Celal Bayar University Journal of Science

Differential Equations for Spacelike Curves According to Light-Cone Frame in L_0^3

Tanju Kahraman[1]*

[1] Department of Mathematics, Faculty of Engineering and Naturel Sciences
Manisa Celal Bayar University, Şehit Prof. Dr. İlhan Varank Campus, Manisa, Türkiye
* tanju.kahraman@cbu.edu.tr
* Orcid No: 0000-0002-0653-712X

Received: 26 August 2024
Accepted: 3 December 2024
DOI: 10.18466/cbayarfbe.1538669

Abstract

In this study, we obtain differential equations of spacelike curves according to components of light-cone frame in light cone L_0^3 in Minkowski 5-space. We find some relations between curvatures of the spacelike curves.

Keywords: Differential equations, Spacelike curves, Light-cone frame, Minkowski 5-space.

1. Introduction

To study differential geometry of curves, we know that is required moving frames along curves. By helping of the frames, many mathematicians are studied differential geometry of curves in many spaces, especially such as associated curves and quaternionic curves [4,6,7].

As known, one of the popular studies of differential geometry and physics is curves on light cones and semi-Riemanian manifolds. However, since the metric is degenerate on light cone in Minkowski spaces. In the light of [1,2,3,5], a new type of frame has established by Wang and He and is called as light-cone frame, then, using the light-cone frame, they have been studied singularities of hypersurfaces along spacelike curves in light cone in Minkowski 5-space [12].

For theory of general relativity, one of the right constructions is five-dimensional space [10]. Einstein and Bergmann said that "… It is much more satisfactory to introduce the fifth dimension not only formally, but to assign it some physical meaning. Nevertheless, there is no contradiction with the empirical four-dimensional character of physical space" [10]. Therefore, studies in five dimensional spaces continue to increase recently.

Dannon showed that spherical curves in can be constructed by Frenet formulae. Then, integral characterizations of spherical curves in were given by her [9]. Kazaz et al. gave similar characterizations of timelike and spacelike spherical curves lying on Lorentzian sphere in the [6]. They have found the differential equation systems characterizing the spherical curves in. Sezer has found the differential equations and integral characterizations of spherical curves in by using differential equation system given by Dannon [8].

In this paper, we study differential equations of spacelike curves in light cone in Minkowski 5-space. Besides, we have relations of differential equations of the curves for some special conditions.

2. Materials and Methods

Minkowski 5-space IR_1^5 is provided with the standart lorentz metric given by

$$\langle\,,\rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2$$

where $(x_1, x_2, x_3, x_4, x_5)$ is a rectangular coordinate system of IR_1^5. According to this metric, an arbitrary vector $\vec{v} = (v_1, v_2, v_3, v_4, v_5)$ in IR_1^5 have one of three Lorentzian causal characters; it can be spacelike if $\langle\vec{v},\vec{v}\rangle > 0$ or $\vec{v} = 0$, timelike if $\langle\vec{v},\vec{v}\rangle < 0$ and null(lightlike) if $\langle\vec{v},\vec{v}\rangle = 0$ and $\vec{v} \neq 0$Similarly, an arbitrary curve $\gamma = \gamma(s)$ can be spacelike, timelike or null (lightlike), if all its velocity vectors $\gamma'(s)$ are spacelike, timelike or null (lightlike), respectively. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively.

In the Minkowski 5-space, there exists the pseudo-inner product $\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5$ of any vectors $x = (x_1\, x_2, x_3, x_4, x_5)$ and $y = (y_1, y_2, y_3, y_4, y_5)$, and the pseudo vector product is defined by

$$x \wedge y \wedge z \wedge w = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{vmatrix},$$

where vectors x, y, z, w are in Minkowski 5-space and $(e_1, e_2, e_3, e_4, e_5)$ is the canonical basis of Minkowski 5-space [5,12].

A pseudo sphere whose vertex is at origin is defined as follows;

$S_1^4 = \{x \in IR_1^5 : -x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1\}$.

The four-dimensional open light cone is defined as

$LC_*^4 = \{x \in IR_1^5 : -x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0\} \backslash \{0\}$.

