delta U = \int_0^L \int_A \sigma_{xx}\delta\varepsilon_{xx} dAdx \quad (14)$$

where δ is the variational operator, A is the cross sectional area, L is the length of the beam, σ_{xx} is the axial stress. The stress resultants can be given by,

$$M_x = \int_A z\sigma_{xx} dA \quad (15a)$$

$$N_x = \int_A \sigma_{xx} dA \quad (15b)$$

Using Eq. (13) and Eq. (15), Eq. (14) can be rewritten as,

$$\delta U = \int_0^L \left(N_x\frac{d\delta u}{dx} - M_x\frac{d^2\delta w}{dx^2}\right)dx \quad (16)$$

The virtual potential energy of the load q(x) is given by

$$\delta V = -\int_0^L q\delta w dx \quad (17)$$

If a body is in equilibrium, $\delta W = \delta U + \delta V$, the total virtual work (δW) done equals zero and is given by,

$$\delta W = \int_0^L \left(N_x\frac{d\delta u}{dx} - M_x\frac{d^2\delta w}{dx^2} - q\delta w\right)dx = 0 \quad (18)$$

After performing integration by parts in Eq. (18) and since δu and δw are arbitrary in (0 < x < L), one can obtain following equilibrium equations

$$-\frac{d^2M_x}{dx^2} = q(x)\ for\ 0 < x < L \quad (19a)$$
$$-\frac{dN_x}{dx} = 0\ for\ 0 < x < L \quad (19b)$$

It is useful to introduce the shear force Q_x and rewrite the Eq. (19a) in the following form

$$-\frac{dM_x}{dx} + Q_x = 0, \quad -\frac{dQ_x}{dx} = q(x) \quad (20)$$

Using Eqs. (10), (13) and (15), the followings can be written,

$$M_x = B\frac{du}{dx} - D\frac{d^2w}{dx^2} \quad (21a)$$

$$N_x = A\frac{du}{dx} - B\frac{d^2w}{dx^2} \quad (21b)$$

where

$$(A,B,D) = \int_{-h/2}^{+h/2} Q_{11}(1,z,z^2)dz \quad (22)$$

The EBT governing equations for a laminated composite beam subjected to the distributed load are given by

$$\frac{d^2}{dx^2}\left(D\frac{d^2w}{dx^2} - B\frac{du}{dx}\right) = q(x) \quad (23a)$$
$$-\frac{d}{dx}\left(A\frac{du}{dx} - B\frac{d^2w}{dx^2}\right) = 0 \quad (23b)$$

4.2 Timoshenko Beam Theory

The following displacement field is given for the TBT,

$$U(x,z) = u(x) + z\phi(x)$$
$$W(x,z) = w(x) \quad (24)$$

where u, ϕ and w are three variables to be determined. Using the Eq. (24), the non zero strains can be given

$$\varepsilon_{xx} = \frac{dU}{dx} = \frac{du}{dx} + z\frac{d\phi}{dx}$$
$$\gamma_{xz} = \frac{dU}{dz} + \frac{dW}{dx} = \phi + \frac{dw}{dx} \quad (25)$$

The virtual strain energy of the beam including the virtual energy associated with the shearing strain can be written as,

$$\delta U = \int_0^L \int_A (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz})dAdx \quad (26)$$

where σ_{xz} is the transverse shear stress and γ_{xz} is the shear strain. The stress resultants can be given by,

$$M_x = \int_A z\sigma_{xx} dA \quad (27a)$$

$$Q_x = \int_A \sigma_{xz} dA \quad (27b)$$

$$N_x = \int_A \sigma_{xx} dA \quad (27c)$$

Using Eqs. (25) and (27), one can rewrite the Eq.(26) as,

$$\delta U = \int_0^L \left[N_x\frac{d\delta u}{dx} + M_x\frac{d\delta\phi}{dx} + Q_x\left(\delta\phi + \frac{d\delta w}{dx}\right)\right]dx \quad (28)$$

The virtual potential energy of the load q(x) is given by

$$\delta V = -\int_0^L q\delta w dx \quad (29)$$

If a body is in equilibrium, $\delta W = \delta U + \delta V$, the total virtual work (δW) done equals zero and is given by,

$$\delta W = \int_0^L \left[N_x\frac{d\delta u}{dx} + M_x\frac{d\delta\phi}{dx} + Q_x\left(\delta\phi + \frac{d\delta w}{dx}\right) - q\delta w\right]dx = 0 \quad (30)$$

