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On Ideal Convergence for Triple Sequences on L-Fuzzy Normed Space

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Abstract

In this paper, we delve into the exploration of ideal convergence within the framework of triple sequences on L -fuzzy normed spaces. Our primary focus is to establish a comprehensive characterization of ideal convergence for these triple sequences, particularly in relation to their convergence in the classical sense. Through rigorous analysis, we demonstrate that the notion of ideal convergence, as developed in this context, exhibits a weaker form of convergence compared to the traditional convergence criteria applied to triple sequences in L - fuzzy normed spaces. This weaker form of convergence, while more generalized, retains significant applicability and provides a broader understanding of the behavior of sequences within these structured spaces. The results presented herein offer new insights into the subtleties of sequence convergence in fuzzy normed spaces, paving the way for further advancements in this area of mathematical analysis.

Keywords: Ideal Convergence, L-Fuzzy Normed Space, Triple Sequences

1. Introduction

In the context of the strong quantum gravity regime, space-time points are determined through a fuzzy framework, where the inherent uncertainty in the fabric of space-time leads to the representation of these points as a sequence of fuzzy numbers. This intrinsic fuzziness, which arises naturally from the quantum gravitational effects, under- mines the applicability of the conventional position space representation used in quantum mechanics. The standard formalism, grounded in precise point-like localization, becomes insufficient to accurately describe physical phenomena at such scales. Consequently, the need arises for a more generalized mathematical structure capable of accommodating the ambiguities inherent in the quantum gravitational domain. This necessitates the development and employment of alternative representational frameworks that extend beyond classical mechanics and quantum mechanics, ensuring a more accurate portrayal of the fundamental nature of space-time at the Planck scale. Such advancements are critical for understanding the intricate interplay between quantum mechanics and gravitational forces in extreme regimes.

A significant body of research has been devoted to the study of statistical convergence, particularly in connection with summability theory, with numerous aspects of this relationship thoroughly examined and characterized [22, 25, 26, 29, 31–37]. These investigations hold a prominent position within the broader field of mathematical analysis, serving as a foundational tool in understanding the behavior of sequences and series through alternative convergence criteria. The concept of statistical convergence has not only expanded the classical notions of con- vergence but has also provided powerful insights into the interplay between summability methods and asymptotic analysis. As a result, this area of study remains a focal point of ongoing research, attracting substantial interest from mathematicians who seek to further explore its applications and implications in both pure and applied mathematics. The continued relevance of these studies underscores their critical role in advancing the theoretical framework of convergence and summability within modern analysis.

The concept of fuzzy sets was first introduced to the mathematical community by Zadeh [38], marking a pivotal development in the study of imprecise data and uncertainty. This was subsequently followed by the introduction of intuitionistic fuzzy sets by Atanassov [3], alongside the development of L-fuzzy sets by Goguen [9]. These foundational works laid the groundwork for extensive research in the subsequent years, as scholars explored various generalizations and applications of these fuzzy structures.

In particular, L-fuzzy normed spaces [4, 28], as natural generalizations of classical normed spaces, fuzzy normed spaces, and intuitionistic fuzzy metric [1, 2] and normed spaces [13, 17, 18, 20, 23], have garnered significant attention. These spaces are based on specific logical and algebraic structures, thereby extending the classical theory of normed spaces and enriching the mathematical understanding of L-fuzzy metric spaces [10–12]. The incorporation of such generalized structures allows for a more nuanced treatment of uncertainty and imprecision, which is particularly valuable in various applications of mathematical analysis.

Moreover, extensive work has been devoted to the study of ideal convergence [5–8, 14–16, 19, 21, 24, 30] within these normed spaces, particularly in relation to L-fuzzy normed spaces. Ideal convergence, as a generalization of classical notions of convergence, offers a broader framework that captures a wider range of convergence behaviors. This has made it a topic of significant interest in the field, leading to valuable contributions by mathematicians in recent years.

The primary objective of the present paper is to introduce and rigorously investigate the concept of ideal convergence for triple sequences within the setting of L-fuzzy normed spaces. By exploring this generalized notion of convergence, we aim to establish new results that not only extend the existing theory but also provide deeper insights into the structure and behavior of triple sequences in L-fuzzy normed spaces. This study contributes to the ongoing development of ideal convergence theory and its applications in fuzzy normed spaces.

2. Preliminaries

In this section, we give some preliminaries on $\mathcal{L}-$ fuzzy normed spaces. n t

Definition 2.1. Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function satisfying the conditions

1. $T(x, y) = T(y, x)$
2. $T(T(x, y), z) = T(x, T(y, z))$
3. $T(x, 1) = T(1, x) = x$
4. If $x \le y, z \le t$, then $T(x, z) \le T(y, t)$

Then, T is called a triangular norm (or shortly t–norm).

