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Weighted Lacunary I-Statistical Convergence

Şükran KONCA[1]

ABSTRACT: In this paper, we define the concepts of weighted lacunary □-statistical convergence or $S_{(R,\theta)}(I)$ -convergence and $[R, p_r, \theta]^I$-summability, and investigate some inclusion relations.

Keywords. I-statistical convergence, ideal convergence, Riesz mean, weighted

Ağırlıklı Lacunary I-İstatistiksel Yakınsaklık

ÖZET: Bu makalede, $S_{(R,\theta)}(I)$ veya ağırlıklı lacunary I-istatistiksel yakınsaklık ile $[R, p_r, \theta]^I$-toplanabilirlik kavramları tanımlanmıştır ve bazı kapsama bağıntıları araştırılmıştır.

Anahtar Kelimeler: Ağırlık, I-istatiksel yakınsaklık, İdeal yakınsaklık, Riesz ortalama

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[1] Bitlis Eren Üniversitesi, Fen Edebiyat Fakültesi, Matematik, Bitlis, Türkiye
Sorumlu yazar/Corresponding Author: Şükran Konca, skonca@beu.edu.tr

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INTRODUCTION

The concept of statistical convergence was formally introduced by Fast (1951), Steinhaus (1951) and later on by Schoenberg (1959). Many years later, it has been discussed in the theory of Fourier analysis, ergodic theory and number theory under different names.

The notion of I-convergence was studied at initial stage by Kostyrko et al. (2000-2001) as a generalization of statistical convergence which was further studied in topological spaces (Lahiri et al., 2005). Kostyrko et al. (2005) gave some of basic properties of I-convergence and dealt with extremal I-limit points. Later on it was studied by Salat et al. (2004), Hazarika and Savaş (2011), Tripathy and Hazarika (2011) and many others. More applications of ideals can be seen in (Lahiri et al., 2005; Tripathy et al., 2009; Das et al., 2011; Hazarika and Savaş, 2011; Kumar et al., 2013; Altundağ and Sözbir, 2015)

Recently, Başarır and Konca (2014) have obtained a new lacunary sequence and a new

concept of statistical convergence which is called weighted lacunary statistical convergence by combining both of the definitions of lacunary sequence and Riesz mean.

In this paper, we define the concepts of weighted lacunary I-statistical convergence or $S_{(R,\theta)}(I)$-convergence and $[R, p_r, \theta]^I$-summability, and investigate some inclusion relations.

MATERIAL AND METHODS

Definitions and Preliminaries

In this section, we present some definitions and notations needed throughout the paper.

Let (p_k) be a sequence of positive real numbers and $P_n = p_1 + p_2 + ... + p_n$ for $n \in \mathbb{N}$ (the set of all natural numbers).

Then the Riesz transformation of $x = (x_k)$ is defined as $t_n = 1/P_n \sum_{k=1}^{n} p_k x_k$. If the transformation sequence (t_n) has a finite limit L then the sequence x is said to be Riesz convergent to L. We denote the set of all Riesz convergent sequences by (R, p_n). Let us note that if $P_n \to \infty$ as $n \to \infty$ then Riesz mean is regular summable method. Throughout the paper, let $P_n \to \infty$ as $n \to \infty$ and $P_0 = p_0 = 0$.

A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

The intervals determined by θ will be defined by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by q_r (for details on lacunary sequence see (Fridy and Orhan, 1993)).