Since the total virtual work done equals zero and the coefficients of $\delta u, \delta\phi$ and δwin $0 < x < L$ are zero, the following governing equations can be given by,

$$-\frac{dM_{xx}}{dx} + Q_x = 0 \quad (31a)$$

$$-\frac{dQ_x}{dx} = q(x) \quad (31b)$$

$$-\frac{dN_x}{dx} = 0 \quad (31c)$$

Using Eqs. (10), (25) and (27) the stress resultants can be written by,

$$M_x = B\frac{du}{dx} + D\frac{d\phi}{dx} \quad (32a)$$

$$Q_x = \kappa_s A_s\left(\phi + \frac{dw}{dx}\right) \quad (32b)$$

$$N_x = A\frac{du}{dx} + B\frac{d\phi}{dx} \quad (32c)$$

where κ_s is the shear correction factor to be used to compensate the error caused by the assumption of a constant transverse shear stress distribution along the beam thickness and

$$A_s = \int_{-h/2}^{+h/2} Q_{55} dz \tag{33}$$

The governing equations of the TBT are given by

$$-\frac{d}{dx}\left(B\frac{du}{dx} + D\frac{d\phi}{dx}\right) + \kappa_s A_s\left(\phi + \frac{dw}{dx}\right) = 0 \tag{34a}$$

$$-\frac{d}{dx}\left[\kappa_s A_s\left(\phi + \frac{dw}{dx}\right)\right] = q(x) \tag{34b}$$

$$-\frac{d}{dx}\left(A\frac{du}{dx} + B\frac{d\phi}{dx}\right) = 0 \tag{34c}$$

5. NUMERICAL RESULTS

The flexure behaviour of the composite beams is investigated by a number of numerical examples considering the EBT and TBT formulations. The numerical results in terms of displacements and stresses of composite beams are obtained by using the SSPH method and considering various lay-ups, aspect ratios and boundary conditions. The results from previous studies [5,8] in terms of dimensionless mid-span deflections, axial and shear stresses are used for comparison purposes. After the verification of the developed code, the number of nodes to be used in the problem domain for the numerical calculations is determined and extensive analysis are performed. The physical parameters of the beam are L=1m and b=0.1m. Three different two aspect ratios (L/h) 5, 10 and 50 are considered. The distributed load q_0 is set to 10000 N/m. The shear correction factor is set to 5/6. The material properties of the problems studied within this paper are given in Table 2.

The following non-dimensional quantities are used for the representation of the results;

Non-dimensional maximum transverse deflection of the beam:

$$\bar{w} = \frac{100E_m bh^3}{q_0 L^4} w(L/2, z) \tag{35}$$

Non-dimensional axial and shear stresses of the beam:

$$\bar{\sigma}_x = \frac{bh^2}{q_0 L^2}\sigma_x\left(\frac{L}{2}, z\right)$$

$$\bar{\sigma}_{xz} = \frac{bh}{q_0 L}\sigma_{xz}(0, z) \tag{36}$$

5.1 Verification, Comparison and Convergence Studies

The developed SSPH code is verified by solving symmetric and anti-symmetric cross-ply composite beams subjected to uniformly distributed load with different boundary conditions (simply supported and cantilever) and aspect ratios. Three types of uniformly node distributions in the problem domain are employed for numerical calculations, 41, 81 and 161 nodes based on the experience of previous studies [22-23]. The following weight function used in [23] is employed for the analysis:

Table 1. Boundary conditions used for the numerical computations.

BC		x=0	x=L
EBT	S-S	$u = 0, w = 0, M_x = 0$	$u = 0$ or $N_x = 0, w = 0, M_x = 0$
	C-S	$u = 0, w = 0, w' = 0$	$u = 0$ or $N_x = 0, w = 0, M_x = 0$
	C-C	$u = 0, w = 0, w' = 0$	$u = 0, w = 0, w' = 0$
	C-F	$u = 0, w = 0, w' = 0$	$N_x = 0, M_x = 0, M_x' = 0$
TBT	S-S	$u = 0, w = 0, M_x = 0$	$u = 0$ or $N_x = 0, w = 0, M_x = 0$
	C-S	$u = 0, w = 0, \phi = 0$	$u = 0$ or $N_x = 0, w = 0, M_x = 0$
	C-C	$u = 0, w = 0, \phi = 0$	$u = 0, w = 0, \phi = 0$
	C-F	$u = 0, w = 0, \phi = 0$	$N_x = 0, M_x = 0, Q_x = 0$

Table 2. Material properties of composite and sandwich beams.