Example 2.2. The functions T_1, T_2, T_3 given with,

1. $T_1(x, y) = min\{x, y\}$,
2. $T_2(x, y) = xy$,
3. $T_3(x, y) = max\{x + y - 1{,}0\}$

are well-known examples of t norms.

Definition 2.3. *G*iven a complete lattice $\mathcal{L} = (L, \preceq)$ and a set X which will be called the universe. A function

$$A: X \to L$$

is called an $L-$fuzzy set, or an $L-$set for short, on X. The family of all $L-$subsets on a set X is denoted by L^X.

Intersection and union of two $L-$sets on X is given by

$$(A \cap B)(x) := A(x) \wedge B(x)$$

and

$$(A \cup B)(x) := A(x) \vee B(x)$$

for all $x \in X$. Similarly union of two $L-$sets and intersection and union of a family $\{A_i : i \in I\}$ of $L-$sets is given by

$$\left(\bigcap_{i \in I} A_i\right)(x) := \bigwedge_{i \in I} A_i(x)$$

and

$$\left(\bigcup_{i \in I} A_i\right)(x) := \bigvee_{i \in I} A_i(x)$$

respectively.

We denote the smallest and the greatest elements of the complete lattice L by 0_L and 1_L. We also use the symbols $\succeq, \prec$ and $\succ$ given a lattice $(L, \preceq)$, in the obvious meanings.

Definition 2.4 *A* triangular norm ($t-$norm) on a complete lattice $\mathcal{L} = (L, \preceq)$ is a function $\mathcal{T}: L \times L \to L$ satisfying the following conditions for all $x, y, z, t \in L$:

1. $\mathcal{T}(x, y) = \mathcal{T}(y, x)$
2. $\mathcal{T}(\mathcal{T}(x, y), z) = \mathcal{T}(x, \mathcal{T}(y, z))$
3. $\mathcal{T}(x, 1_L) = \mathcal{T}(1_L, x) = x$
4. if $x \preceq y$ and $z \preceq t$, then $\mathcal{T}(x, z) \preceq \mathcal{T}(y, t)$.

A $t-$norm $\mathcal{T}$ on a complete lattice $\mathcal{L} = (L, \preceq)$ is called continuous, if for every pair of sequences (x_n) and (y_n) on L such that $(x_n) \to x \in L$ and $(y_n) \to y \in L$, one have the property that the sequence $\mathcal{T}(x_n, y_n) \to \mathcal{T}(x, y)$ with respect to the order topology on L.

Definition 2.5 *A* mapping $\mathcal{N}: L \to L$ is called
a negator on $\mathcal{L} = (L, \preceq)$ if,
N_1) $\mathcal{N}(0_L) = 1_L$
N_2) $\mathcal{N}(1_L) = 0_L$
N_3) $x \preceq y$ implies $\mathcal{N}(y) \preceq \mathcal{N}(x)$ for all $x, y \in L$.
In addition, if
N_4) $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$,
then the negator $\mathcal{N}$ is said to be involutive.

On the lattice $([0,1], \le)$ the function $\mathcal{N}_s: [0,1] \to [0,1]$ defined as $\mathcal{N}_s(x) = 1 - x$ is an example of an involutive negator, called standart negator on $[0,1]$, which is used in the theory of fuzzy sets. On the other hand, given the lattice $([0,1]^2, \preceq)$ with the order

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$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \Leftarrow \mu_1 \leq \mu_2 \; and \; \nu_1 \geq \nu_2$$

for all $(\mu_i, \nu_i) \in [0,1]^2$, $i = 1,2$. Then, the mapping $\mathcal{N}_1: [0,1]^2 \to [0,1]^2$, is an involutive negator used in the theory of intuitionistic fuzzy sets in the sense of Atanassov[4]. A possible candidate for a non-involutive negator on $([0,1]^2, \preceq)$ would be given by

$$\mathcal{N}_2(\mu, \nu) = \left(\frac{1-\mu+\nu}{2}, \frac{1+\mu-\nu}{2}\right).$$

Remark 2.6 In general, for any given continuous t −norm $\mathcal{T}$ and a negator $\mathcal{N}$, it is not always possible to find for each given $\varepsilon \in L - \{0_L, 1_L\}$, an element $r \in L - \{0_L, 1_L\}$ such that $\mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \succ \mathcal{N}(\varepsilon)$. In this study, a continuous t −norm and an involutive negator $\mathcal{N}$ such that for each $\varepsilon \in L - \{0_L, 1_L\}$, there exists an $r \in L - \{0_L, 1_L\}$ satisfying $\mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \succ \mathcal{N}(\varepsilon)$, is supposed to be given and fixed.