The three-dimensional open light cone L_0^3 in LC_*^4 is defined as

$L_0^3 = \{x \in LC_*^4 : x_2 = 0, -x_1^2 + x_3^2 + x_4^2 + x_5^2 = 0\} \backslash \{0\}$.

Let $\gamma: I \to L_0^3$ be a unit speed spacelike curve in light cone in Minkowski 5-space. The moving Frénet frame along the curve γ is denoted by $\{\gamma(s), \gamma_L(s), t(s), b(s), e_2\}$and is called the light-cone frame. Differential formulae according to the light-cone frame is given as follows

$$\gamma'(s) = t(s)$$

$$\gamma_L'(s) = k(s)t(s) + \tau(s)b(s)$$

$$t'(s) = \frac{k(s)}{2}\gamma(s) + \frac{1}{2}\gamma_L(s)$$

$$b'(s) = \frac{\tau(s)}{2}\gamma(s),$$

where

$\langle t(s), t(s)\rangle = \langle b(s), b(s)\rangle = \langle e_2, e_2\rangle = 1,$

$\langle \gamma(s), \gamma_L(s)\rangle = -2,$

$\langle \gamma(s), \gamma(s)\rangle = \langle \gamma_L(s), \gamma_L(s)\rangle = \langle t(s), \gamma(s)\rangle = \langle t(s), \gamma_L(s)\rangle$

$= \langle t(s), b(s)\rangle = \langle t(s), e_2\rangle = 0,$

$\langle b(s), \gamma(s)\rangle = \langle b(s), \gamma_L(s)\rangle$

$= \langle e_2, \gamma(s)\rangle = \langle e_2, \gamma_L(s)\rangle = \langle e_2, b(s)\rangle = 0,$

and light-cone curvature $k(s)$ and light-cone torsion $\tau(s)$ are given by

$k(s) = -\langle \gamma''(s), \gamma''(s)\rangle$,

$\tau(s) = -2\, det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s), e_2)$

respectively [12].

Suppose that $\gamma(\varphi)$ be another parametrization of the curve with parameter $\varphi = \int k(s)ds$. Then, light-cone formulae can be written as follows

$$\begin{bmatrix} \frac{d\gamma}{d\varphi} \\ \frac{d\gamma_L}{d\varphi} \\ \frac{dt}{d\varphi} \\ \frac{db}{d\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & f(\varphi) & 0 \\ 0 & 0 & 1 & g(\varphi) \\ \frac{1}{2} & \frac{1}{2}f(\varphi) & 0 & 0 \\ \frac{1}{2}g(\varphi) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma_L \\ t \\ b \end{bmatrix} \tag{1}$$

where $f(\varphi) = \frac{1}{k(\varphi)}$ and $g(\varphi) = \frac{\tau(\varphi)}{k(\varphi)}$. Each derivative equations given by the Light-cone formulae is expressed with the equations (1_1), (1_2), (1_3) and (1_4) respectively.

In following section, we obtain differential equations of spacelike curves according to components of light-cone frame in light cone L_0^3 in Minkowski 5-space.

3. Differential Equations of Spacelike Curves according to Light-Cone Frame

Theorem 1. Let $\gamma(s)$ be a spacelike curve parametrized by arclength s in light cone L_0^3 in Minkowski 5-space. Suppose that $\gamma(\varphi)$ be another parametrization of the curve with parameter $\varphi = \int k(s)ds$. Then,$\gamma(s)$ satisfies differential equation of fourth order according to Light-cone frame $\gamma(s)$ given by

$$\frac{d}{d\varphi}\left[\frac{2}{f^2 g}\frac{d^3\gamma}{d\varphi^3} - \frac{6}{f^3 g}\frac{df}{d\varphi}\frac{d^2\gamma}{d\varphi^2} + \left(\frac{6}{f^4 g}\left(\frac{df}{d\varphi}\right)^2 - \frac{2}{fg} - \frac{2}{f^3 g}\frac{d^2 f}{d\varphi^2}\right)\frac{d\gamma}{d\varphi} + \frac{1}{f^2 g}\frac{df}{d\varphi}\gamma\right] = \frac{1}{2}g\gamma. \tag{2}$$