Problem	Structure	Material Properties
1	Type A	$E_1/E_2 = 25; E_3 = E_2;\ G_{12} = G_{13} = 0.5E_2;\ G_{23} = 0.2E_2$ $\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0.25$
2	Type B	Face Layer: Type A
		Core Layer: $E_1/E_2 = 1; E_3 = E_2;\ G_{12} = G_{13} = 1.5E_2;\ G_{23} = 0.4E_2$ $\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0.25$

$$W(x,\xi) = \begin{cases} \left(1-\frac{d}{\rho}\right)^7 35\left(\frac{d}{\rho}\right)^6 + 245\left(\frac{d}{\rho}\right)^5 + 720\left(\frac{d}{\rho}\right)^4 + 1120\left(\frac{d}{\rho}\right)^3 + 928\left(\frac{d}{\rho}\right)^2 + 336\left(\frac{d}{\rho}\right) + 48 & 0 \le d \le \rho \\ 0 & d > \rho \end{cases} \quad (37)$$

where $d = |x - \xi|/h$ is the radius of the compact support domain, h is the smoothing length.

The numerical calculations are performed according to the following meshless parameters; the radius of the

Table 3. Verification and convergence studies of the SSPH code, dimensionless mid-span deflections for different number of nodes.

Theory	Reference	Symmetric (0°/90°/0°)			Anti-symmetric (0°/90°)		
		L/h=5	10	50	L/h=5	10	50
a.	Simply Supported Beams (S-S)						
EBT	Khdeir and Reddy [5]	0.646	0.646	0.646	3.322	3.322	3.322
TBT		2.146	1.021	0.661	5.036	3.750	3.339
EBT	SSPH - 41 nodes	0.6464	0.6464	0.6464	3.3216	3.3216	3.3216
	SSPH - 81 nodes	0.6464	0.6464	0.6464	3.3216	3.3216	3.3216
	SSPH - 161 nodes	0.6464	0.6464	0.6464	3.3216	3.3216	3.3216
TBT	SSPH - 41 nodes	2.1464	1.0214	0.6614	5.0359	3.7502	3.3387
	SSPH - 81 nodes	2.1464	1.0214	0.6614	5.0359	3.7502	3.3387
	SSPH - 161 nodes	2.1464	1.0214	0.6614	5.0359	3.7502	3.3387
b.	Cantilever Beams (C-F)						
EBT	Khdeir and Reddy [5]	2.198	2.198	2.198	11.293	11.293	11.293
TBT		6.698	3.323	2.243	16.436	12.579	11.345
EBT	SSPH - 41 nodes	2.1978	2.1978	2.1978	11.2934	11.2934	11.2934
	SSPH - 81 nodes	2.1978	2.1978	2.1978	11.2934	11.2934	11.2934
	SSPH - 161 nodes	2.1978	2.1978	2.1978	11.2934	11.2934	11.2934
TBT	SSPH - 41 nodes	6.6978	3.3228	2.2428	16.4362	12.5791	11.3448
	SSPH - 81 nodes	6.6978	3.3228	2.2428	16.4362	12.5791	11.3448
	SSPH - 161 nodes	6.6978	3.3228	2.2428	16.4362	12.5791	11.3448

Table 4. Verification and convergence studies of the SSPH code, dimensionless axial $\bar{\sigma}_x\left(\frac{L}{2},\frac{h}{2}\right)$ and shear $\bar{\sigma}_{xz}(0,0)$ stresses of S-S beams for different number of nodes.

Theory	Reference	Symmetric (0°/90°/0°)			Anti-symmetric (0°/90°)		
		L/h=5	10	50	L/h=5	10	50
a.	Axial (Normal) Stress						
EBT	Zenkour [8]	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
TBT		0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
EBT	SSPH - 41 nodes	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
	SSPH - 81 nodes	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
	SSPH - 161 nodes	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
TBT	SSPH - 41 nodes	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
	SSPH - 81 nodes	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
	SSPH - 161 nodes	0.7776	0.7776	0.7776	0.2336	0.2336	0.2336
b.	Shear Stress						
TBT	Zenkour [8]	0.2994	0.2994	0.2994	0.8553	0.8553	0.8553
TBT	SSPH - 41 nodes	0.3000	0.3000	0.3000	0.8571	0.8571	0.8571
	SSPH - 81 nodes	0.3000	0.3000	0.3000	0.8571	0.8571	0.8571
	SSPH - 161 nodes	0.3000	0.3000	0.3000	0.8571	0.8571	0.8571

support domain (d) is chosen as 8 and the smoothing length (*h*) equals to 1.3Δ. Δ can be defined as the minimum distance between two adjacent nodes. The meshless parameters are selected by performing trial and error method.

Based on the various node distributions, aspect ratios, symmetric and anti-symmetric beam structures, nondimensional mid-span deflections, axial and shear stresses are obtained by using different beam theories. The results are given in Table 3-4 along with the results from previous studies. It is clear that the results obtained by using the SSPH method agree completely with those of previous papers [5,8]. The computed results obtained by using the EBT and TBT, the mid-span deflections, axial and transverse shear stresses are almost the same with those obtained from various authors. Due to this agreement, the verification of the developed code is established. For the sake of accuracy, uniformly distributed 161 nodes will be used in the problem domain for the extensive analysis.