Definition 2.7 Let V be a real vector space, $\mathcal{L} = (L, \preceq)$ be a complete lattice, $\mathcal{T}$ be a continuous t −norm on $\mathcal{L}$ and ρ be an $\mathcal{L}$ −set on $V \times (0, \infty)$ satisfying the following:

1. $\rho(x,t) > 0_L$ for all $x \in V$, $t > 0$
2. $\rho(x,t) = 1_L$ for all $t > 0$, if and only if $x = \theta$
3. $\rho(\alpha x, t) = \rho(x, \frac{t}{|\alpha|})$ for all $x \in V$, $t > 0$ and $\alpha \in \mathbb{R} - \{0\}$
4. $\mathcal{T}(\rho(x,s), \rho(y,t)) \preceq \rho(x+y, s+t)$ for all $x, y \in V$ and $s, t > 0$
5. $\lim_{t\to\infty} \rho(x,t) = 1_L$ and $\lim_{t\to 0} \rho(x,t) = 0_L$ for all $x \in V \backslash \{\theta\}$
6. The mappings $f_x: (0,\infty) \to L$ given by $f(t) = \rho(x,t)$ are continuous.

In this case, the triple $(V, \rho, \mathcal{T})$ is called a $\mathcal{L}$ −fuzzy normed space or $\mathcal{L}$ −normed space, for short.

Definition 2.8 *A* sequence (x_n) in a $\mathcal{L}$ −fuzzy normed space $(V, \rho, \mathcal{T})$ is said to be convergent to $x \in V$, if for each $\varepsilon \in L - \{0_L\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$

$$\rho(x_n - x, t) \succ \mathcal{N}(\varepsilon).$$

Definition 2.9 *A* sequence (x_n) in a $\mathcal{L}$ −fuzzy normed space $(V, \rho, \mathcal{T})$ is said to be a Cauchy sequence, if for each $\varepsilon \in L - \{0_L\}$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\rho(x_n - x_m, t) \succ \mathcal{N}(\varepsilon)$$

for all $m, n > n_0$.

Definition 2.10 Let $(V, \rho, \mathcal{T})$ be a $\mathcal{L}$ −fuzzy normed space. Then, a sequence $x = (x_k)$ is statistically convergent to $l \in V$ with respect to ρ fuzzy norm, provided that, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$,

$$\delta\{k \in \mathbb{N}: \rho(x_k - l, t) \nsucc \mathcal{N}(\varepsilon)\} = 0$$

or equivalently

$$\lim_m \frac{1}{m}\{j \leq m: \rho(x_k - l, t) \nsucc \mathcal{N}(\varepsilon)\} = 0.$$

In this scenario, we will write $st_{\mathcal{L}} - \lim x = l$.

Definition 2.11 Let $(V, \rho, \mathcal{T})$ be a $\mathcal{L}$ −fuzzy normed space. Then, a sequence $x = (x_k)$ is said to be statistically Cauchy with respect to fuzzy norm ρ, provided that

$$\delta\{k \in \mathbb{N}: \rho(x_k - x_m, t) \nsucc \mathcal{N}(\varepsilon)\} = 0$$

for each $\epsilon \in L - \{0_L\}$, $m \in \mathbb{N}$ and $t > 0$.

Definition 2.12 Let $(V, \rho, \mathcal{T})$ be a $\mathcal{L}$ −fuzzy normed space. Then, a sequence $x = (x_k)$ is said to be statistically bounded with respect to fuzzy norm ρ, provided that there exists $r \in L - \{0_L, 1_L\}$ and $t > 0$ such that

$$\delta\{k \in \mathbb{N}: \rho(x_k, t) \nsucc \mathcal{N}(r)\} = 0$$

for each positive integer k.

For any given $\varepsilon > 0$, if there exists an integer N such that $|x_{jk} - l| < \varepsilon$ whenever $j, k > N$, a double sequence $x = (x_{jk})$ is said to be Pringsheim's convergent or shortly $P-$ convergent. This will be written as

$$\lim_{j,k\to\infty} x_{jk} = l$$

with j and k tending to infinity independently of one another.

Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers, and let $K(m,n)$ be the numbers of (j,k) in K such that $j \leq m$ and $k \leq n$. Then, we can define the two-dimensional analogue of natural density as follows: The lower asymptotic density of the set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as

$$\delta_2(K) = \liminf_{m,n} \frac{K(m,n)}{mn}$$

and if the sequence $\left(\frac{K(m,n)}{mn}\right)$ has a limit in the sense of Pringsheim, we say it has a double natural density, and it is defined as

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

In the following, statistical convergence of double sequences in $\mathcal{L}$ −fuzzy normed space is given.