Let $\theta = (k_r)$ be a lacunary sequence, (p_k) be a sequence of positive real numbers such that

$$H_r := \sum_{k \in I_r} p_k \ , \ P_{k_r} := \sum_{k \in (0, k_r]} p_k \ , \ P_{k_{r-1}} := \sum_{k \in (0, k_{r-1}]} p_k \ , \ Q_r = \frac{P_{k_r}}{P_{k_{r-1}}} \ , \ P_0 = 0 \ , \ I_r' = \left(P_{k_{r-1}}, P_{k_r}\right]$$

. It is easy to see that $H_r = P_{k_r} - P_{k_{r-1}}$. If we take $p_k = 1$ for all $k \in \mathbb{N}$ then $H_r, P_{k_r}, P_{k_{r-1}}, Q_r$ and I_r' reduce to h_r, k_r, k_{r-1}, q_r and I_r respectively. Throughout the paper by $lim_k x_k$ we mean $lim_{k \to \infty} x_k$ for brevity, we assume that $P_n \to \infty$ as $n \to \infty$ such that $H_r \to \infty$ as $r \to \infty$.

The sequence $x = (x_k)$ is said to be (R, p_r, θ)-summable to L in $lim_{r \to \infty} \omega_r(x) = L$. In this case, we write $x_k \to L(R, p_r, \theta)$ (Başarır and Konca, 2014).

Definition 2.1. (Fast, 1951) Recall that a number sequence $x=(x_k)$ is said to be statistically convergent to a number L (denoted by $S-lim_k\, x_k = L$) provided that for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : |x_k - L| \ge \varepsilon\right\}\right| = 0$$

where the vertical bars denote the cardinality of the enclosed set. Let S denotes the set of all statistically convergent sequences of numbers.

Definition 2.2. (Fridy and Orhan, 1993) Let $\theta=(k_r)$ be a lacunary sequence. A sequence $x=(x_k)$ of numbers is said to be lacunary statistically convergent to a number L (denoted by $S_\theta - lim_k\, x_k = L$) if for each $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r : |x_k - L| \ge \varepsilon\right\}\right| = 0$$

and S_θ denotes the set of all lacunary statistically convergent sequences of numbers.

Definition 2.3. (Fast, 1951) Recall that a number sequence $x=(x_k)$ is said to be weighted statistically convergent to a number L (denoted by $S_R - lim_k\, x_k = L$) provided that for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{P_n}\left|\left\{k \le P_n : |x_k - L| \ge \varepsilon\right\}\right| = 0.$$

By S_R, we denote the set of all statistically convergent sequences of numbers. Moricz and Orhan (2004) have defined the concept of statistical summability (R, p_n) as follows:

A sequence $x=(x_k)$ is statistically summable to L by the weighted mean method determined by the sequence (p_k) or briefly statistically summable (R, p_n) to L if $S-lim_n\, t_n(x) = L$. In this case, we write $S_R\text{-}\,lim\, x = L$.

Definition 2.4. (Başarır and Konca, 2014) A sequence $x=(x_k)$ is said to be weighted lacunary statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{H_r}\left|\left\{k \in I'_r : p_k |x_k - L| \ge \varepsilon\right\}\right| = 0 .$$

In this case, we write $S_{(R,\theta)}\text{-}lim_k\, x_k = L$. We denote the set of all weighted lacunary statistically convergent sequences by $S_{(R,\theta)}$.

Definition 2.5. (Kostyrko et al., 2000-2001) For any non-empty set X, $P(X)$ denotes the power set of X. A family of sets $I \subset 2^X = P(X)$ is called an ideal in X if and only if

1) $\varnothing \in F$

2) For each $A, B \in I$ we have $A \cup B \in I$,

3) For $A \in I,\ B \subseteq A$ we have $B \in I$,

A non-empty family of sets $F \subset P(X)$ is called a filter in X if and only if

1) $\varnothing \notin F$

2) For each $A, B \in I$ we have $A \cup B \in \mathrm{I}$,

3) For $A \in I,\ B \subseteq A$ we have $B \in I$.

It immediately implies that $I \subset P(X)$ is a non-trivial ideal if and only if the class $F = F(I) = \{X - A : A \in I\}$ is a filter on X. The filter $F = F(I)$ is called the filter associated with the ideal I. An ideal I is called non-trivial ideal if $\emptyset \notin I$ and $X \notin I$. Also, a non-trivial ideal I is called an admissible ideal in X if and only if it contains $\{\{x\} : x \in X\}$. Throughout the paper, I and w are considered as a non-trivial admissible ideal and the spaces of all sequences, respectively, unless otherwise stated. By I_{fin}, we mean the ideal of all subsets of ℕ.