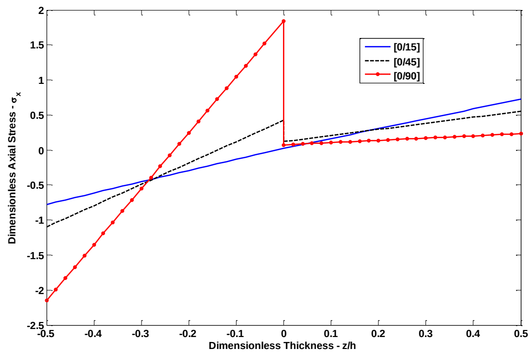
5.2 Elastostatic Analysis of Laminated Composite and Sandwich Beams

Four different boundary conditions, SS, CS, CC and CF are considered respectively for the bending analysis of laminated composite and sandwich beams subjected to uniformly distributed load. The mid-span deflections, axial and shear stresses are computed based on the various beam theories, lay-ups, fiber angles and aspect ratios.

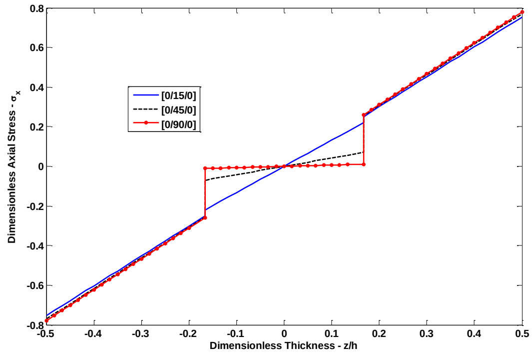
5.2.1 Laminated Composite Beams: Type A

By extending the problem which is solved for the convergence and verification analysis, symmetric [0°/θ/0°] and unsymmetric [0°/θ] composite beams are considered. In Tables 5 and 6, variations of mid-span displacements, axial and shear stresses respect to the fiber angle are given. As the fiber angle increases, mid-span deflections and axial stress values increase for all type of boundary conditions and aspect ratios. With the increasing of the aspect ratio, the mid-span deflections decrease. It is found that the axial stresses computed based on the EBT and TBT formulation are almost same. As it is expected, the difference between the EBT and TBT in terms of mid-span deflections is negligible for a thin beam.

As it is seen from Figs. 3 and 4, as the fiber angle increases, the dimensionless axial and shear stresses increase for all type of boundary conditions and aspect ratios. The discontinuities are visible for all types of composite beam structures.

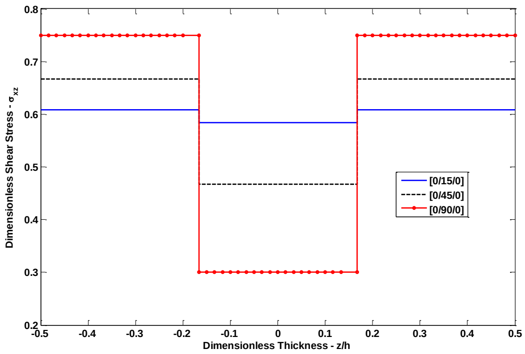


a. Symmetric lay-up

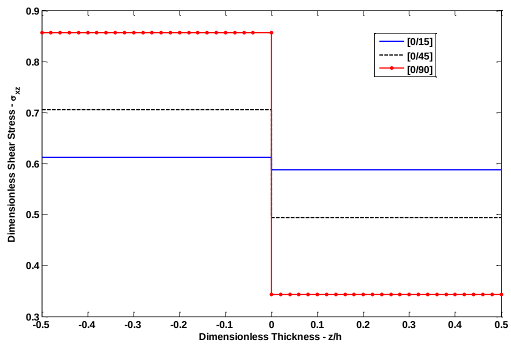


b. Anti-symmetric lay-up

Figure 3. Axial stress distribution through the thickness of symmetric and anti-symmetric beams with S-S boundary condition based on TBT



a. Symmetric lay-up



b. Anti-symmetric lay-up

Figure 4. Shear stress distribution through the thickness of symmetric and anti-symmetric beams with S-S boundary condition based on TBT, Type A, L/h=5

Table 5. Dimensionless mid-span deflections of [0°/θ/0°] and [0°/θ] beams for various boundary conditions under a uniformly distributed load, Type A.