Definition 2.13 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space. Then, a double sequence $x=(x_{jk})$ is statistically convergent to $l\in V$ with respect to ρ provided that, for each $\varepsilon\in L-\{0_L\}$ and $t>0$,

$$\delta_2\{(j,k)\in\mathbb{N}\times\mathbb{N}: \rho(x_{jk}-l,t)\nsucc\mathcal{N}(\varepsilon)\}=0$$

or equivalently

$$\lim_{m,n}\frac{1}{mn}\{j\leq m,k\leq n: \rho(x_{jk}-l,t)\nsucc\mathcal{N}(\varepsilon)\}=0.$$

In this case, we write $st_{2^{\mathcal{L}}}-\lim x=l$.

Definition 2.14 *[41]* If X is a non-empty set then a family $\mathcal{I}$ of subsets of X is called an ideal in X if and only if
1. $\emptyset\in\mathcal{I}$,
2. $A,B\in I$ implies $A\cup B\in\mathcal{I}$,
3. For each $A\in I$ and $B\subset A$ we have $B\in\mathcal{I}$,

where $P(X)$ is the power set of X. $\mathcal{I}$ is called nontrivial ideal if $X\notin\mathcal{I}$.

Definition 2.15 *[41]* Let X be a non-empty set. A non-empty family of sets $F\subset P(X)$ is called a filter on X if and only if
1. $\emptyset\notin\mathcal{I}$,
2. $A,B\in F$ implies $A\cap B\in F$,
3. For each $A\in F$ and $A\subset B$ we have $B\in F$.

Definition 2.16 *[41]* A nontrivial ideal $\mathcal{I}$ in X is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletions, i.e., $\{x\}\in\mathcal{I}$ for each $x\in X$.
Let $\mathcal{I}\subset P(X)$ be a nontrivial ideal. Then, a class $F(I)=\{M\subset X: M=X\backslash A,$ for some $A\in\mathcal{I}\}$ is a filter on X, called the filter associated with the ideal $\mathcal{I}$.

Definition 2.17 *[41]* An admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}$ there are sets $B_n\subset\mathbb{N}$, $n\in\mathbb{N}$, such that the symmetric difference $A_n\triangle B_n$ is a finite set for every n and $\bigcap_{n\in B_n}\in\mathcal{I}$.

Definition 2.18 *[41]* Let $\mathcal{I}\subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then, a sequence $x=(x_k)$ is said to be $\mathcal{I}-$convergent to L if, for every $\epsilon>0$, the set

$$\{k\in\mathbb{N}: |x_k-L|\geq\epsilon\}\in I.$$

In this case, we write $\mathcal{I}-lim x=L$.

3 Ideal Convergence for Triple Sequences on $\mathcal{L}-$ Fuzzy Normed Space

In this section, we will look into ideal convergence on $\mathcal{L}-$fuzzy normed spaces. Throughout the paper we take $\mathcal{I}_3$ as a nontrivial ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

Definition 3.1 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space and $\mathcal{I}_3$ be a nontrivial ideal in $\mathbb{N}$. Then a sequence $x=(x_{mnk})$ is $\mathcal{I}_3$ convergent to $\ell\in V$ with respect to ρ fuzzy norm, provided that, for each $\varepsilon\in L-\{0_L\}$ and $t>0$,

$$\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(x_{mnk}-\ell,t)\nsucc\mathcal{N}(\varepsilon)\}\in\mathcal{I}_3.$$

In this scenario, we will write $\mathcal{I}_3^{\mathcal{L}}-\lim x=\ell$.

Lemma 3.2 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space. Then, the following statements are equivalent, for every $\varepsilon\in L-\{0_L\}$ and $t>0$:
1. $\mathcal{I}_3^{\mathcal{L}}-\lim x=\ell$.
2. $\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(x_{mnk}-l,t)\nsucc\mathcal{N}(\varepsilon)\}\in\mathcal{I}_3$.
3. $\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(x_{mnk}-l,t)\succ\mathcal{N}(\varepsilon)\}\in F(\mathcal{I}_3)$.
4. $\mathcal{I}_3^{\mathcal{L}}-\lim\rho(x_{mnk}-\ell,t)=1_L$.