Definition 2.6. (Kostyrko et al., 2000-2001) Given $\mathrm{I} \subset 2^{\mathbb{N}}$ is a nontrivial ideal in ℕ. The sequence $(x_n)_{n \in \mathbb{N}}$ in w is said to be I-convergent to the number L, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to I. We write $\mathrm{I} - lim_k\, x_k = L$.

Recently, Das et al. (2011) unified the ideas of statistical convergence and ideal convergence to introduce new concepts of I-statistical convergence and I-lacunary statistical convergence as follows.

Definition 2.7. (Das et al., 2011) A sequence $x = (x_k)$ of numbers is said to be I-statistical convergent or $S(\mathrm{I})$-convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathrm{I}.$$

In this case, we write $x_k \to L(S(\mathrm{I}))$ or $S(\mathrm{I}) - lim_k\, x_k = L$. Let $S(\mathrm{I})$ denotes the set of all I-statistically convergent sequences of numbers. For $I = I_{fin}$, I-statistical convergence coincides with statistical convergence.

Definition 2.8. (Das et al., 2011) Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be I-lacunary statistically convergent or $S_\theta(\mathrm{I})$-convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in \mathrm{I}_r : |x_k - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathrm{I}.$$

In this case, we write $x_k \to L(S_\theta(\mathrm{I}))$ or $S_\theta(\mathrm{I}) - lim\, x = L$ The set of all I-lacunary statistically convergent sequences will be denoted by $S_\theta(\mathrm{I})$.

Definition 2.9. (Altundağ and Sözbir, 2015) A sequence $x = (x_k)$ is said to be weighted I-statistically convergent or $S_R(\mathrm{I})$-convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{P_n} \left| \left\{ k \leq P_n : p_k |x_k - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathrm{I}$$

In this case, we write $x_k \to L(S_R(\mathrm{I}))$ or $S_\theta(\mathrm{I}) - lim\, x = L$. Let $S_R(\mathrm{I})$ denotes the set of all weighted I-statistically convergent sequences coincides with S_R.

RESULTS AND DISCUSSION

Main Results

Let I be an admissible ideal of □. A sequence $x=(x_k)$ is said to be $[R, p_r, \theta]^I$- summable to L, if for any $\varepsilon>0$, $\left\{r\in □ : \frac{1}{H_r}\sum_{k\in I_r} p_k|x_k-L|\geq\varepsilon\right\}\in I$

In this case, we write $[R, p_r, \theta]^I - lim\, x = L$ or $x_k \to L\left([R, p_r, \theta]^I\right)$.

Now, we give a new concept which will be called weighted lacunary □-statistical convergence. Let us investigate some relations related with this notion.

Definition 3.1. A sequence $x=(x_k)$ is said to be weighted lacunary I-statistical convergent (or $S_{(R,q)}(I)$ -convergent to L), if for every $\varepsilon>0$ and $\delta>0$,

$$\left\{r\in □ : \frac{1}{H_r}\left|\left\{k\in I'_r : p_k|x_k-L|\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$$

In this case, we write $S_{(R,\theta)}(I)-lim\, x=L$ or $x_k\to L\left(S_{(R,q)}(I)\right)$. The class of weighted lacunary I-statistical convergent sequences will be denoted by $S_{(R,\theta)}(I)$. For $I=I_{fin}$, $S_{(R,\theta)}(I)$ convergence coincides with $S_{(R,\theta)}$. If $p_k=1$ for all $k\in □$, then $S_{(R,q)}(I)$ convergence reduces to $S_q(I)$-convergence.