Aspect Ratio (L/h)	Theory	Lay-ups	0°	15°	30°	45°	60°	75°	90°
a.	Simply supported beams (S-S)								
5	EBT	[0°/θ]	0.6234	0.6665	0.8303	1.2639	2.2352	3.1802	3.3216
		[0°/θ/0°]	0.6234	0.6263	0.6332	0.6404	0.6448	0.6463	0.6464
10		[0°/θ]	0.6234	0.6665	0.8303	1.2639	2.2352	3.1802	3.3216
		[0°/θ/0°]	0.6234	0.6263	0.6332	0.6404	0.6448	0.6463	0.6464
50		[0°/θ]	0.6234	0.6665	0.8303	1.2639	2.2352	3.1802	3.3216
		[0°/θ/0°]	0.6234	0.6263	0.6332	0.6404	0.6448	0.6463	0.6464
5	TBT	[0°/θ]	1.8234	1.8910	2.1276	2.6757	3.7836	4.8467	5.0359
		[0°/θ/0°]	1.8234	1.8426	1.8964	1.9737	2.0566	2.1216	2.1464
10		[0°/θ]	0.9234	0.9726	1.1547	1.6169	2.6223	3.5968	3.7502
		[0°/θ/0°]	0.9234	0.9304	0.9490	0.9737	0.9978	1.0151	1.0214
50		[0°/θ]	0.6354	0.6787	0.8433	1.2780	2.2507	3.1969	3.3387
		[0°/θ/0°]	0.6354	0.6385	0.6458	0.6537	0.6590	0.6610	0.6614
b.	Clamped simply supported beams (C-S)								
5	EBT	[0°/θ]	0.2494	0.2666	0.3321	0.5056	0.8941	1.2721	1.3286
		[0°/θ/0°]	0.2494	0.2505	0.2533	0.2562	0.2579	0.2585	0.2587
10		[0°/θ]	0.2494	0.2666	0.3321	0.5056	0.8941	1.2721	1.3286
		[0°/θ/0°]	0.2494	0.2505	0.2533	0.2562	0.2579	0.2585	0.2587
50		[0°/θ]	0.2494	0.2666	0.3321	0.5056	0.8941	1.2721	1.3286
		[0°/θ/0°]	0.2494	0.2505	0.2533	0.2562	0.2579	0.2585	0.2587
5	TBT	[0°/θ]	1.5899	1.6371	1.7929	2.1135	2.6811	3.2070	3.3197
		[0°/θ/0°]	1.5899	1.6087	1.6623	1.7409	1.8269	1.8953	1.9216
10		[0°/θ]	0.5983	0.6229	0.7107	0.9194	1.3500	1.7637	1.8345
		[0°/θ/0°]	0.5983	0.6041	0.6203	0.6432	0.6674	0.6860	0.6931
50		[0°/θ]	0.2636	0.2811	0.3475	0.5223	0.9125	1.2919	1.3490
		[0°/θ/0°]	0.2636	0.2649	0.2683	0.2720	0.2747	0.2760	0.2764
c.	Cantilever beams (C-F)								
5	EBT	[0°/θ]	2.1197	2.2660	2.8232	4.2973	7.5998	10.8128	11.2934
		[0°/θ/0°]	2.1197	2.1294	2.1529	2.1774	2.1925	2.1974	2.1978
10		[0°/θ]	2.1197	2.2660	2.8232	4.2973	7.5998	10.8128	11.2934
		[0°/θ/0°]	2.1197	2.1294	2.1529	2.1774	2.1925	2.1974	2.1978
50		[0°/θ]	2.1197	2.2660	2.8232	4.2973	7.5998	10.8128	11.2934
		[0°/θ/0°]	2.1197	2.1294	2.1529	2.1774	2.1925	2.1974	2.1978
5	TBT	[0°/θ]	5.7197	5.9398	6.7150	8.5326	12.2449	15.8121	16.4363
		[0°/θ/0°]	5.7197	5.7783	5.9424	6.1774	6.4278	6.6232	6.6978
10		[0°/θ]	3.0197	3.1844	3.7961	5.3561	8.7611	12.0626	12.5791
		[0°/θ/0°]	3.0197	3.0416	3.1002	3.1774	3.2513	3.3038	3.3228
50		[0°/θ]	2.1557	2.3027	2.8621	4.3397	7.6462	10.8628	11.3448
		[0°/θ/0°]	2.1557	2.1659	2.1908	2.2174	2.2348	2.2416	2.2428
d.	Clamped clamped beams (C-C)								
5	EBT	[0°/θ]	0.1247	0.1333	0.1661	0.2528	0.4470	0.6360	0.6643
		[0°/θ/0°]	0.1247	0.1253	0.1266	0.1281	0.1290	0.1293	0.1293
10		[0°/θ]	0.1247	0.1333	0.1661	0.2528	0.4470	0.6360	0.6643
		[0°/θ/0°]	0.1247	0.1253	0.1266	0.1281	0.1290	0.1293	0.1293
50		[0°/θ]	0.1247	0.1333	0.1661	0.2528	0.4470	0.1293	0.6643
		[0°/θ/0°]	0.1247	0.1253	0.1266	0.1281	0.1290	0.6360	0.1293
5	TBT	[0°/θ]	1.3247	1.3579	1.4634	1.6645	1.9954	2.3025	2.3786
		[0°/θ/0°]	1.3247	1.3416	1.3898	1.4614	1.5407	1.6046	1.6293
10		[0°/θ]	0.4247	0.4394	0.4904	0.6057	0.8341	1.0527	1.0929
		[0°/θ/0°]	0.4247	0.4293	0.4424	0.4614	0.4819	0.4981	0.5043
50		[0°/θ]	0.1367	0.1455	0.1790	0.2669	0.4625	0.6527	0.6815
		[0°/θ/0°]	0.1367	0.1374	0.1393	0.1414	0.1431	0.1440	0.1443

