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Results of Convergence, Stability, and Data Dependency for an Iterative Algorithm

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Abstract — In this study, we reconstruct an existing result related to the strong convergence of a recently introduced iterative algorithm by removing certain restrictions on the coefficient sequences. We then obtain some new results on the stability and data dependency of this algorithm. To validate our results, we provide a series of nontrivial complex examples, demonstrating the significance and accuracy of our theoretical contributions.

Keywords *Data dependency, fixed point, iterative algorithm, stability, strong convergence*

Mathematics Subject Classification (2020) 47H09, 47H10

1. Introduction

Fixed point theory provides a powerful tool for solving various problems encountered in fields such as engineering, economics, biology, physics and chemistry [1, 2]. Let X be a non-empty set and S a mapping from X to X. If $Su = u$, for an element u in X, then u is called a fixed point of S. Fixed point theory has been studied on various spaces, including metric spaces, finite dimensional spaces, infinite dimensional Banach spaces, and Hilbert spaces. Various theories have been developed to determine the existence and uniqueness of fixed points of a mapping. However, finding the value of a fixed point is not easy in general. To approximate the fixed point, many effective iterative algorithms have been defined and studied, such as the Mann iterative algorithm [3], Ishikawa iterative algorithm [4], two step Mann iterative algorithm [5], Suantai-Phuengrattana (SP) iterative algorithm [6]. The convergence speed, stability, and data dependency of an iterative algorithm are significant factors in determining the performance of one algorithm compared to another. There are many studies [7–12] in the literature that deal with these factors.

Chauhan et al. [13] introduced a new iterative algorithm inspired by the Karakaya et al. [14], providing better results than the Karakaya iterative algorithm in terms of convergence speed. They named this new algorithm the Surjeet-Naveen-Imdad-Asim (SNIA) iterative algorithm (Naveen et al. iterative algorithm) and proved that the iterative sequence $(\sigma_n)_n$ generated by this algorithm converges strongly to the fixed point of S if the coefficient sequences $(\alpha_n^i)_{n=1}^{\infty}$ are in $(\frac{1}{2}, 1)$, for $i \in \{1, 2, 3, 4, 5\}$, and the mapping S satisfies quasi contraction condition. We denote that the proof of Theorem 2.1 in [13] was done under the assumptions $1 - \alpha_n^2 - \alpha_n^3 \geq 0$ and $1 - \alpha_n^4 - \alpha_n^5 \geq 0$, for all $n \in \mathbb{N}$, the set of all the

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natural numbers. However, under the condition $(\frac{1}{2}, 1)$ on the coefficient sequences, $1 - \alpha_n^2 - \alpha_n^3 < 0$ and $1 - \alpha_n^4 - \alpha_n^5 < 0$, for all $n \in \mathbb{N}$.

The aim of this paper is to reconstruct the convergence result in Theorem 2.1 in [13], removing the restricting conditions $(\frac{1}{2}, 1)$ on the coefficient sequences and obtain the convergence results for some algorithms. The another aim is to prove the stability and data dependency of the SNIA iterative algorithm generated by quasi-contractive mappings. Nontrivial examples will be presented to confirm the validity and applicability of all obtained theoretical results.

2. Preliminaries

We remind the basic terminology that is connected to our study. Let (X, d) be a metric space and S a mapping from X to X. Osilike [15] considered the mapping S having a fixed point and satisfying the contractive condition:

$$\forall x_1, x_2 \in X,\ d(Sx_1, Sx_2) \leq Ld(x_1, Sx_1) + \delta d(x_1, x_2) \tag{2.1}$$

where $\delta \in [0, 1)$ and $L \geq 0$. He obtained stability results for some iterative algorithms generated with the mapping S satisfying (2.1). Imoru and Olatinwo [16] defined a more general conctractive condition than (2.1) as follows:

$$\forall x_1, x_2 \in X,\ d(Sx_1, Sx_2) \leq \varphi(d(x_1, Sx_1)) + \delta d(x_1, x_2) \tag{2.2}$$

where $\delta \in [0, 1)$ and $\varphi : R^+ \to R^+$ is monotone increasing such that $\varphi(0) = 0$. They proved some stability results using mappings satisfying (2.2). If $\varphi(x) = Lx$ is taken in (2.2), which $L \geq 0$ is a constant, then the condition (2.2) is reduced to condition (2.1). Thus, (2.2) is more general than (2.1). Bosede and Rhoades [17] made an assumption which makes all generalizations of the form (2.2) meaningless and implied by (2.1). In their assumption, S is a self mapping on a complete metric space that has a fixed point x^* and satisfies the following quasi contractive condition:

$$\forall x \in X,\ d(Sx, x^*) \leq \delta d(x, x^*) \tag{2.3}$$

where $\delta \in [0, 1)$. Bosede and Rhoades [17] obtained some stability results using mappings satisfying (2.3). It is clear that, if X is a normed space, then the quasi contractive condition (2.3) turns into

$$\forall x \in X,\ \|Sx - x^*\| \leq \delta \|x - x^*\| \tag{2.4}$$

Throughout this paper, we denote the set of all the fixed points of a mapping S by F_S.

Let C be a nonempty convex subset of a normed space E. Karakaya et al. [14] have described a three-step iterative algorithm that can be used to generate several types of iterative algorithms by choosing specific coefficient sequences as follows:

Karakaya iterative algorithm

Input: Self mapping S on C, initial point s_1, $(\alpha_n^i)_{n=1}^\infty \subset [0, 1]$, $i \in \{1, 2, 3, 4, 5\}$, such that $(\alpha_n^2 + \alpha_n^3)_{n=1}^\infty \subset [0, 1], (\alpha_n^4 + \alpha_n^5)_{n=1}^\infty \subset [0, 1]$, and $N \in \mathbb{N}$.

1: **for** $n \in \{1, 2, \cdots, N\}$ **do**

2: $p_n = (1 - \alpha_n^1)\, s_n + \alpha_n^1 S s_n$

$r_n = (1 - \alpha_n^2 - \alpha_n^3)\, p_n + \alpha_n^2 S p_n + \alpha_n^3 S s_n$

$s_{n+1} = (1 - \alpha_n^4 - \alpha_n^5)\, r_n + \alpha_n^4 S r_n + \alpha_n^5 S p_n$

3: **end for**

Output: Approximate solution s_N

Some iterative algorithms obtained by special choosing of the coefficient sequences in Karakaya iterative algorithm are given below.

If $\alpha_n^1 = 1$, for all $n \in \mathbb{N}$, and the other coefficient sequences are zero, then Karakaya iterative algorithm turns into Picard iterative algorithm. If all coefficient sequences except for $(\alpha_n^4)_n$ are zero, then it turns into Mann iterative algorithm. If $\alpha_n^5 = \alpha_n^3 = 0$ for all $n \in \mathbb{N}$, then it turns into SP iterative algorithm. If $\alpha_n^5 = \alpha_n^3 = \alpha_n^1 = 0$, for all $n \in \mathbb{N}$, then it turns into two-step Mann iterative algorithm [14].