Proof. The equivalences between (a), (b) and (c) follow directly from the definitions. $(a)\Leftarrow(d)$: Note that $\mathcal{I}_3^{\mathcal{L}}-\lim x=\ell$ means that, for all $\varepsilon\in L-\{0_L\}$ and $t>0$ we have

$$\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(x_{mnk}-\ell,t)\nsucc\mathcal{N}(\varepsilon)\}\in\mathcal{I}_3.$$

On the other hand, a local base for the open neighborhoods of $1_L\in L$ with respect to the order topology on the lattice $\mathcal{L}=(L,\leq)$, are the sets

$$(a,1_L]=\{x\in L: a<x\leq 1_L\}$$

for each $a\in L-\{1_L\}$. $\mathcal{I}_3^{\mathcal{L}}-\lim\rho(x_{mnk}-\ell,t)=1_L$ if and only if, for any given $a\in L-\{1_L\}$,

$$\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times: \rho(x_{mnk}-\ell,t)\notin(a,1_L]\}\in\mathcal{I}_3$$

or equivalently

$$\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times: \rho(x_{mnk}-\ell,t)\notin a\})\in\mathcal{I}_3.$$

Note that, the two statements $\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(x_{mnk}-\ell,t)\nsucc\mathcal{N}(\varepsilon)\}\in\mathcal{I}_3$ for all $\varepsilon\in L-\{0_L\}$ and $\{(m,n,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(x_{mnk}-\ell,t)\notin a\})\in\mathcal{I}_3$ are equivalent since for each $\varepsilon\in L-\{0_L\}$ we can choose $a\in L-\{1_L\}$ as $a=\mathcal{N}(\varepsilon)$ and conversely for each $a\in L-\{1_L\}$ we can choose $\varepsilon\in L-\{0_L\}$ as $\varepsilon=\mathcal{N}(a)$, so that $a=\mathcal{N}(\mathcal{N}(a))=\mathcal{N}(\varepsilon)$. This proves that (a) is equivalent to (d).

Theorem 3.3 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space. If $\lim x=\ell$ for a triple sequence $x=(x_{mnk})$ then $\mathcal{I}_3^{\mathcal{L}}-\lim x=\ell$.

Proof. Suppose that $\lim x=\ell$. Then, for every $\varepsilon\in L-\{0_L\}$ and $t>0$, there exists a positive integer N such that

$$\rho(x_{mnk}-l,t)\succ\mathcal{N}(\varepsilon)$$

for all $k\geq N$.

Consider the set $A := \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times : \rho(x_{mnk} - l, t) \nsucc \mathcal{N}(\varepsilon)\}$. Then, since the all singletions belong to the admissible ideal $\mathcal{I}_3$. We also have $A \in \mathcal{I}_3$, since it is a subset of a finite union of sets in $\mathcal{I}_3$ and accordingly $\mathcal{I}_3^{\mathcal{L}} - limx = \ell$.

As seen in the following example, the converse of the theorem is not true in general.

Example 3.4 Let $\mathcal{L} = [0, \infty]$ be the lattice of nonnegative extended real numbers with the usual order. Also given the triangular norm $\mathcal{T}(\alpha, \beta) = min\{\alpha, \beta\}$ and the negator $\mathbb{N}(\alpha) = \alpha^{-1}$. Then, the triple $(\mathbb{R}, \rho, \mathcal{T})$ is an $\mathcal{L}-$ fuzzy normed space, where ρ is given by $\rho(x,t) = \frac{t}{|x|}$ for all $x \in \mathbb{R}$ and $t > 0$. Note that, by the rules of arithmetics in extended real line, 0 is allowed in the denominator and $\lim_{n \to 0} \frac{t}{n} = \infty$ for all $t > 0$.
Consider the admissible ideal $\mathcal{I}_3$ consisting of the small subsets of the set of positive integers, that is

$$\mathcal{I}_3 = \{A \subset \mathbb{N}: \sum_{k \in A} \frac{1}{k} < \infty\}$$

and the sequence $x = (x_{mnk})$, given by $x_{mnk} = \kappa(\log_2^{mnk})$, where κ stands for the Dirichlet function. Thus, (x_{mnk}) is 1, for powers of 2 and 0, for other values of (m,n,k). Then, x does not converge in the classical sense, since it gets both the values 0 and 1, for arbitrary large values of (m,n,k).

However, it $\mathcal{I}_3 -$ converges to 0, since $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{((2^m)^n)^k} = 1 < \infty$, suggesting that

$$\begin{aligned}
&(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho(x_{mnk} - 0, t) \nsucc \mathcal{N}(\varepsilon) \\
&= \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho(x_{mnk}, t) \nsucc \varepsilon^{-1}\} \\
&= \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \frac{t}{|x_{mnk}|} \leq \frac{1}{\varepsilon}\} \\
&= \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \varepsilon t \leq |x_{mnk}|\} \\
&\subseteq \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: x_{mnk} \neq 0\} \\
&= \{1,2,4,8,16, \dots\} \in \mathcal{I}_3
\end{aligned}$$

for all $\varepsilon \in (0, \infty]$ and $t > 0$. Hence $\mathcal{I}_3^{\mathcal{L}} - limx = 0$.

Theorem 3.5 Let $(V, \rho, \mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space. If a sequence $x = (x_{mnk})$ is $\mathcal{I}_3$ convergent with respect to the $\mathcal{L}-$fuzzy norm ρ, then $\mathcal{I}_3^{\mathcal{L}} -$limit is unique.