Proof. Let $\gamma(\varphi)$ be a spacelike curve parametrized by arclength φ in light cone L_0^3. Then, taking derivation of equality (1_1) twice, we obtain

$$\frac{d^2\gamma}{d\varphi^2} = \frac{1}{2}f\gamma + \frac{df}{d\varphi}t + \frac{1}{2}f^2\gamma_L \tag{3}$$

and

$$\frac{d^3\gamma}{d\varphi^3} = \frac{df}{d\varphi}\gamma + \frac{1}{2}f\frac{d\gamma}{d\varphi} + \frac{3}{2}f\frac{df}{d\varphi}\gamma_L + \left(\frac{d^2 f}{d\varphi^2} + \frac{1}{2}f^2\right)t + \frac{1}{2}f^2 gb \tag{4}$$

respectively.

If equality (1_1) is arranged and is derivatived, we get

$$\frac{dt}{d\varphi} = \frac{-1}{f^2}\frac{df}{d\varphi}\frac{d\gamma}{d\varphi} + \frac{1}{f}\frac{d^2\gamma}{d\varphi^2}. \tag{5}$$

By substituting (5) into (1_3), we obtain

$$\gamma_L = \frac{-1}{f}\gamma - \frac{2}{f^3}\frac{df}{d\varphi}\frac{d\gamma}{d\varphi} + \frac{2}{f^2}\frac{d^2\gamma}{d\varphi^2}. \tag{6}$$

Then, by substituting (1_3) and (6) into (4), we have

$$b = \frac{2}{f^2 g}\frac{d^3\gamma}{d\varphi^3} - \frac{6}{f^2 g}\frac{df}{d\varphi}\frac{d^2\gamma}{d\varphi^2} + \left(\frac{6}{f^4 g}\left(\frac{df}{d\varphi}\right)^2 - \frac{2}{fg} - \frac{2}{f^3 g}\frac{d^2 f}{d\varphi^2}\right)\frac{d\gamma}{d\varphi} + \frac{1}{f^2 g}\frac{df}{d\varphi}\gamma. \tag{7}$$

Using the equations (1_4) and (7), we obtain desired equation (2).

Corollary 1. Let $\gamma(\varphi)$ be a spacelike curve in light cone L_0^3. If the function f of $\gamma(\varphi)$ is constant, then following equation is satisfied:

$$\frac{2}{f^2 g}\frac{d^4\gamma}{d\varphi^4}-\frac{2g'}{f^2g^2}\frac{d^3\gamma}{d\varphi^3}-\frac{2}{fg}\frac{d^2\gamma}{d\varphi^2}+\frac{2g'}{fg^2}\frac{d\gamma}{d\varphi}-\frac{1}{2}g\gamma=0.$$

Theorem 2. Let $\gamma(\varphi(s))$ be a spacelike curve parametrized by $\varphi=\int k(s)ds$ in light cone L_0^3 in Minkowski 5-space. Then,$\gamma(\varphi)$ satisfies differential equation of fourth order according to Light-cone frame b given by

$$\begin{aligned}\frac{d^4b}{d\varphi^4}=\frac{3}{fg}\frac{d(fg)}{d\varphi}\frac{d^3b}{d\varphi^3}&+\left[\frac{3}{g}\frac{d^2g}{d\varphi^2}+f+\frac{9}{fg}\frac{df}{d\varphi}\frac{dg}{d\varphi}+\frac{1}{f}\frac{d^2f}{d\varphi^2}\right.\\&\left.+\frac{6}{g^2}\left(\frac{dg}{d\varphi}\right)^2+\frac{3}{f^2}\left(\frac{df}{d\varphi}\right)^2+\frac{3}{fg}\frac{df}{d\varphi}\frac{dg}{d\varphi}\right]\frac{d^2b}{d\varphi^2}\\+\left[\begin{matrix}\frac{3}{f^2g}\left(\frac{df}{d\varphi}\right)^2\frac{dg}{d\varphi}+\frac{6}{fg^2}\frac{df}{d\varphi}\left(\frac{dg}{d\varphi}\right)^2+\frac{6}{g^3}\left(\frac{dg}{d\varphi}\right)^3\\-\frac{3}{fg}\frac{df}{d\varphi}\frac{d^2g}{d\varphi^2}-\frac{3}{g^2}\frac{dg}{d\varphi}\left(1+\frac{d^2g}{d\varphi^2}\right)-\frac{f}{g}\frac{dg}{d\varphi}-\frac{1}{fg}\frac{dg}{d\varphi}\frac{d^2f}{d\varphi^2}\end{matrix}\right]&\frac{db}{d\varphi}\\+\frac{1}{4}f^2g^2b\,.\end{aligned}\tag{8}$$