Let E be a Banach space. SNIA iterative algorithm is defined by Chauhan et al. [13] as follows:

SNIA iterative algorithm

Input: Self mapping S on E, initial point σ_1, $(\alpha_n^i)_{n=1}^{\infty} \subset (\frac{1}{2}, 1)$, $i \in \{1,2,3,4,5\}$, and $N \in \mathbb{N}$.

1: **for** $n \in \{1, 2, \cdots, N\}$ **do**

2: $\varphi_n = S\left[\left(1-\alpha_n^1\right)\sigma_n + \alpha_n^1 S\sigma_n\right]$

$\tau_n = S\left[\left(1-\alpha_n^2-\alpha_n^3\right)\varphi_n + \alpha_n^2 S\varphi_n + \alpha_n^3 S\sigma_n\right]$

$\sigma_{n+1} = S\left[\left(1-\alpha_n^4-\alpha_n^5\right)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n\right]$

3: **end for**

Output: Approximate solution σ_N

Karakaya iterative algorithm is obtained if S is taken as the identity operator in SNIA iterative algorithm. Therefore, SNIA iterative algorithm is more general than Karakaya iterative algorithm [13].

The following definitions and lemmas are important in obtaining the findings stated in this study.

Definition 2.1. [18] Let $(a_n)_n$ be a sequence in a (X, d) metric space. The sequence $(b_n)_n \subset X$ is called the approximate sequence of the sequence $(a_n)_n$ if, for all $m \in \mathbb{N}$, there exists an $\zeta = \zeta(m)$ such that

$$\forall i \geq m, \ d(a_i, b_i) \leq \zeta$$

Lemma 2.2. [18] The sequence $(b_n)_n$ is an approximate sequence of the sequence $(a_n)_n$ if and only if there is a decreasing sequence of positive numbers $(c_n)_n$ converging to some $\eta \geq 0$ such that

$$\forall n \geq k \text{ (fixed)}, \ d(a_n, b_n) \leq c_n$$

Definition 2.3. [18] Let $S : X \to X$ be a mapping, in which (X, d) is a metric space. Let $a_{n+1} = f(S, a_n)$ be an iterative algorithm such that $(a_n)_n$ converges to the fixed point x^* of S. Let $(b_n)_n \subset X$ be an approximate sequence of $(a_n)_n$ and $\varepsilon_n := d(b_{n+1}, f(S, b_n))$, for all $n \in \mathbb{N}$. The iterative algorithm $a_{n+1} = f(S, a_n)$ is said to be weakly S-stable if

$$\lim_{n\to\infty} \varepsilon_n = 0 \ \Rightarrow \ \lim_{n\to\infty} b_n = x^*$$

Definition 2.4. [18] Let $S, \widetilde{S} : X \to X$ be two mappings, where (X, d) is a metric space. $\widetilde{S}$ is referred to as an approximate mapping for S if there exists a suitable $\varepsilon > 0$ such that $d(Sx, \widetilde{S}x) \leq \varepsilon$, for all $x \in X$.

Lemma 2.5. [19] Let $(p_n)_n$ and $(t_n)_n$ be nonnegative real number sequences and $\theta \in [0, 1)$ such that $p_{n+1} \leq \theta p_n + t_n$, for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

3. Main Results

In this section, we reconstruct the strong convergence result in [13] by removing the restriction on the coefficient sequences and provide some convergence results. We then obtain new results related to stability and data dependency for the SNIA iterative algorithm.

The following theorem is a reformulated version of Theorem 2.1 in [13], with the restriction on the coefficient sequences removed.

Theorem 3.1. Let C be a non-empty convex and closed subset of a Banach space E and $S : C \to C$ be a mapping satisfying (2.4) with $F_S \neq \emptyset$. For all $\sigma_1 \in C$, let $(\sigma_n)_n$ be a sequence generated by SNIA iterative algorithm with $(\alpha_n^i)_{n=1}^{\infty} \subset [0,1]$, $i \in \{1,2,3,4,5\}$, such that $(\alpha_n^2 + \alpha_n^3)_{n=1}^{\infty} \subset [0,1]$ and $(\alpha_n^4 + \alpha_n^5)_{n=1}^{\infty} \subset [0,1]$. Then, the sequence $(\sigma_n)_n$ converges strongly to the fixed point of S.

Proof. Assume that x^* is a fixed point of S. It can be observed from (2.4) that x^* is unique fixed point of S. Using (2.4) and $(\alpha_n^1)_n \subset [0,1]$,

$$\|\varphi_n - x^*\| \leq \delta[1 - \alpha_n^1(1-\delta)]\|\sigma_n - x^*\| \tag{3.1}$$

and by (2.4), $1 - \alpha_n^2 - \alpha_n^3 \geq 0$, $\alpha_n^2 \geq 0$, and $\alpha_n^3 \geq 0$, for all $n \in \mathbb{N}$, and $\delta < 1$,

$$\|\tau_n - x^*\| \leq \delta(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\|\varphi_n - x^*\| + \delta^2\alpha_n^3\|\sigma_n - x^*\| \tag{3.2}$$

If (3.1) is used in (3.2), then the following inequality are valid:

$$\begin{aligned} \|\tau_n - x^*\| &\leq \delta(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\delta[1 - \alpha_n^1(1-\delta)]\|\sigma_n - x^*\| + \alpha_n^3\delta^2\|\sigma_n - x^*\| \\ &= \delta^2\left[(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\left(1 - \alpha_n^1(1-\delta)\right) + \alpha_n^3\right]\|\sigma_n - x^*\| \end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned} \|\sigma_{n+1} - x^*\| &= \|S\left[\left(1 - \alpha_n^4 - \alpha_n^5\right)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n\right] - x^*\| \\ &\leq \delta(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta)\|\tau_n - x^*\| + \delta^2\alpha_n^5\|\varphi_n - x^*\| \end{aligned} \tag{3.4}$$

If (3.1) and (3.3) are used in (3.4), then

$$\begin{aligned} \|\sigma_{n+1} - x^*\| \leq &\{\delta^3(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta)\left[(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\left(1 - \alpha_n^1(1-\delta)\right) + \alpha_n^3\right] \\ &+\delta^3\alpha_n^5[1 - \alpha_n^1(1-\delta)]\}\|\sigma_n - x^*\| \end{aligned} \tag{3.5}$$