5.2.2 Laminated Composite Sandwich Beams: Type B

Cross-ply sandwich beams (Type B) under uniformly distributed load with the top and bottom face thickness (h_1) and core thickness (h_2) are studied, Fig. 5. The dimensionless mid-span deflections and stresses are computed by using different beam theories for various thickness and aspect ratios.

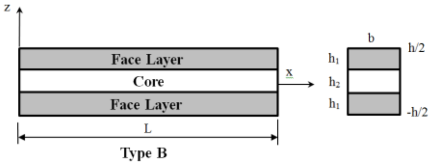


Figure 5. Geometry of the laminated composite sandwich beam (Type B)

Table 6. Dimensionless axial $\bar{\sigma}_x\left(\frac{L}{2},\frac{h}{2}\right)$ and shear $\bar{\sigma}_{xz}(0,0)$ stresses of [0°/θ/0°] and [0°/θ] S-S beams under a uniformly distributed load, Type A.

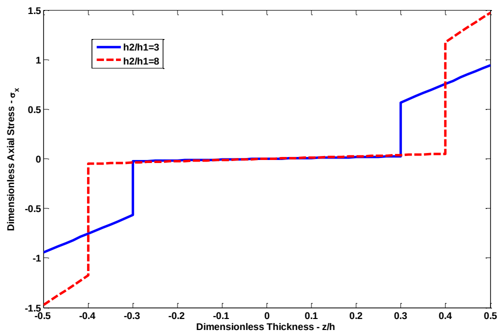
Aspect Ratio (L/h)	Theory	Lay-ups	0°	15°	30°	45°	60°	75°	90°
a. Axial stress									
5	EBT	[0°/θ]	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336
		[0°/θ/0°]	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776
10		[0°/θ]	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336
		[0°/θ/0°]	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776
50		[0°/θ]	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336
		[0°/θ/0°]	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776
5	TBT	[0°/θ]	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336
		[0°/θ/0°]	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776
10		[0°/θ]	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336
		[0°/θ/0°]	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776
50		[0°/θ]	0.7500	0.7261	0.6597	0.5538	0.3921	0.2538	0.2336
		[0°/θ/0°]	0.7500	0.7534	0.7617	0.7704	0.7758	0.7775	0.7776
b. Shear stress									
5	TBT	[0°/θ]	0.6000	0.6123	0.6486	0.7059	0.7742	0.8332	0.8571
		[0°/θ/0°]	0.6000	0.5837	0.5368	0.4667	0.3882	0.3247	0.3000
10		[0°/θ]	0.6000	0.6123	0.6486	0.7059	0.7742	0.8332	0.8571
		[0°/θ/0°]	0.6000	0.5837	0.5368	0.4667	0.3882	0.3247	0.3000
50		[0°/θ]	0.6000	0.6123	0.6486	0.7059	0.7742	0.8332	0.8571
		[0°/θ/0°]	0.6000	0.5837	0.5368	0.4667	0.3882	0.3247	0.3000

Table 7. Dimensionless mid-span deflections of [0°/90°/0°] beams for various boundary conditions under a uniformly distributed load, Type B.