Proof. Let $\gamma(\varphi)$ be a spacelike curve parametrized by arclength φ in light cone L_0^3. Then, taking derivation of equality (1_4) three times, we obtain

$$\frac{d^2b}{d\varphi^2}=\frac{1}{2}\frac{dg}{d\varphi}\gamma+\frac{1}{2}fgt,\tag{9}$$

$$\frac{d^3b}{d\varphi^3}=\left(\frac{1}{2}\frac{d^2g}{d\varphi^2}+\frac{1}{4}fg\right)\gamma+\frac{1}{4}f^2g\gamma_L+\left(\frac{dg}{d\varphi}f+\frac{1}{2}g\frac{df}{d\varphi}\right)\tag{10}$$

and

$$\begin{aligned}\frac{d^4b}{d\varphi^4}&=\left(\frac{1}{2}\frac{d^3g}{d\varphi^3}+\frac{3}{4}f\frac{dg}{d\varphi}+\frac{1}{2}\frac{df}{d\varphi}g\right)\gamma+\left(\frac{3}{4}fg\frac{df}{d\varphi}+\frac{3}{4}f^2\frac{dg}{d\varphi}\right)\gamma_L\\&+\left(\frac{3}{2}f\frac{d^2g}{d\varphi^2}+\frac{1}{2}f^2g+\frac{3}{2}\frac{df}{d\varphi}\frac{dg}{d\varphi}+\frac{1}{2}g\frac{d^2f}{d\varphi^2}\right)t+\left(\frac{1}{4}f^2g^2\right)b.\end{aligned}\tag{11}$$

Using the equations (1_1) and (1_4), we have

$$t=\frac{-2}{fg^2}\frac{dg}{d\varphi}\frac{db}{d\varphi}+\frac{2}{fg}\frac{d^2b}{d\varphi^2}.\tag{12}$$

By substituting (1_4) into (1_3), we get

$$2\frac{dt}{d\varphi}=\frac{2}{g}\frac{db}{d\varphi}+f\gamma_L\tag{13}$$

and, by substituting (12) into (13), we obtain

$$\gamma_L=\frac{2}{f}\frac{d}{d\varphi}\left(\frac{-2}{fg^2}\frac{dg}{d\varphi}\frac{db}{d\varphi}+\frac{2}{fg}\frac{d^2b}{d\varphi^2}\right)-\frac{2}{fg}\frac{db}{d\varphi}.\tag{14}$$

By using the equations (1_4), (12) and (14) into (11), the equation (8) is obtained.

Corollary 2. Let $\gamma(\varphi)$ be a spacelike curve in light cone L_0^3. If the function f of $\gamma(\varphi)$ is constant, then following equation is obtained.

$$\begin{aligned}\frac{d^4b}{d\varphi^4}-\frac{3}{g}\frac{dg}{d\varphi}\frac{d^3b}{d\varphi^3}-\left[\frac{3}{g}\frac{d^2g}{d\varphi^2}+f+\frac{6}{g^2}\left(\frac{dg}{d\varphi}\right)^2\right]\frac{d^2b}{d\varphi^2}\\-\left[\frac{6}{g^3}\left(\frac{dg}{d\varphi}\right)^2-\frac{3}{g^2}-\frac{3}{g^2}\frac{d^2g}{d\varphi^2}-\frac{f}{g}\right]\frac{dg}{d\varphi}\frac{db}{d\varphi}\\-\frac{1}{4}f^2g^2b=0\end{aligned}$$