Since $1 - \alpha_n^1(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, by (3.5),

$$\begin{aligned} \|\sigma_{n+1} - x^*\| &\leq \{\delta^3\left(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta\right)\left(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta + \alpha_n^3\right) + \delta^3\alpha_n^5\}\|\sigma_n - x^*\| \\ &= \delta^3[(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta)\left(1 - \alpha_n^2 + \alpha_n^2\delta\right) + \alpha_n^5]\|\sigma_n - x^*\| \end{aligned}$$

and if $1 - \alpha_n^2(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, is used in the last inequality, then

$$\|\sigma_{n+1} - x^*\| \leq \delta^3(1 - \alpha_n^4 - \alpha_n^5 + \alpha_n^4\delta + \alpha_n^5)\|\sigma_n - x^*\| \leq \delta^3\left(1 - \alpha_n^4(1-\delta)\right)\|\sigma_n - x^*\| \tag{3.6}$$

is obtained. Using that $1 - \alpha_n^4(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, in (3.6),

$$\|\sigma_{n+1} - x^*\| \leq \delta^3\|\sigma_n - x^*\|$$

Since $\delta \in [0,1)$, by Lemma 2.5, $\sigma_n \to x^*$ as $n \to \infty$. □

Remark 3.2. Chauhan et al. stated that the main results of Karakaya [14] could be obtained by assuming $S(x) = 0$, for all $x \in C$, in Theorem 2.1 of [13]. However, if $S(x) = 0$ is taken in Theorem 2.1 of [13], then the SNIA iterative algorithm does not denote Karakaya iterative algorithm. Therefore, the main results of Karakaya [14] can not be obtained. We denote that if S is taken as the identity operator in Theorem 2.1 of [13], then the Karakaya iterative algorithm can be obtained but in this case the necessary hypotheses (2.4) on S is not provided. Thus, the main result(s) of Karakaya [14] can not be obtained from Theorem 2.1 in [13].

We observed that if the condition on S in Theorem 3 of [14] is replaced by the quasi contractive condition (2.4), then this theorem is satisfied under the same hypotheses. In the following theorem, we will consider this situation by an extra condition on the sequence $(\alpha_n^4)_n$. It means that if the sequence $(\sigma_n)_n$ in Theorem 3.1 is replaced by the sequence $(s_n)_n$ generated by Karakaya iterative algorithm, then an extra condition is required to the hypotheses in Theorem 3.1. The proof of the theorem will be done by following similar steps in the proof of Theorem 3 in [14].

Theorem 3.3. Let C be a non-empty convex and closed subset of a Banach space E and $S : C \to C$ be a mapping satisfying (2.4) with $F_S \neq \emptyset$. For all $s_1 \in C$, let $(s_n)_n$ be a sequence generated by Karakaya iterative algorithm with $(\alpha_n^i)_{n=1}^\infty \subset [0,1]$, $i \in \{1,2,3,4,5\}$, such that $(\alpha_n^2 + \alpha_n^3)_{n=1}^\infty \subset [0,1]$, $(\alpha_n^4 + \alpha_n^5)_{n=1}^\infty \subset [0,1]$, and $\sum_{n=1}^\infty \alpha_n^4 = \infty$. Then, the sequence $(s_n)_n$ converges strongly to the fixed point of S.

Proof. If similar steps in the proof of Theorem 3 in [14] are followed using (2.4), then the below inequalities are obtained, for all $n \in \mathbb{N}$:

$$\|p_n - x^*\| = \|(1-\alpha_n^1)s_n + \alpha_n^1 S s_n - x^*\| \leq [1 - \alpha_n^1(1-\delta)]\|s_n - x^*\|$$

$$\|r_n - x^*\| \leq [(1 - \alpha_n^2(1-\delta) - \alpha_n^3)(1 - \alpha_n^1(1-\delta)) + \delta\alpha_n^3]\|s_n - x^*\|$$

and

$$\begin{aligned}\|s_{n+1} - x^*\| &\leq \{(1 - \alpha_n^4(1-\delta) - \alpha_n^5)\left[(1 - \alpha_n^2 - \alpha_n^3 + \alpha_n^2\delta)\left(1 - \alpha_n^1(1-\delta)\right) + \delta\alpha_n^3\right] \\ &\Rightarrow +\delta\alpha_n^5(1 - \alpha_n^1(1-\delta))\}\|s_n - x^*\| \\ &\leq \{(1 - \alpha_n^4(1-\delta) - \alpha_n^5) + \delta\alpha_n^5(1 - \alpha_n^1(1-\delta))\}\|s_n - x^*\| \\ &\leq [1 - \alpha_n^4(1-\delta)]\|s_n - x^*\|\end{aligned} \tag{3.7}$$

Thus, by using induction,

$$\|s_{n+1} - x^*\| \leq \|s_1 - x^*\| \prod_{k=1}^n \{(1 - \alpha_k^4(1-\delta)\}$$

It is well known that $1 - t \leq e^{-t}$, for all $t \in [0,1]$. Therefore,

$$\|s_{n+1} - x^*\| \leq \|s_1 - x^*\| e^{-(1-\delta)\sum_{i=1}^n \alpha_i^4} \tag{3.8}$$

By using the condition $\sum\limits_{i=1}^{\infty} \alpha_i^4 = \infty$ in (3.8), we obtain $\|s_{n+1} - x^*\| \to 0$ as $n \to \infty$. Thus, the proof is completed. □

Remark 3.4. We observe that the condition $\sum\limits_{n=1}^{\infty} \alpha_n^4 = \infty$ can be replaced by $\sum\limits_{n=1}^{\infty} \alpha_n^5 = \infty$ in Theorem 3.3. In this case, the proof of Theorem 3.3 is followed by rearranging the inequality in (3.7) as follows:

$$\|s_{n+1} - x^*\| \leq [1 - \alpha_k^5(1-\delta)]\|s_n - x^*\|$$

Corollary 3.5. Assume that all the hypotheses in Theorem 3.3 are satisfied. Then, we get the following results, which possibly existing in the literature.

i. Mann iterative algorithm generated by S satisfying quasi contraction condition (2.4) converges strongly to the fixed point of S if taken $\alpha_n^5 = \alpha_n^2 = \alpha_n^3 = \alpha_n^1 = 0$, for all $n \in \mathbb{N}$, in Theorem 3.3.

ii. SP iterative algorithm generated by S satisfying quasi contraction condition (2.4) converges strongly to the fixed point of S if taken $\alpha_n^5 = \alpha_n^3 = 0$, for all $n \in \mathbb{N}$, in Theorem 3.3.

iii. Two-step Mann iterative algorithm generated by S satisfying quasi contraction condition (2.4) converges strongly to the fixed point of S if taken $\alpha_n^5 = \alpha_n^3 = \alpha_n^1 = 0$, for all $n \in \mathbb{N}$, in Theorem 3.3.