Theory	$h_2/h_1 = 3$			$h_2/h_1 = 8$		
	L/h=5	10	50	L/h=5	10	50
a. Simply supported beams (S-S)						
EBT	0.7860	0.7860	0.7860	1.2229	1.2229	1.2229
TBT	2.1496	1.1269	0.7996	2.6515	1.5801	1.2372
b. Clamped simply supported beams (C-S)						
EBT	0.3144	0.3144	0.3144	0.4892	0.4892	0.4892
TBT	1.8438	0.7116	0.3306	2.1140	0.9077	0.5061
c. Cantilever beams (C-F)						
EBT	2.6723	2.6723	2.6723	4.1580	4.1580	4.1580
TBT	6.7634	3.6951	2.7133	8.4437	5.2294	4.2008
d. Clamped clamped beams (C-C)						
EBT	0.1572	0.1572	0.1572	0.2446	0.2446	0.2446
TBT	1.5208	0.4981	0.1708	1.6732	0.6017	0.2589

Table 8. Dimensionless axial $\bar{\sigma}_x\left(\frac{L}{2},\frac{h}{2}\right)$ and shear $\bar{\sigma}_{xz}(0,0)$ stresses of [0°/90°/0°] S-S beams under a uniformly distributed load, Type B.

Theory	$h_2/h_1 = 3$			$h_2/h_1 = 8$		
	L/h=5	10	50	L/h=5	10	50
a. Axial stress						
EBT	0.9455	0.9455	0.9455	1.4712	1.4712	1.4712
TBT	0.9455	0.9455	0.9455	1.4712	1.4712	1.4712
b. Shear stress						
TBT	0.5455	0.5455	0.5455	0.5714	0.5714	0.5714



a. Axial Stress

b. Shear Stress

Figure 6. Axial and shear stress distribution through the thickness of symmetric sandwich beams with S-S boundary condition based on TBT, Type B, L/h=5.

It is clear that from Tables 7 and 8, the dimensionless mid-span deflections and the stresses increase as the thickness ratio changes from 3 to 8. As the aspect ratio increase, the difference between the EBT and TBT in terms of mid-span deflections decreases. The maximum axial and shear stresses are obtained for the thickness value at 8 as it is seen from Figure 6.

6. CONCLUSION

The flexure behaviour of the laminated composite and sandwich beams are presented by using the EBT and TBT formulation employing the SSPH basis functions with strong formulation of the problem. The EBT and TBT formulations are developed regarding to different types of composite beam structures to evaluate the mid-span deflections, axial and shear stresses. The verification of the developed code is established by solving symmetric and anti-symmetric cross-ply composite beams subjected to uniformly distributed load with different boundary conditions (simply supported and cantilever) and aspect ratios. The numerical calculations are performed by using 161 nodes uniformly distributed in the problem domain and by employing 7 terms in the TSEs. The numerical results based on the TBT formulation are compared with those obtained by other authors and the computed results based on the EBT formulation to show the validity of the SSPH method.

Composite and sandwich beams with various configurations are considered. The following results can be drawn from the computed results based on the TBT:

• For the fiber angle value 0°, all coupling effects from material vanish. Thus, the axial displacement u cannot be obtained.

• Bending behavior can be controlled to meet the desired goals by choosing suitable fiber angle.

• The importance of the shear effect increases as the fiber angle increases for the anti-symmetric laminated composite beams (Type A) for all type of boundary conditions and aspect ratios.

• The mid-span deflections, axial and shear stresses of the symmetric and anti-symmetric laminated composite beams (Type A) are affected by the fiber angles for all type of boundary conditions and aspect ratios.

• The difference in terms of mid-span deflections, axial and shear stresses between the symmetric and anti-symmetric laminated composite beams (Type A) increase as the fiber angle increases.

• C-F laminated composite sandwich beam (Type B) is much more sensitive to the thickness ratio change than the other sandwich beam models.

• For S-S laminated composite sandwich beam (Type B), the difference in terms of axial stress values between the studied thickness ratios is more obvious than shear stress values.

It is found that the SSPH method provides satisfactory and expected results at least for the problems studied here. Based on the results obtained within the scope of the study, it is recommended that the SSPH method can be applied for solving linear laminated composite and sandwich beam problems by employing different shear deformation theories and strong form formulation.

REFERENCES

[1] Nguyen T.K., Nguyen N.D., Vo T.P., Thai H.T., "Trigonometric-series solution for analysis of laminated composite beams", ***Compos Struct***, 160:142-151, (2017).

[2] Timoshenko S.P., Goodier J.C., "Theory of Elasticity", ***McGraw-Hill Co. Inc.***, New York, 1970.

[3] Wang C.M., Reddy J.N., Lee, K.H., "Shear Deformable Beams and Plates Relations with Classical Solutions", ***Elsevier Science Ltd.***, Oxford 2000.

[4] Kant T., Manjunath B.S., "Refined theories for composite and sandwich beams with C0 finite elements", ***Comput Struct***, 33(3):755–764, (1989).

[5] Khdeir A.A., Reddy J.N., "An exact solution for the bending of thin and thick cross-ply laminated beams", ***Compos Struct*** 37(2):195–203, (1997).

[6] Soldatos K.P., Watson P., "A general theory for the accurate stress analysis of homogeneous and laminated composite beams", ***Int J Solids Struct.*** 34(22): 2857–2885, (1997).