Proof. Suppose that $\mathcal{I}_3^{\mathcal{L}} - \lim x = \ell_1$ and $\mathcal{I}_3^{\mathcal{L}} - \lim x = \ell_2$. For any given $\varepsilon \in L - \{0_L\}$ and $t > 0$, $r \in L - \{0_L\}$ such that

$$\mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \succ \mathcal{N}(\varepsilon).$$

Define the following sets

$$K_0 = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho(\ell_1 - \ell_2, t) \nsucc \mathcal{N}(\varepsilon)\},$$

$$K_1 = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho(x_{mnk} - \ell_1, t) \nsucc \mathcal{N}(r)\}$$

and

$$K_2 = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho(x_{mnk} - \ell_2, t) \nsucc \mathcal{N}(r)\}$$

for any $t > 0$. Then $K_1, K_2 \in I_3$. Since for elements of the set $\mathbb{N}/(K_1 \cup K_2)$, we have

$$\begin{aligned}
\rho(\ell_1 - \ell_2, t) &\geq \mathcal{T}\left(\rho\left(x_{mnk} - \ell_1, \frac{t}{2}\right), \rho\left(x_{mnk} - l_2, \frac{t}{2}\right)\right) \\
&\succ \mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \\
&\succ \mathcal{N}(\varepsilon),
\end{aligned}$$

so that $\mathbb{N}/(K_1 \cup K_2) \subset \mathbb{N}/K_0$, or equivalently $K_0 \subset K_1 \cup K_2$. Since the expression $\rho(\ell_1 - \ell_2, r) \nsucc \mathcal{N}(\varepsilon)$ independent of $(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ either $K_0 = \emptyset$ or $K_0 = \mathbb{N}$, but $K_0 \in I_3$ enforces $K_0 = \emptyset$. Hence $\rho(\ell_1 - \ell_2, t) \succ \mathcal{N}(\varepsilon)$ for all $\varepsilon \in L/\{0_L\}$. Thus $\rho(\ell_1 - \ell_2, t) = 1_L$ which proves that $\ell_1 = \ell_2$.

Theorem 3.6 Let $(V, \rho, \mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space and $\mathcal{I}_3$ be an admissible ideal. Then,

1. If $\mathcal{I}_3^{\mathcal{L}} - \lim x_{mnk} = \ell_1$ and $\mathcal{I}_3^{\mathcal{L}} - \lim y_{mnk} = \ell_2$ then $\mathcal{I}_3^{\mathcal{L}} - \lim(x_{mnk} + y_{mnk}) = (\ell_1 + \ell_2)$

2. If $\mathcal{I}_3^{\mathcal{L}} - \lim x_{mnk} = \ell$ then $\mathcal{I}_3^{\mathcal{L}} - \lim \alpha x_{mnk} = \alpha\ell$.

Proof.
1. Let $\mathcal{I}_3^{\mathcal{L}} - \lim x_{mnk} = \ell_1$ and $\mathcal{I}_3^{\mathcal{L}} - \lim y_{mnk} = \ell_2$. For any given $\varepsilon \in L - \{0_L\}$ and $t > 0$, we can choose $r > 0$ such that $\mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \succ \mathcal{N}(\varepsilon)$. Define the sets,

$$K_1 = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho(x_{mnk} - \ell_1, t) \nsucc \mathcal{N}(r)\}$$

and

$$K_2 = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho(y_{mnk} - \ell_2, t) \nsucc \mathcal{N}(r)\}$$

for any $t > 0$. Since $\mathcal{I}_3^{\mathcal{L}} - \lim x_{mnk} = \ell_1$ and $\mathcal{I}_3^{\mathcal{L}} - \lim y_{mnk} = \ell_2$, we get $K_1, K_2 \in \mathcal{I}_3$. If we define the set

$$K = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: \rho((x_{mnk} + y_{mnk}) - (\ell_1 + \ell_2), t) \nsucc \mathcal{N}(\varepsilon)\}$$

then we have to show that $K \in \mathcal{I}_3$. Since $K_1, K_2 \in \mathcal{I}_3$, from the definition of filter, we know that $K_1^c, K_2^c \in F(\mathcal{I}_3)$. Therefore,

$$\begin{aligned}
\rho((x_{mnk} + y_{mnk}) - (\ell_1 + \ell_2), t) &\geq \mathcal{T}(\rho(x_{mnk} - \ell_1, \frac{t}{2}), \rho(y_{mnk} - \ell_2, \frac{t}{2})) \\
&\succ \mathcal{T}(\mathcal{N}(r), \mathcal{N}(r)) \\
&\succ \mathcal{N}(\varepsilon).
\end{aligned}$$

This shows that,

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$$K^c \subset \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho((x_{mnk}+y_{mnk}) - (\ell_1+\ell_2),t) \succ \mathcal{N}(\varepsilon)\}.$$

Thus, $K^c \in F(\mathcal{I}_3)$. In other words $K \in \mathcal{I}_3$. As a result of this, $\mathcal{I}_3^{\mathcal{L}} - \lim(x_{mnk}+y_{mnk}) = (\ell_1+\ell_2)$.