3.1. Stability Results

An iterative algorithm that converges to a unique fixed point is stable if the numerical errors that occur in each step have no effect on the convergence of algorithm. In this part, we show the stability of SNIA iterative algorithm for quasi contractive mappings.

Theorem 3.6. Let C be a non-empty convex and closed subset of a Banach space E and $S : C \to C$ be a mapping satisfying (2.4) with $F_S \neq \emptyset$ and $\sigma_1,\ c_1 \in C$. Let $(\sigma_n)_n$ be a sequence generated by SNIA iterative algorithm with $(\alpha_n^i)_{n=1}^\infty \subset [0,1]$, $i \in \{1,2,3,4,5\}$, such that $(\alpha_n^2 + \alpha_n^3)_{n=1}^\infty \subset [0,1]$, $(\alpha_n^4 + \alpha_n^5)_{n=1}^\infty \subset [0,1]$, and $(y_n)_{n=1}^\infty \subset C$ be an approximate sequence of $(\sigma_n)_n$. Define a sequence $(\varepsilon_n)_{n=1}^\infty \subset R^+$ by

$$v_n = S\left[(1-\alpha_n^1)y_n + \alpha_n^1 S y_n\right]$$

$$u_n = S\left[\left(1-\alpha_n^2-\alpha_n^3\right)v_n + \alpha_n^2 S v_n + \alpha_n^3 S y_n\right]$$

$$f(S,y_n) = S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right]$$

and

$$\varepsilon_n = \|y_{n+1} - f(S,y_n)\|,\ n \in \mathbb{N}$$

Then, $\lim\limits_{n\to\infty} \varepsilon_n = 0$ implies $\lim\limits_{n\to\infty} y_n = x^*$. In other words, SNIA iterative algorithm is weakly S-stable.

Proof. By Theorem 3.1, the sequence $(\sigma_n)_n$ generated by SNIA iterative algorithm converges the fixed point x^* of S. Assume that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. We will prove that $\lim\limits_{n\to\infty} y_n = x^*$.

$$\begin{aligned}\|y_{n+1} - x^*\| &\leq \|y_{n+1} - S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right]\| \\ &\quad + \|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\| + \|\sigma_{n+1} - x^*\| \\ &= \varepsilon_n + \|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\| + \|\sigma_{n+1} - x^*\|\end{aligned} \tag{3.9}$$

By (2.4),

$$\|Sx - Sy\| \leq \delta\|x - x^*\| + \delta\|y - x^*\|,\ \text{for all } x,y \in C \tag{3.10}$$

If (3.10), (2.4), and the definition of SNIA iterative algorithm are used and operations are continued as in Theorem 3.1, then

$$\begin{aligned}\|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\| &\leq \left[\delta(1-\alpha_n^4-\alpha_n^5) + \delta^2\alpha_n^4\right]\left[\|u_n - x^*\| + \|\tau_n - x^*\|\right] \\ &\quad + \delta^2\alpha_n^5\left[\|v_n - x^*\| + \|\varphi_n - x^*\|\right] \\ &\leq \delta^2\left[(1-\alpha_n^2(1-\delta)-\alpha_n^3)(1-\alpha_n^1(1-\delta)) + \alpha_n^3\right] \\ &\quad \times \left[\delta\left(1-\alpha_n^4(1-\delta)-\alpha_n^5\right)\right]\left[\|y_n - x^*\| + \|\sigma_n - x^*\|\right] \\ &\quad + \delta^3\alpha_n^5\left(1-\alpha_n^1(1-\delta)\right)\left[\|y_n - x^*\| + \|\sigma_n - x^*\|\right]\end{aligned} \tag{3.11}$$

Using $\delta < 1$ and $1-\alpha_n^1(1-\delta) \leq 1$, $1-\alpha_n^2(1-\delta) \leq 1$, and $1-\alpha_n^4(1-\delta) \leq 1$, for all $n \in \mathbb{N}$, in (3.11),

$$\left\|S\left[\left(1-\alpha_n^4-\alpha_n^5\right)u_n + \alpha_n^4 S u_n + \alpha_n^5 S v_n\right] - \sigma_{n+1}\right\| \leq \delta\left[\|y_n - x^*\| + \|\sigma_n - x^*\|\right] \tag{3.12}$$

If (3.12) is used in (3.9), then it is obtained

$$\|y_{n+1} - x^*\| \leq \delta\|y_n - x^*\| + \varepsilon_n + \delta\|\sigma_n - x^*\| + \|\sigma_{n+1} - x^*\|$$

Let $t_n := \varepsilon_n + \delta\|\sigma_n - x^*\| + \|\sigma_{n+1} - x^*\|$. By hypotheses, $\lim_{n\to\infty} t_n = 0$. Thus, by Lemma 2.5, $\lim_{n\to\infty} y_n = x^*$. This completes the proof. □

3.2. Data Dependency Results

In this part, we give a result regarding the data dependency of SNIA iterative algorithm for mappings satisfying quasi contractive condition (2.4).

Theorem 3.7. Let E, C, and S be as in Theorem 3.1. Let $\widetilde{S}$ be an approximate mapping of S as in Definition 2.4 with a suitable error ε. Let $(\sigma_n)_n$ be the sequence generated by SNIA iterative algorithm and let the sequence $(\widetilde{\sigma}_n)_n$ be as follows:

$$\begin{aligned}
&\widetilde{\sigma}_1 \in C \\
&\widetilde{\varphi}_n = \widetilde{S}[(1-\alpha_n^1)\,\widetilde{\sigma}_n + \alpha_n^1 \widetilde{S}\widetilde{\sigma}_n] \\
&\widetilde{\tau}_n = \widetilde{S}[(1-\alpha_n^2-\alpha_n^3)\,\widetilde{\varphi}_n + \alpha_n^2 \widetilde{S}\widetilde{\varphi}_n + \alpha_n^3 \widetilde{S}\widetilde{\sigma}_n] \\
&\widetilde{\sigma}_{n+1} = \widetilde{S}[(1-\alpha_n^4-\alpha_n^5)\,\widetilde{\tau}_n + \alpha_n^4 \widetilde{S}\widetilde{\tau}_n + \alpha_n^5 \widetilde{S}\widetilde{\varphi}_n], \quad n\in\mathbb{N}
\end{aligned} \tag{3.13}$$

where $(\alpha_n^i)_{n=1}^{\infty} \subset [0,1]$, for $i \in \{1,2,3,4,5\}$, are sequences satisfying the conditions in Theorem 3.1. If $Sx^* = x^*$ and $\widetilde{S}\widetilde{x}^* = \widetilde{x}^*$ such that $\lim_{n\to\infty} \widetilde{\sigma}_n = \widetilde{x}^*$, then

$$\|x^* - \widetilde{x}^*\| \le \frac{1+\delta}{1-\delta}\,\varepsilon$$

Proof. By Definition 2.4 and (2.4), the mapping S satisfies the below inequality, for all $x, \widetilde{x} \in C$:

$$\|Sx - \widetilde{S}\widetilde{x}\| \le \|Sx - x^*\| + \|S\widetilde{x} - x^*\| + \varepsilon \le 2\delta\|x - x^*\| + \delta\|x - \widetilde{x}\| + \varepsilon \tag{3.14}$$