Theorem 3.2. Let $I\subseteq P(□)$ be an admissible ideal, θ be a double lacunary sequence and □, □ □, .Then $x_k\to L\left([R, p_r, \theta]^I\right)$ implies $x_k\to L\left(S_{(R,\theta)}(I)\right)$.

Proof. Suppose $x_k\to L\left([R, p_r, \theta]^I\right)$ and let

$$K_r(\varepsilon):=\left\{k\in I_r' : p_k|x_k-L|\geq\varepsilon\right\}. \tag{3.1}$$

Then we have

$$\begin{aligned}\frac{1}{H_r}\sum_{k\in I_r}p_k|x_k-L| &\geq \frac{1}{H_r}\sum_{k\in I'_r}p_k|x_k-L|\\ &\geq \frac{1}{H_r}\sum_{\substack{k\in I'_r\\ k\in K_r(e)}}p_k|x_k-L|\\ &\geq \varepsilon\frac{1}{H_r}\left|\left\{k\in I'_r : p_k|x_k\ L|\ \varepsilon\right\}\right|\end{aligned}$$

which implies $\frac{1}{\varepsilon}\frac{1}{H_r}\sum_{k\in I_r}p_k|x_k-L|\geq\frac{1}{H_r}\left|\left\{k\in I_r' : p_k|x_k-L|\geq e\right\}\right|$. Thus for any $\delta>0$ we have the following

$$\left\{r\in N : \frac{1}{H_r}\left|\left\{k\in I_r : p_k|x_k-L|\geq\varepsilon\right\}\right|\geq\delta\right\}\subseteq\left\{r\in N : \frac{1}{H_r}\sum_{k\in I_r}p_k|x_k-L|\geq\varepsilon\delta\right\}.$$

Since $x_k \to L\left(\left[R, p_r, \theta\right]^I\right)$, it follows that the latter set belongs to $\mathcal{I}$ and hence the result is obtained. In order to establish that the inclusion is strict; for $I = I_{fin}$ let $\theta = (k_r)$ be given and $x = (x_k)$ be defined as $1, 2, \ldots, \sqrt{h_r}$ at the first $\sqrt{h_r}$ integers in I_r and $x_k = 0$ for all other $k \in \mathbb{N}$. Let $(p_k) = 1^2, 2^2, 3^2, \ldots, h_r$ for $k \in I_r$ and $p_k = 0$ otherwise. Since $\mathrm{I}_r{}' \subset \mathrm{I}_r$, we can see that $x \in S_{(R,q)}(\mathrm{I})$ but $x \notin \left[R, p_r, \theta\right]^I$.

Theorem 3.3. Let $p_k \left|x_k - L\right| \leq M$ for all $k \in \mathbb{N}$ and $\mathrm{I}_r \subseteq \mathrm{I}_r{}'$. If $x_k \to L\left(S_{(R,q)}(\mathrm{I})\right)$ then $x_k \to L\left(\left[R, p_r, \theta\right]^I\right)$.

Proof. Suppose that $p_k \left|x_k - L\right| \leq M$ for all $k \in \mathbb{N}$ and $\mathrm{I}_r \subset \mathrm{I}_r'$. Let $x_k \to L\left(S_{(R,\theta)}(\mathrm{I})\right)$ and $K_r(\varepsilon)$ be defined as in (3.1). For each $\varepsilon > 0$ we have

$$\begin{aligned}
\frac{1}{H_r}\sum_{k\in \mathrm{I}_r} p_k \left|x_k - L\right| &\leq \frac{1}{H_r}\sum_{k\in \mathrm{I}'_r} p_k \left|x_k - L\right| \\
&\leq \frac{1}{H_r}\sum_{\substack{k\in \mathrm{I}'_r \\ k\in K_r(e)}} p_k \left|x_k - L\right| + \frac{1}{H_r}\sum_{\substack{k\in \mathrm{I}'_r \\ k\notin K_r(e)}} p_k \left|x_k - L\right| \\
&\leq \mathrm{M}\frac{1}{H_r}\left|\mathrm{K}_r(\varepsilon)\right| + \varepsilon.
\end{aligned}$$