Corollary 3. Let $\gamma(\varphi)$ be a spacelike curve in light cone L_0^3. If the function g of $\gamma(\varphi)$ is constant, then following differential equation is satisfied:

$$\frac{d^4b}{d\varphi^4}-\frac{3}{f}\frac{df}{d\varphi}\frac{d^3b}{d\varphi^3}-\left[f+\frac{1}{f}\frac{d^2f}{d\varphi^2}+\frac{3}{f^2}\left(\frac{df}{d\varphi}\right)^2\right]\frac{d^2b}{d\varphi^2}-\frac{1}{4}f^2g^2b=0$$

Theorem 3. Let $\gamma(\varphi(s))$ be a spacelike curve parametrized by $\varphi=\int k(s)ds$ in light cone L_0^3 in Minkowski 5-space. Then,$\gamma(\varphi)$ satisfies differential equation of fourth order according to Light-cone frame t given by

$$\begin{aligned}\frac{d}{d\varphi}\left[\begin{matrix}\left(\frac{4f^2g}{f^3g^3+4gf'^2+2ff'g'-2fgf''}\right)\frac{d^3t}{d\varphi^3}-\left(\frac{8fgf'+4f^2g'}{f^3g^3+4gf'^2+2ff'g'-2fgf''}\right)\frac{d^2t}{d\varphi^2}\\+\left(\frac{8gf'^2+4ff'g'-4fgf''-4f^3g}{f^3g^3+4gf'^2+2ff'g'-2fgf''}\right)\frac{dt}{d\varphi}+\left(\frac{2f^2gf'+4f^3g'}{f^3g^3+4gf'^2+2ff'g'-2fgf''}\right)t\end{matrix}\right]\\-ft=0.\end{aligned}\tag{15}$$

Proof. Let $\gamma(\varphi)$ be a spacelike curve parametrized by arclength φ in light cone L_0^3. Then, by editing equation (1_3), we can write

$$\gamma_L=\frac{2}{f}\frac{dt}{d\varphi}-\frac{1}{f}\gamma.\tag{16}$$

Using the equations (16) and (1_2), we have

$$b=\frac{-2}{f^2}\frac{df}{d\varphi}\frac{dt}{d\varphi}+\frac{2}{fg}\frac{d^2t}{d\varphi^2}+\frac{1}{f^2g}\frac{df}{d\varphi}\gamma-\frac{2}{g}t.\tag{17}$$

By taking derivation of (17) and using the (1_4), we get

$$\begin{aligned}\left(\frac{g}{2}+\frac{2f'^2}{f^3g}+\frac{f'g'}{f^2g^2}-\frac{f''}{f^2g}\right)\gamma=\left(\frac{2}{fg}\right)\frac{d^3t}{d\varphi^3}-\left(\frac{4f'}{f^2g}+\frac{2g'}{fg^2}\right)\frac{d^2t}{d\varphi^2}\\+\left(\frac{4f'^2}{f^3g}+\frac{2f'g'}{f^2g^2}-\frac{2f''}{f^2g}-\frac{2}{g}\right)\frac{dt}{d\varphi}+\left(\frac{f'}{fg}+\frac{2g'}{g^2}\right)t\end{aligned}\tag{18}$$

and by taking derivation of (18) again, we obtain desired result.

Corollary 4. Let $\gamma(\varphi)$ be a spacelike curve in light cone L_0^3. If the function f of $\gamma(\varphi)$ is constant, then following differential equation is given by

$$\begin{aligned}\frac{4}{fg^2}\frac{d^4t}{d\varphi^4}-\frac{12g'}{fg^3}\frac{d^3t}{d\varphi^3}-\left(\frac{4g''}{fg^3}-\frac{12g'^2}{fg^4}+\frac{4}{g^2}\right)\frac{d^2t}{d\varphi^2}\\+\left(\frac{8g'}{g^3}+\frac{4g'}{g^6}\right)\frac{dt}{d\varphi}+\left(\frac{4g''}{g^6}-\frac{12g'^2}{g^4}-f\right)t\\=0\end{aligned}$$