By the definition of SNIA iterative algorithm, (3.13), and (3.14),

$$\begin{aligned}
\|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| \le{}& \delta \left\| \left[\left(1-\alpha_n^4-\alpha_n^5\right)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n \right] - \left(1-\alpha_n^4-\alpha_n^5\right)\widetilde{\tau}_n - \alpha_n^4\widetilde{S}\widetilde{\tau}_n - \alpha_n^5\widetilde{S}\widetilde{\varphi}_n] \right\| \\
&+ 2\delta \left\| \left[\left(1-\alpha_n^4-\alpha_n^5\right)\tau_n + \alpha_n^4 S\tau_n + \alpha_n^5 S\varphi_n \right] - x^* \right\| + \varepsilon
\end{aligned}$$

and, by using (3.14) and (2.4),

$$\begin{aligned}
\|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| \le{}& \delta\left(1-\alpha_n^4-\alpha_n^5\right)\|\tau_n - \widetilde{\tau}_n\| + \delta\alpha_n^4\|S\tau_n - \widetilde{S}\widetilde{\tau}_n\| + +\delta\alpha_n^5\|S\varphi_n - \widetilde{S}\widetilde{\varphi}_n\| \\
&+ 2\delta\left(1-\alpha_n^4-\alpha_n^5\right)\|\tau_n - x^*\| + 2\delta\alpha_n^4\|S\tau_n - x^*\| + 2\delta\alpha_n^5\|S\varphi_n - x^*\| + \varepsilon \\
\le{}& \delta\left(1-\alpha_n^4-\alpha_n^5\right)\|\tau_n - \widetilde{\tau}_n\| + \delta^2\alpha_n^4\|\tau_n - \widetilde{\tau}_n\| + 2\delta^2\alpha_n^4\|\tau_n - x^*\| + \delta\alpha_n^4\varepsilon \\
&+ \delta^2\alpha_n^5\|\varphi_n - \widetilde{\varphi}_n\| + 2\delta^2\alpha_n^5\|\varphi_n - x^*\| + \delta\alpha_n^5\varepsilon + 2\delta\left(1-\alpha_n^4-\alpha_n^5\right)\|\tau_n - x^*\| \\
&+ 2\delta^2\alpha_n^4\|\tau_n - x^*\| + 2\delta^2\alpha_n^5\|\varphi_n - x^*\| + \varepsilon
\end{aligned}$$

By arranging the last inequality,

$$\begin{aligned}
\|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| \le{}& \{\delta\left(1-\alpha_n^4-\alpha_n^5\right) + \delta^2\alpha_n^4\}\|\tau_n - \widetilde{\tau}_n\| + \delta^2\alpha_n^5\|\varphi_n - \widetilde{\varphi}_n\| \\
&+2\delta\{1-\alpha_n^4-\alpha_n^5+2\delta\alpha_n^4\}\|\tau_n - x^*\| + 4\delta^2\alpha_n^5\|\varphi_n - x^*\| + \delta\alpha_n^4\varepsilon + \delta\alpha_n^5\varepsilon + \varepsilon
\end{aligned} \tag{3.15}$$

By following similar steps above,

$$\begin{aligned}
\|\tau_n - \widetilde{\tau}_n\| \le{}& \{\delta\left(1-\alpha_n^2-\alpha_n^3\right) + \delta^2\alpha_n^2\}\|\varphi_n - \widetilde{\varphi}_n\| + \delta^2\alpha_n^3\|\sigma_n - \widetilde{\sigma}_n\| \\
&+2\delta\{1-\alpha_n^2-\alpha_n^3+2\delta\alpha_n^2\}\|\varphi_n - x^*\| + 4\delta^2\alpha_n^3\|\sigma_n - x^*\| + \delta\alpha_n^2\varepsilon + \delta\alpha_n^3\varepsilon + \varepsilon
\end{aligned} \tag{3.16}$$

and

$$\|\varphi_n - \widetilde{\varphi}_n\| \le \delta\left\{1-\alpha_n^1(1-\delta)\right\}\|\sigma_n - \widetilde{\sigma}_n\| + 2\delta\{1-\alpha_n^1+2\delta\alpha_n^1\}\|\sigma_n - x^*\| + \delta\alpha_n^1\varepsilon + \varepsilon \tag{3.17}$$

If (3.16) and (3.17) are used in (3.15), we obtain the following inequality:

$$\|\sigma_{n+1} - \widetilde{\sigma}_{n+1}\| \leq A\|\sigma_n - \widetilde{\sigma}_n\| + B\|\sigma_n - x^*\| + C\|\varphi_n - x^*\| + D\|\tau_n - x^*\| + E \tag{3.18}$$

where

$$\begin{aligned} A :=& \; \delta(1-\alpha_n^1(1-\delta))\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &+\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\} \\ B :=& \; 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &+4\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\} \end{aligned}$$

$$C := 2\delta(1-\alpha_n^2-\alpha_n^3+2\delta\alpha_n^2)[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4]+4\delta^2\alpha_n^5$$

$$D := 2\delta(1-\alpha_n^4-\alpha_n^5+2\delta\alpha_n^4)$$

and

$$\begin{aligned} E :=& \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon) \\ &+[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta\alpha_n^2\varepsilon+\delta\alpha_n^3\varepsilon+\varepsilon]+\delta\alpha_n^4\varepsilon+\delta\alpha_n^5\varepsilon+\varepsilon \end{aligned}$$

Arrange the number A,

$$\begin{aligned} A =& \left[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\right]\left\{\left[\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2\right]\left[\delta(1-\alpha_n^1(1-\delta))\right]+\delta^2\alpha_n^3\right\} \\ &+\delta^3\alpha_n^5(1-\alpha_n^1(1-\delta)) \end{aligned}$$

Since $\delta \in [0,1)$ and $1-\alpha_n^1(1-\delta) \leq 1$, for all $n \in \mathbb{N}$,

$$\begin{aligned} A \leq& \; [\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4]\left\{\delta(1-\alpha_n^3)\delta(1-\alpha_n^1(1-\delta))+\delta^2\alpha_n^3\right\}+\delta^3\alpha_n^5 \\ \leq& \; [\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4]\left\{\delta(1-\alpha_n^3)\delta+\delta^2\alpha_n^3\right\}+\delta^3\alpha_n^5 \\ \leq& \; [\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4]\,\delta^2+\delta^3\alpha_n^5=\delta^3 \end{aligned} \tag{3.19}$$