Consequently, we obtain

$$\left\{ r \in N : \frac{1}{H_r}\sum_{k\in \mathrm{I}_r} p_k \left|x_k - L\right| \geq \varepsilon \right\} \subseteq \left\{ r \in N : \frac{1}{H_r}\left|\left\{ k \in \mathrm{I}_r' : p_k \left|x_k - L\right| \geq \varepsilon \right\}\right| \geq \frac{e}{M} \right\}.$$

Since $x_k \to L\left(S_{(R,\theta)}(\mathrm{I})\right)$, it follows that the latter set belongs to $\mathcal{I}$, which immediately implies

$\left\{ r \in N : \frac{1}{H_r}\sum_{k\in \mathrm{I}_r} p_k \left|x_k - L\right| \geq \varepsilon \right\} \in \mathrm{I}$. This shows that $x_k \to L\left(\left[R, p_r, \theta\right]^I\right)$.

If anyone wants to show that the converse of the previous theorem is strict, then for $I = I_{fin}$, $p_k = 1$ can be taken for all $k \in \mathbb{N}$ and $\theta = (k_r) = 2^r$ for all $r > 0$. Consider the sequence $x = (x_k) = (1, 0, 1, 0, \ldots)$ of course the inequality $p_k \left|x_k - L\right| \leq M$ holds for all $k \in \mathbb{N}$. The sequence $x \in \left[R, p_r, \theta\right]^I$ but $x \notin S_{(R,\theta)}(\mathrm{I})$.

Definition 3.4. A sequence $x = (x_k)$ is said to be $(R, p_r, \theta)^I$ summable to L, if $\mathrm{I} - lim_r\, \omega_r(x) \to L$

i.e for any $\varepsilon > 0$, $\{r \in \mathbb{N} : |\omega_r(x) - L| \geq \varepsilon\} \in \mathrm{I}$ where $\omega_r(x) := 1/H_r \sum_{k \in I_r} p_k x_k$. In this case we write $(R, p_r, \theta)^I\text{-}lim\, x = L$ or $x_k \to L\left((R, p_r, \theta)^I\right)$. For $I = I_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, $(R, p_r, \theta)^I$ summability becomes (R, p_r, θ) summability. In the following theorem, we examine the relation between $S_{(R,\theta)}(\mathrm{I})$-convergence and $(R, p_r, \theta)^I$-summability.

Theorem 3.5. Let $p_k|x_k - L| \leq M$ for all $k \in \mathbb{N}$ and $\mathrm{I}_r \subseteq \mathrm{I}'_r$. If a sequence $x = (x_k)$ is $S_{(R,\theta)}(\mathrm{I})$-convergent to L, then it is $(R, p_r, \theta)^I$- summable to L.

Proof. Let $p_k|x_k - L| \leq M$ for all $k \in \mathbb{N}$ and $\mathrm{I}_r \subseteq \mathrm{I}'_r$. Suppose that $x_k \to L\left(S_{(R,\theta)}(\mathrm{I})\right)$.

Then we have the following, where $K_r(\varepsilon)$ is defined as in (3.1)

$$|\omega_r - L| = \left|\frac{1}{H_r}\sum_{k \in \mathrm{I}_r} p_k(x_k - L)\right| \leq \left|\frac{1}{H_r}\sum_{k \in \mathrm{I}_r'} p_k(x_k - L)\right|$$

$$\leq \frac{1}{H_r}\sum_{k \in \mathrm{I}_r'} p_k|x_k - L| + \frac{1}{H_r}\sum_{k \in \mathrm{I}_r'} p_k|x_k - L|$$

$$= M\frac{1}{H_r}\left|\{k \in \mathrm{I}_r' : p_k|x_k - L| \geq \varepsilon\}\right| + \varepsilon.$$