Corollary 5. Let $\gamma(\varphi)$ be a spacelike curve in light cone L_0^3. If the function g of $\gamma(\varphi)$ is constant, then following differential equation is given by

$$\left(\frac{4f^2}{f^3g^2+2f'^2-ff''}\right)\frac{d^4t}{d\varphi^4}+\left[\frac{-12f^4g^2f'-12f^2f'f''+4f^3f'''}{(f^3g^2+2f'^2-ff'')^2}\right]\frac{d^3t}{d\varphi^3}$$
$$-\left(\frac{8f'^2+8ff''}{f^3g^2+2f'^2-ff''}-\frac{8ff'(3f^2g^2f'+3f'f''-ff''')}{(f^3g^2+2f'^2-ff'')^2}\right.$$
$$\left.-\frac{8f'^2-4ff''-2f^3}{f^3g^2+2f'^2-ff''}\right)\frac{d^2t}{d\varphi^2}$$
$$+\left[\frac{d}{d\varphi}\left(\frac{8f'^2-4ff''-4f^3}{f^3g^2+4f'^2-2ff''}\right)+\frac{2f^2}{f^3g^2+4f'^2-2ff''}\right]\frac{dt}{d\varphi}$$
$$+\left[\frac{d}{d\varphi}\left(\frac{2f^2}{f^3g^2+4f'^2-2ff''}\right)-f\right]t=0.$$

Author’s Contributions

Tanju Kahraman: Drafted and wrote the manuscript, performed the experiment and result analysis.

Ethics

There are no ethical issues after the publication of this manuscript.

References

[1]. B. Sahin, On a Submersion Between Reinhart Lightlike Manifolds and Semi-Riemannian Manifolds, Mediterr. J. Math. 5(2008), 273–284.

[2]. F. J. Palomo, F. J. Rodrguez, A. Romero, New characterizations of compact Totally umbilical spacelike surfaces in 4-Dimensional Lorentz Minkowski spacetime through a lightcone, Mediterr. J. Math.11(2014), 1229–1240.

[3]. G. Ganchev, V. Milousheva, An invariant theory of spacelike surfaces in the four-dimensional Minkowski space, Mediterr. J. Math.9(2012), 267–294.

[4]. H. Liu, J. Miao, D. Pei, Curves and surfaces of spacelike curves according to Bishop frame and their singularities, J. Nonlinear Sci. Appl., 9, 5020–5037 (2017).

[5]. K. L. Duggal, B. Sahin, Differential Geometry of Lightlike Submanifolds, Birkh¨auser, Boston, 2010.

[6]. M. Kazaz, H.H. Uğurlu, A. Özdemir, Integral Characterizations for Timelike and Spacelike Curves on Lorentzian Sphere , Iranian Journal of Science and Technology, Transaction A, Vol. 32, No. A1, 2008

[7]. M. Önder, T. Kahraman, H.H. Uğurlu, Differential Equations and Integral Characterizations of Timelike and Spacelike Spherical Curves in the Minkowski Space-time Matematychni Studii, V.40, No.1, 2013, pp. 30-37.

[8]. M. Sezer, Differential Equations and Integral Characterizations for Spherical Curves, Turkish J. Math., Vol. 13, No. 3, 1989.

[9]. V. Dannon, Integral Characterizations and the Theory of Curves, Proceedings of the American Mathematical Society, Volume 81, Number 4, (1981), 600-602.

[10]. W. Edward, A Note On Einstein, Bergmann, and the Fifth Dimension, arXiv:1401.8048 [physics.hist-ph] (2014).

[11]. J. Walrave, Curves and surfaces in Minkowski space, PhD. thesis, K.U. Leuven, Fac. of Science, Leuven, 1995.

[12]. Z. Wang, M. He, Singularities dual hypersurfaces and hyperbolic focal surfaces along spacelike curves in light cone in Minkowski 5-space, Mediterr. J. Math. (2019) 16(4):96 https://doi.org/10.1007/s00009-019-1355-5