Since $\delta \in [0,1)$, and $1-\alpha_n^3 \leq 1$ and $1-\alpha_n^5 \leq 1$, for all $n \in \mathbb{N}$,

$$\begin{aligned} B =& \; 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &+4\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4\} \\ \leq& \; 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta\alpha_n^2]+\delta^2\alpha_n^5\right\} \\ &+4\delta^2\alpha_n^3\{\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4\} \\ =& \; 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{\delta(1-\alpha_n^5)\delta(1-\alpha_n^3)+\delta^2\alpha_n^5\right\}+4\delta^3\alpha_n^3(1-\alpha_n^5) \\ \leq& \; 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)\left\{\delta^2+\delta^2\alpha_n^5\right\}+4\delta^3\alpha_n^3 \end{aligned} \tag{3.20}$$

Using $1-\alpha_n^1 \leq 1$, $\alpha_n^5 \leq 1$ and $\alpha_n^3 \leq 1$, for all $n \in \mathbb{N}$ in (3.20),

$$B \leq 2\delta(1-\alpha_n^1+2\delta\alpha_n^1)2\delta^2+4\delta^3 \leq 2\delta(1+2\delta)2\delta^2+4\delta^3 = 8\delta^3(1+\delta) \tag{3.21}$$

Since $1-\alpha_n^2-\alpha_n^3 \leq 1$, $\alpha_n^2 \leq 1$, $\delta < 1$, and $\alpha_n^5 \leq 1$, $1-\alpha_n^5 \leq 1$, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} C =& \; 2\delta(1-\alpha_n^2-\alpha_n^3+2\delta\alpha_n^2)[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4]+4\delta^2\alpha_n^5 \\ \leq& \; 2\delta(1+2\delta)[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4]+4\delta^2\alpha_n^5 \leq 2\delta(1+2\delta)\delta(1-\alpha_n^5)+4\delta^2 \leq 2\delta^2(3+2\delta) \end{aligned} \tag{3.22}$$

Since $1-\alpha_n^4-\alpha_n^5 \leq 1$ and $\alpha_n^4 \leq 1$, for all $n \in \mathbb{N}$,

$$D = 2\delta(1-\alpha_n^4-\alpha_n^5+2\delta\alpha_n^4) \leq 2\delta(1+2\delta) \tag{3.23}$$

Using $\delta \in [0,1)$ and $\alpha_n^2 + \alpha_n^3 \leq 1$ and $\alpha_n^4 + \alpha_n^5 \leq 1$, for all $n \in \mathbb{N}$,

$$\begin{aligned}
E &= \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^2-\alpha_n^3)+\delta^2\alpha_n^2]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon) \\
&\quad + [\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta\alpha_n^2\varepsilon+\delta\alpha_n^3\varepsilon+\varepsilon]+\delta\alpha_n^4\varepsilon+\delta\alpha_n^5\varepsilon+\varepsilon \\
&\leq \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta(1-\alpha_n^3)]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon) \\
&\quad + [\delta(1-\alpha_n^4-\alpha_n^5)+\delta^2\alpha_n^4][\delta\varepsilon(\alpha_n^2+\alpha_n^3)+\varepsilon]+\delta\varepsilon(\alpha_n^4+\alpha_n^5)+\varepsilon \\
&\leq \left\{[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4][\delta(1-\alpha_n^3)]+\delta^2\alpha_n^5\right\}(\delta\alpha_n^1\varepsilon+\varepsilon)+[\delta(1-\alpha_n^4-\alpha_n^5)+\delta\alpha_n^4][\delta\varepsilon+\varepsilon]+\delta\varepsilon+\varepsilon \\
&= \left\{\delta(1-\alpha_n^5)(1-\alpha_n^3)+\delta\alpha_n^5\right\}\delta\varepsilon(\delta\alpha_n^1+1)+\delta(1-\alpha_n^5)[\delta\varepsilon+\varepsilon]+\delta\varepsilon+\varepsilon
\end{aligned}$$

Since $\delta \in [0,1)$ and $1-\alpha_n^3 \leq 1$, $1-\alpha_n^5 \leq 1$, and $\alpha_n^1 \leq 1$, for all $n \in \mathbb{N}$, we get the following inequality for the number E:

$$\begin{aligned}
E &\leq \left\{\delta(1-\alpha_n^5)+\delta\alpha_n^5\right\}\delta\varepsilon(\delta\alpha_n^1+1)+(\delta\varepsilon+\varepsilon)[\delta(1-\alpha_n^5)+1] \\
&= \delta^2\varepsilon(\delta\alpha_n^1+1)+(\delta\varepsilon+\varepsilon)[\delta(1-\alpha_n^5)+1] \\
&\leq \delta^2\varepsilon(\delta+1)+(\delta\varepsilon+\varepsilon)(\delta+1)=\varepsilon(\delta+1)(\delta^2+\delta+1)
\end{aligned} \tag{3.24}$$

Therefore, using (3.19) and (3.21)-(3.24) in (3.18),

$$\begin{aligned}
\|\sigma_{n+1}-\widetilde{\sigma}_{n+1}\| \leq\ & \delta^3\|\sigma_n-\widetilde{\sigma}_n\|+8\delta^3(1+\delta)\|\sigma_n-x^*\|+2\delta^2(3+2\delta)\|\varphi_n-x^*\| \\
&+2\delta(1+2\delta)\|\tau_n-x^*\|+\varepsilon(\delta+1)(\delta^2+\delta+1)
\end{aligned} \tag{3.25}$$

By (3.1) and (3.3),

$$\|\varphi_n-x^*\| \leq \|\sigma_n-x^*\| \quad \text{and} \quad \|\tau_n-x^*\| \leq \|\sigma_n-x^*\|$$

Besides, under hypotheses, by Theorem 3.1, since $\lim\limits_{n\to\infty}\|\sigma_n-x^*\|=0$,

$$\lim_{n\to\infty}\|\varphi_n-x^*\|=\lim_{n\to\infty}\|\tau_n-x^*\|=0$$

Thus, taking the limit for $n\to\infty$ in (3.25),

$$\|x^*-\widetilde{x}^*\| \leq \frac{\varepsilon(\delta+1)(\delta^2+\delta+1)}{1-\delta^3}=\frac{1+\delta}{1-\delta}\varepsilon$$

□

4. Numerical Examples

In this section, we provide some numerical examples that support our theoretical results.