If we take $A(\varepsilon) = \left\{r \in N : \frac{1}{H_r}\left|\{k \in \mathrm{I}_r' : p_k|x_k - L| \geq \varepsilon\}\right| \geq \frac{\varepsilon}{M}\right\}$ then for $r \in (A(\varepsilon))^C$ we take $|\omega_r - L| < 2\varepsilon$. Hence $\{r \in \mathbb{N} : |\omega_r - L| \geq 2\varepsilon\} \subset A(\varepsilon)$ and belongs to $\mathbb{I}$. This shows that $\mathrm{I} - lim_r w_r = L$, hence $x_k \to L\left((R, p_r, \theta)^I\right)$.

Theorem 3.6. The following statements are true:

1) If $p_k \leq 1$ for all $k \in \mathbb{N}$ and $x_k \to L\left(S_\theta(\mathrm{I})\right)$ then $x_k \to L\left(S_{(R,\theta)}(\mathrm{I})\right)$.

2) Let $\frac{H_r}{h_r}$ be upper bounded. If $p_k \geq 1$ for all $k \in \mathbb{N}$ and $x_k \to L\left(S_{(R,\theta)}(\mathrm{I})\right)$ then $x_k \to L\left(S_\theta(\mathrm{I})\right)$.

Proof. 1) If $p_k \leq 1$ for all $k \in \mathbb{N}$ then $H_r \leq h_r$ for all $r \in \mathbb{N}$. So, there exist M_1 and M_2 constants such that $0 < M_1 \leq \frac{H_r}{h_r} \leq M_2 \leq 1$

for all $r \in \mathbb{N}$. Let $x_k \to L\left(S_q\left(\mathrm{I}\right)\right)$ then for an arbitrary $\varepsilon > 0$ we have

$$\frac{1}{H_r}\left|\left\{k \in \mathrm{I}_r{}' : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| = \frac{1}{H_r}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{M_1 h_r}\left|\left\{P_{k_{r-1}} \leq k_{r-1} < k \leq P_{k_r} \leq k_r : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{M_1}.\frac{1}{h_r}\left|\left\{k_{r-1} < k \leq k_r : \left|x_k - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{M_1}.\frac{1}{h_r}\left|\left\{k \in \mathrm{I}_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| .$$

Thus for a given $\delta > 0$,

$$\frac{1}{H_r}\left|\left\{k \in I_r{}' : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta \Rightarrow \frac{1}{h_r}\left|\left\{k \in I_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| \geq M_1\delta. \quad .$$

Hence

$$\left\{r \in \mathbb{N} : \frac{1}{H_r}\left|\left\{k \in \mathrm{I}_r{}' : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in \mathrm{I}_r : \left|x_k - L\right| \geq \varepsilon\right\}\right| \geq M_1\delta\right\} .\text{Since}$$

$x_k \to L\left(S_\theta\left(\mathrm{I}\right)\right)$, the set on the right hand side belongs to $\mathcal{I}$ and so it follows that $x_k \to L\left(S_{(R,\theta)}\left(\mathrm{I}\right)\right)$.

2) Let $\frac{H_r}{h_r}$ be upper bounded, so there exist M_1 and M_2 constants such that $1 \leq M_1 \leq \frac{H_r}{h_r} \leq M_2 < \infty$ for all $r \in \mathbb{N}$. If $p_k \geq 1$ for all $k \in \mathbb{N}$ then we have $H_r \geq h_r$ for all $r \in \mathbb{N}$. Assume that $x = (x_k)$ converges to the limit L in $\left(S_{(R,\theta)}\left(\mathrm{I}\right)\right)$, then for an arbitary $\varepsilon > 0$ we have

$$\frac{1}{h_r}\left|\left\{k\in \mathrm{I}_r : \left|x_k-L\right|\geq \varepsilon\right\}\right| = \frac{1}{h_r}\left|\left\{k_{r-1}<k\leq k_r : \left|x_k-L\right|\geq \varepsilon\right\}\right|$$

$$\leq \frac{M_2}{H_r}\left|\left\{k_{r-1}\leq P_{k_{r-1}}<k\leq k_r\leq P_{k_r} : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|$$

$$= M_2 \cdot \frac{1}{H_r}\left|\left\{P_{k_{r-1}}<k\leq P_{k_r} : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|$$

$$= M_2\frac{1}{H_r}\left|\left\{k\in \mathrm{I}_r' : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|.$$