[7] Shi G., Lam K.Y., Tay T.E., "On efficient finite element modeling of composite beams and plates using higher-order theories and an accurate composite beam element", ***Compos Struct***, 41(2):159–165, (1998).

[8] Zenkour A. M., "Transverse shear and normal deformation theory for bending analysis of laminated and sandwich elastic beams", ***Mechanics of Composite Materials & Structures***, 6(3): 267-283 (1999).

[9] Karama M., Afaq K.S., Mistou S., "Mechanical behaviour of laminated composite beam by the new multi-layered laminated composite structures model with transverse shear stress continuity", ***Int J Solids Struct.***, 40(6):1525–1546, (2003).

[10] Murthy M.V.V.S., Mahapatra D.R., Badarinarayana K., Gopalakrishnan S., "A refined higher order finite element for asymmetric composite beams", ***Compos Struct***, 67(1):27–35, (2005).

[11] Vidal P., Polit O., "A family of sinus finite elements for the analysis of rectangular laminated beams", ***Compos Struct***, 84(1):56–72, (2008).

[12] Aguiar R.M., Moleiro F., Soares C.M.M., "Assessment of mixed and displacement-based models for static analysis of composite beams of different cross-sections", ***Compos Struct***, 94 (2):601–616, (2012).

[13] Nallim L.G., Oller S., Onate E., Flores F.G., "A hierarchical finite element for composite laminated beams using a refined zigzag theory", ***Compos Struct***, 163:168–184, (2017).

[14] Vo T.P., Thai H.T., Nguyen T.K., Lanc D., Karamanli A., "Flexural analysis of laminated composite and sandwich beams using a four-unknown shear and normal deformation theory", ***Compos Struct***, 176:388-397, (2017).

[15] Donning B.M., Liu W.K., "Meshless methods for shear-deformable beams and plates", ***Computer Methods in Applied Mechanics and Engineering***, 152:47-71, (1998).

[16] Gu Y.T., Liu G.R., "A local point interpolation method for static and dynamic analysis of thin beams", ***Computer Methods in Applied Mechanics and Engineering***, 190(42):5515-5528, (2001).

[17] Ferreira A.J.M., Roque C.M.C., Martins P.A.L.S., "Radial basis functions and higher-order shear deformation theories in the analysis of laminated composite beams and plates" ***Compos Struct***, 66:287-293, (2004).

[18] Ferreira A.J.M., Fasshauer G.E., "Computation of natural frequencies of shear deformable beams and plates by an RBF-pseudospectral method", ***Computer Methods in Applied Mechanics and Engineering***, 196:134-146, (2006).

[19] Moosavi M.R., Delfanian F., Khelil A., "The orthogonal meshless finite volume method for solving Euler–Bernoulli beam and thin plate problems", ***Finite Elements in Analysis and Design***, 49:923-932, (2011).

[20] Wu C.P., Yang S.W., Wang Y.M., Hu H.T., "A meshless collocation method for the plane problems of functionally graded material beams and plates using the DRK interpolation", ***Mechanics Research Communications***, 38:471-476, (2011).

[21] Roque C.M.C., Figaldo D.S., Ferreira A.J.M., Reddy J.N., "A study of a microstructure-dependent composite laminated Timoshenko beam using a modified couple stress theory and a meshless method", ***Compos Struct***, 96:532-537, (2013).

[22] Karamanli A., "Elastostatic analysis of two-directional functionally graded beams using various beam theories and Symmetric Smoothed Particle Hydrodynamics method", ***Compos Struct***, 160:653-669, (2017).

[23] Karamanli A., "Bending behaviour of two directional functionally graded sandwich beams by using a quasi-3d shear deformation theory", ***Compos Struct***, 160:653-669, (2017).

[24] Ferreira A.J.M., Roque C.M.C., Martins P.A.L.S., "Radial basis functions and higher order shear deformation theories in the analysis of laminated composite beams and plates", ***Compos Struct***, 66:287-293, (2004).

[25] Ferreira A.J.M., "Thick composite beam analysis using a global meshless approximation based on radial basis functions", ***Mech Adv Mater Struct***, 10:271–84, (2003).

[26] Roque C.M.C., Fidalgo D.S., Ferreira A.J.M., Reddy J.N., "A study of a microstructure dependent composite laminated Timoshenko beam using a modified couple stress theory and a meshless method", ***Compos Struct***, 96:532-537, (2013).

[27] Liew K.M., Lim H.K., Tan M.J., He X.Q., "Analysis of laminated composite beams and plates with piezoelectric patches using the element-free Galerkin method", ***Computational Mechanics***, 29:486-497, (2002).