The first example, built on an infinite dimensional Banach space and satisfying the conditions of Theorem 3.1 and Theorem 3.3, shows that the SNIA iterative algorithm is more effective than Karakaya, SP, and two-step Mann iterative algorithms in terms of convergence.

Example 4.1. Let E be the Banach space $l_1 = \{(x_i)_{i=1}^{\infty} \subset \mathbb{K} : \sum_{i=1}^{\infty}|x_i| < \infty\}$ endowed with norm $\|(x_i)_i\|_1 = \sum\limits_{i=1}^{\infty}|x_i|$ and be defined a sequence $(x_i)_i$ as follows:

$$\forall i \in \mathbb{N}, \quad x_i = (x_n^i)_{n=1}^{\infty}, \quad x_n^i = \begin{cases} 0, n \neq i \\ \dfrac{1}{i}, n = i \end{cases}$$

It is clear that $(x_i)_i$ is a sequence in E. Moreover, $\lim_{i\to\infty}\|x_i - 0\|_1 = 0$. We define the set

$$C := \left\{ \sum_{k=1}^{\infty} \mu_k x_k : (\mu_k)_{k=1}^{\infty} \in B_{l_1} \right\}$$

where B_{l_1} is the closed unit ball of l_1. Since $(x_n)_n$ is a null sequence in E, it is well known in the literature that C is a convex and closed subset in E [20, 21]. Moreover, by Grothendieck's characterization [22], we can say that C is a proper subset of B_E. Using the above definition of the sequence $(x_i)_i$, we get the set C as follows:

$$C = \left\{ \left(\frac{\mu_k}{k}\right)_{k=1}^{\infty} : (\mu_k)_{k=1}^{\infty} \in B_{l_1} \right\}$$

We define a mapping $S : C \to C$ by

$$S\left(\left(\frac{\mu_k}{k}\right)_{k=1}^{\infty}\right) := \left(\frac{k}{4}\left(\frac{\mu_k}{k}\right)^2\right)_{k=1}^{\infty}$$

It can be observed that the mapping S is well defined and S has a unique fixed point $x^* = (0, 0, 0, 0, \cdots)$. We show that there exist a number $\delta \in [0, 1)$ such that $\|Sx - x^*\|_1 \leq \delta\|x - x^*\|_1$, for all $x \in C$. If $x \in C$, then there is a $(\mu_k)_{k=1}^{\infty} \in B_{l_1}$ such that $x = \left(\frac{\mu_k}{k}\right)_{k=1}^{\infty}$. Thus,

$$\|Sx - x^*\|_1 = \left\| \left(\frac{1}{4}\frac{\mu_k^2}{k}\right)_k \right\|_1 = \frac{1}{4}\sum_{k=1}^{\infty}\frac{|\mu_k^2|}{k} \leq \frac{1}{4}\sum_{k=1}^{\infty}\frac{|\mu_k|}{k} = \frac{1}{4}\|x - x^*\|_1$$

This shows that $\delta = \frac{1}{4}$. That is, S satisfies quasi contractive condition (2.4). However, we denote that for all $x, y \in C$, $\|Sx - Sy\|_1 \not\leq \frac{1}{4}\|x - y\|_1$. For example, for $x = (1, 0, 0, 0, \cdots)$ and $y = \left(\frac{1}{2}, 0, 0, 0, \cdots\right)$, $\|Sx - Sy\|_1 \not\leq \frac{1}{4}\|x - y\|_1$. Let the initial terms of all mentioned algorithms be $s_0 = \sigma_0 = \left(\frac{1}{n2^n}\right)_n$, $\alpha_n^4 = \alpha_n^2 = \alpha_n^1 = 1 - \frac{1}{n^5 + 1}$, and $\alpha_n^5 = \alpha_n^3 = \frac{1}{2(n^5 + 1)}$, for all $n \in \mathbb{N}$, satisfying $(\alpha_n^4 + \alpha_n^5)_n \subset [0, 1]$ and $(\alpha_n^2 + \alpha_n^3)_n \subset [0, 1]$. Figure 1 manifests that the sequence generated by SNIA iterative algorithm converges the fixed point $x^* = 0$ of S faster than the sequences generated by Karakaya, Mann, SP, and two-step Mann iterative algorithms.

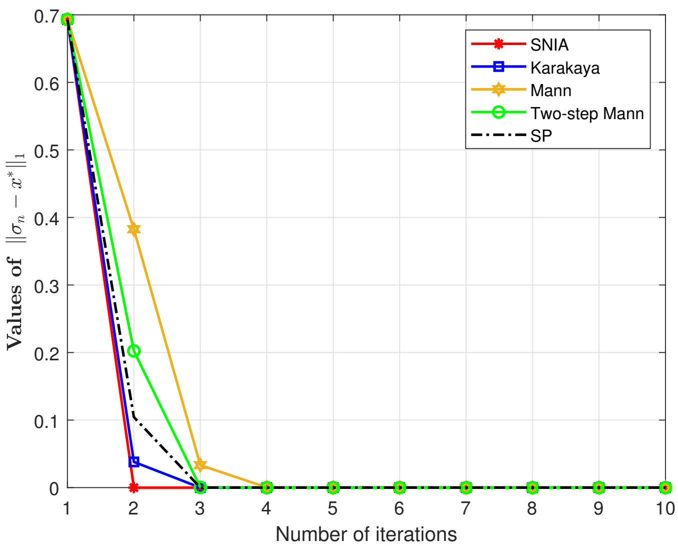


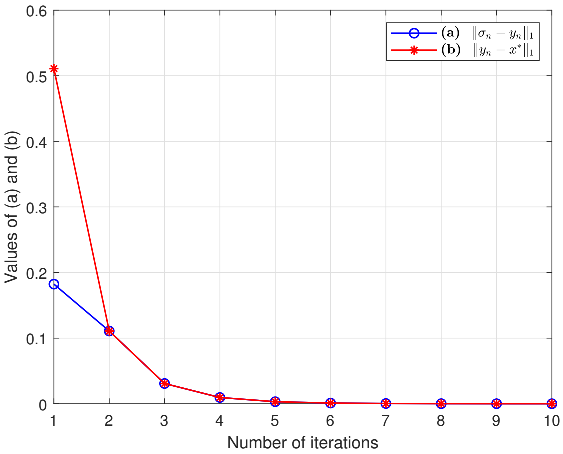
Figure 1. Convergence behaviors of algorithms in Example 4.1

The following example, which supports the accuracy of the result in Theorem 3.6 shows that SNIA iterative algorithm in Example 4.1 is weakly S-stable.