2. It is obvious for $\alpha = 0$. Now let $\alpha \neq 0$. Then for $\varepsilon \in L - \{0_L\}$ and $t > 0$,

$$A = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(x_{mnk}-\ell,t) \succ \mathcal{N}(\varepsilon)\} \in F(\mathcal{I}_3).$$

It is sufficient to prove that, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$, it can be written

$$A \subset \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(\alpha x_{mnk}-\alpha l,|\alpha|t) \succ \mathcal{N}(\varepsilon)\}.$$

Since also $|\alpha|t > 0$. Then we have,

$$\rho(\alpha x_{mnk}-\alpha\ell,|\alpha|t) = \rho(x_{mnk}-\ell,\frac{|\alpha|t}{|\alpha|}) = \rho(x_{mnk}-\ell,t) \succ \mathcal{N}(\varepsilon).$$

Therefore, we have

$$A = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \rho(\alpha x_{mnk}-\alpha l,|\alpha|t) \succ \mathcal{N}(\varepsilon)\},$$

and $A^c \in \mathcal{I}_3$. Obviously $I_3^{\mathcal{L}} - \lim\alpha x_{mnk} = \alpha\ell$.

4 $I_3^* -$ Convergence for Triple Sequences on $\mathcal{L}-$ Fuzzy Normed Space

In this section, we give the notion of the $\mathcal{I}_3^* -$ convergence on $\mathcal{L} -$ fuzzy normed space.

Definition 4.1 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space. A sequence of $x = (x_{mnk})$ of elements in X is said to be $\mathcal{I}_3^* -$ convergent to $\ell \in X$ with respect to the $\mathcal{L} -$ fuzzy norm if there exist a subset $K = \{(m_p,n_p,k_p): m_1,m_2,\dots,n_1 < n_2 <\dots,k_1,k_2,\dots\}$ of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $K \in F(\mathcal{I}_3)$ $(i.e. \mathbb{N}\times\mathbb{N}\times\mathbb{N}/K \in \mathcal{I}_3)$ and $\mathcal{L} - \lim_p x_{m_pn_pk_p} = \ell$.
In this case we write $\mathcal{I}_3^{*,\mathcal{L}} - \lim x = \ell$, and ℓ is called the $\mathcal{I}_3^* -$ limit of the sequence $x = (x_{mnk})$ with respect to $\mathcal{L} -$ fuzzy norm.

Theorem 4.2 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space and $\mathcal{I}_3$ be an admissible ideal. If $\mathcal{I}_3^{*,\mathcal{L}} - \lim x = \ell$, then $\mathcal{I}_3^{\mathcal{L}} - \lim x = \ell$.

Proof. Let $\mathcal{I}_3^{*,\mathcal{L}} - \lim x = \ell$. Then $K = \{(m_p,n_p,k_p): m_1,m_2,\dots,n_1 < n_2 <\dots,k_1,k_2,\dots\} \in F(\mathcal{I}_3)(i.e. \mathbb{N}\times\mathbb{N}\times\mathbb{N}/K = H \in \mathcal{I}_3)$ such that $\mathcal{L} - \lim_p x_{m_pn_pk_p} = \ell$. Then for each $\varepsilon \in L - \{0_L\}$ and $t > 0$ there exists a positive integer n_0 such that

$$\rho(x_{m_pn_pk_p}-\ell,t) \succ \mathcal{N}(\varepsilon)$$

for all $p > n_0$. Since the ideal is admissible and from the definition of the convergence, we have
$\{(m_p,n_p,k_p) \in K: \rho(x_{m_pn_pk_p}-\ell,t) \nsucc \mathcal{N}(\varepsilon)\} \in \mathcal{I}_3$
and

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times: \rho(x_{mnk}-\ell,t) \nsucc \mathcal{N}(\varepsilon)\} \subset \cup\{m_1,\dots,m_{p_0-1},n_1,\dots,n_{p_0-1},k_1 <\dots< k_{p_0-1}\} \in \mathcal{I}_3$$

for all $\varepsilon \in L - \{0_L\}$ and $t > 0$. Therefore, $\mathcal{I}_3^{\mathcal{L}} - \lim x = \ell$.

Remark 4.3 From the above example we have seen that $\mathcal{I}_3^* -$ convergence implies $\mathcal{I}_3 -$ convergence but not conversely. Now the question arises under what condition the converse may hold. For this we define the condition (AP) and see that under this condition the converse holds.