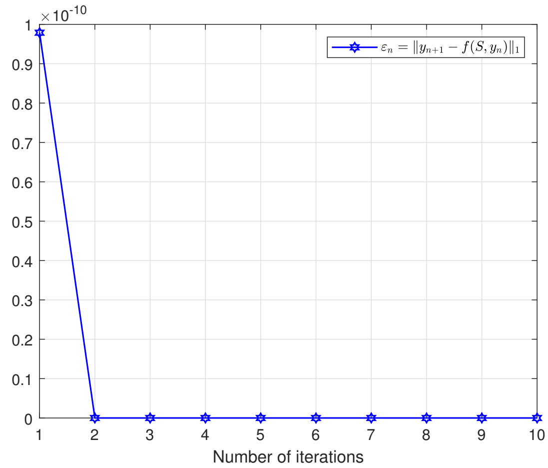
Example 4.2. Let E, C, and S be as in Example 4.1. We define the sequence $(y_n)_n$ in C as follows:

$$\forall n \in \mathbb{N}, \quad y_n = (y_i^n)_{i=1}^{\infty}, \quad y_i^n = \begin{cases} 0, i < n \\ \dfrac{2^i}{i5^i}, i \geq n \end{cases}$$

Figure 2 (a) shows that the $(y_n)_n$ is an approximate sequence of the sequence $(\sigma_n)_n$ generated by SNIA iterative algorithm. Further, Figure 2 (a)-(b) manifests that $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = x^*$. In other words, SNIA iterative algorithm is weakly S-stable.



(a)



(b)

Figure 2. Graphs showing the convergence states of the sequences $(y_n - \sigma_n)_n$, $(y_n - x^*)_n$, and $(\varepsilon_n)_n$

The following example deals with the data dependency of the sequence $(\sigma_n)_n$ generated by SNIA iterative algorithm in Example 4.1.

Example 4.3. Let E, C, and S be as in Example 4.1. We define a mapping $\widetilde{S} : C \to C$ as in the following:

$$\widetilde{S}\left(\left(\frac{\alpha_k}{k}\right)_k\right) := (\beta_k)_k, \quad \beta_k = \begin{cases} 1/4, & k = 1 \\ \dfrac{\alpha_{k-1}}{k3^{k-1}}, & k \geq 2 \end{cases}$$

where $(\alpha_k)_{k=1}^{\infty} \in B_E$. Then, $\widetilde{S} : C \to C$ is well defined, and for all $x = \left(\frac{\alpha_k}{k}\right)_k \in C$,

$$\begin{aligned} \|Sx - \widetilde{S}x\|_1 &= \frac{1}{4}\left|\alpha_1^2 - 1\right| + \sum_{k=2}^{\infty} \frac{1}{k}\left|\frac{\alpha_k^2}{4} - \frac{\alpha_{k-1}}{3^{k-1}}\right|, \quad ((\alpha_k)_k \in B_{l_1}) \\ &\leq \frac{1}{4} + \frac{1}{8}\sum_{k=2}^{\infty}|\alpha_k^2| + \frac{1}{6}\sum_{k=2}^{\infty}|\alpha_{k-1}|, \quad ((\alpha_k)_k \in B_{l_1}) \\ &\leq \frac{1}{4} + \frac{1}{8} + \frac{1}{6} = 0.5416666 = \varepsilon \end{aligned}$$

Thus, we can consider the mappings S and $\widetilde{S}$ as approximate operators in Definition 2.4. If $\widetilde{S}$ has a fixed point $\widetilde{x}^*$ and the sequence $(\widetilde{\sigma}_n)_n$ generated by (3.13) with the choice of the coefficient sequences satisfying the conditions in Theorem 3.7, converges to $\widetilde{x}^*$, then without knowing and calculating $\widetilde{x}^*$, we can determine an upper bound for $\widetilde{x}^*$ by (3.7) as follows:

$$\|x^* - \widetilde{x}^*\| \leq \frac{1+\delta}{1-\delta}\varepsilon = \frac{1+1/4}{1-1/4}(0.5416666) = 0.902730$$

We get that the fixed point of $\widetilde{S}$ as $\widetilde{x}^* = \left(\frac{1}{3^{\frac{k(k-1)}{2}} 4k}\right)_k$. Figure 3 shows that the sequence $(\widetilde{\sigma}_n)_n$ generated by (3.13) converges to $\widetilde{x}^*$. In addition, $\|x^* - \widetilde{x}^*\| = \frac{1177}{3992} = 0.2948$. That is, (3.7) is satisfied.

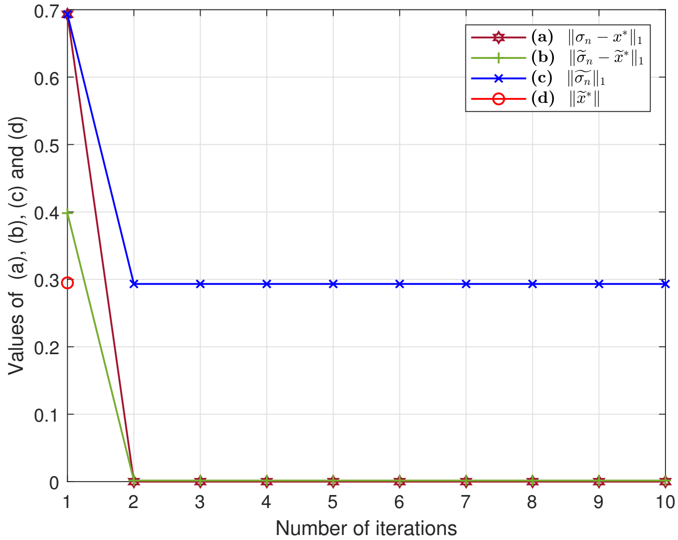


Figure 3. Graphs showing the values of $\|\sigma_n - \widetilde{x}^*\|_1$, $\|\widetilde{\sigma}_n - \widetilde{x}^*\|_1$, $\|\widetilde{\sigma}_n\|_1$, and $\|\widetilde{x}^*\|_1$, for $n \in \{1, 2, ..., 10\}$

5. Conclusion

In this study, the convergence result of the SNIA iterative algorithm introduced by Chauhan et al. [13] has been revised and improved while simultaneously obtaining its weak stability and data dependency. The findings of this study are substantiated by nontrivial examples in an infinite dimensional Banach space, thereby bridging the gap between practice and theory. Based on the graphs presented, it has been observed that the algorithm yields superior results in numerical examples. Furthermore, the algorithm's convergence, which does not necessitate additional conditions (except for convexity) on coefficient sequences, sets it apart from the aforementioned algorithms. Consequently, it can be concluded that the algorithm with the stability and data dependency properties is more effective for quasi-contractive mappings when compared to the algorithms discussed in this study, based on both theoretical and practical outcomes. In future studies, researchers can examine the convergence of the SNIA iterative algorithm for different mapping classes under appropriate conditions. Moreover, they can compare the algorithm's performance speed with existing algorithms in the literature for these mapping classes.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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