Definition 4.4 An admissible ideal $\mathcal{I}_3 \subset P(\mathbb{X})$ is said to satisfy the condition (AP) if for every sequence$(A_n)_n \in \mathbb{N}$ of pairwise disjoint sets from $\mathcal{I}_3$ there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \triangle B_n$ is a finite set for every n and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}_3$.

It is worth noting that both the ideal consisting of finite subsets of $\mathbb{N}$ with zero asymptotic density satisfy the condition (AP). However the ideal $\mathcal{I}_p$ given in Example 4.4. does not have this property. To see this, define the sets

$$A_n := \{p_n^k: k \in \mathbb{N}\}$$

here p_n denotes the $n - th$ prime number. Clearly for all $n \in \mathbb{N}$, $Prime(A_n) = \{p_n\}$ is finite so that $A_n \in I_p$. Moreover this sets are pairwise disjoint. If $A_n \triangle B_n$ is finite, then there at most finite number of elements of A_n, which are not in B_n. Since A_n is an infinite set, this means that B_n has elements of the form p_n^k. Then $p_n \in Prime(B_n)$. Then for the set $B := \bigcup_{n\in\mathbb{N}} B_n$, we have $Prime(B) = \mathbb{P}$, which is an infinite set. So in this situation it is not possible to form a sequence $(B_n)_{n\in\mathbb{N}}$ of such sets with $B \in \mathcal{I}_p$.

Proposition 4.5 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space and the ideal $\mathcal{I}_3$ satisfy the condition (AP). If $x = (x_{mnk})$ is a sequence in X such that $\mathcal{I}_3^{\mathcal{L}} - \lim x = \ell$, then $\mathcal{I}_3^{*,\mathcal{L}} - \lim x = \ell$.

Proposition 4.6 Let $(V,\rho,\mathcal{T})$ be a $\mathcal{L}-$fuzzy normed space. Then the following conditions are equivalent:
1. $\mathcal{I}_3^{*,\mathcal{L}} - \lim x = \ell$.
2. There exist two sequences $y = (y_{mnk})$ and $z = (z_{mnk})$ in X such that $x = y + z$, $\mathcal{L} - limy = \ell$ and the set $\{(m,n,k): z_{mnk} \neq \theta\} \in \mathcal{I}_3$, where θ denotes the zero element of X.

5 Conclusion

In this manuscript, we explore certain fundamental characteristics of the ideal convergence of sequences within the framework of $\mathcal{L}$-fuzzy normed spaces. This structure offers a versatile and expansive generalization of several classical spaces, including normed spaces, fuzzy normed spaces, and intuitionistic fuzzy (IF) normed spaces. The $\mathcal{L}$-fuzzy normed spaces provide a more flexible platform for analyzing convergence behaviors, particularly in the presence of uncertainty and imprecision, thereby extending the applicability of convergence theory to a broader class of spaces.

Moreover, several novel theoretical ideas have been developed and systematically outlined in this context. These innovations are further illustrated through examples that highlight the intricate connections between the various forms of ideal convergence. The results obtained in this study are made possible by leveraging both the lattice structure inherent in $\mathcal{L}$-fuzzy normed spaces and the underlying normed space framework. By synthesizing these elements, we introduce a more generalized interpretation of the norm, which enables us to extend the concept to a larger family of topological spaces within vector space theory.

The flexibility and richness of this generalized norm structure, coupled with the lattice-theoretic approach, facilitate a deeper understanding of the convergence properties of sequences in $\mathcal{L}$-fuzzy normed spaces. These results not only advance the current theory of ideal convergence but also open up new avenues for research within the broader context of functional analysis and topology.

The results of this study highlight that ideal convergence for triple sequences in $\mathcal{L}-$ fuzzy normed spaces can yield a more nuanced understanding of sequence convergence, particularly in environments where uncertainty and imprecision are inherent. These findings have far-reaching applications, especially in disciplines such as information theory, where data structures and processing often involve fuzzy or uncertain elements, as well as in artificial intelligence, where algorithmic learning benefits from a robust handling of ambiguous data patterns. Moreover, these principles are highly relevant in fields like quantum computing and physics, where multi-dimensional convergence within imprecise spaces can aid in modeling and predicting phenomena within non-deterministic frameworks. In economic modeling and financial analysis, where projections and assessments often rely on fuzzy data inputs, the application of triple sequence ideal convergence could improve predictive analytics and risk assessment models. Overall, this study establishes a valuable foundation for future research and interdisciplinary applications, illustrating the utility of $\mathcal{L}-$fuzzy normed spaces in accommodating more sophisticated forms of convergence essential for modern mathematical and applied sciences.

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