Thus for a given $\delta > 0$,

$$\frac{1}{h_r}\left|\left\{k\in \mathrm{I}_r : \left|x_k-L\right|\geq \varepsilon\right\}\right|\geq \delta \Rightarrow \frac{1}{H_r}\left|\left\{k\in \mathrm{I}_r' : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|\geq \frac{\delta}{M_2}.$$

Hence

$$\left\{r\in □ : \frac{1}{h_r}\left|\left\{k\in \mathrm{I}_r : \left|x_k-L\right|\geq \varepsilon\right\}\right|\geq \delta\right\}\subset\left\{r\in □ : \frac{1}{H_r}\left|\left\{k\in \mathrm{I}_r' : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|\geq \frac{\delta}{M_2}\right\}.$$

Since $x_k \to L\left(S_{(R,\theta)}(\mathrm{I})\right)$, the set on the right-hand side belongs to □ and so it follows that $x_k \to L\left(S_\theta(\mathrm{I})\right)$.

Theorem 3.7. For any lacunary sequence if $liminf_r\, Q_r > 1$ and $x_k \to L\left(S_R(\mathrm{I})\right)$, then

$x_k \to L\left(S_{(R,\theta)}(\mathrm{I})\right)$.

Proof. Suppose that $liminf_r\, Q_r > 1$. Then there exists a $\gamma > 0$ such that $Q_r \geq 1+\gamma$ for sufficiently large values of r, which implies that $\frac{H_r}{P_{k_r}} \geq \frac{\gamma}{1+\gamma}$. Let $x=(x_k)\in S_R(\mathrm{I})$ with $S_R(\mathrm{I})-lim\, x = L$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{P_{k_r}}\left|\left\{k\leq P_{k_r} : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|\geq \frac{1}{P_{k_r}}\left|\left\{P_{k_{r-1}}<k\leq P_{k_r} : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|$$

$$= \frac{H_r}{P_{k_r}}\left(\frac{1}{H_r}\left|\left\{P_{k_{r-1}}<k\leq P_{k_r} : p_k\left|x_k-L\right|\geq \varepsilon\right\}\right|\right)$$

$$\geq \frac{\gamma}{1+\gamma}\left(\frac{1}{H_r}\left|\left\{k \in I_r{}': p_k\left|x_k - L\right| \geq \varepsilon\right\}\right|\right).$$

$$\frac{1}{H_r}\left|\left\{k \in I_r{}': p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta \Rightarrow \frac{1}{P_{k_r}}\left|\left\{k \leq P_{k_r} : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \frac{\gamma}{1+\gamma}\delta.$$

Hence

$$\left\{r \in \mathbb{N}: \frac{1}{H_r}\left|\left\{k \in I_r{}': p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N}: \frac{1}{P_{k_r}}\left|\left\{k \leq P_{k_r} : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \frac{\gamma}{1+\gamma}\delta\right\}.$$

Since $x_k \to L\left(S_R\left(\mathrm{I}\right)\right)$, the set on the right-hand side belongs to I and so it follows that $x_k \to L\left(S_{(R,\theta)}\left(\mathrm{I}\right)\right)$.

Theorem 3.8. Let $\theta = \left(k_r\right)$ be a lacunary sequence with $\limsup_r Q_r < \infty$ and

$x_k \to L\left(S_{(R,\theta)}\left(\mathrm{I}\right)\right)$ then $x_k \to L\left(S_R\left(\mathrm{I}\right)\right)$.

Proof. If $\limsup_r Q_r < \infty$, then there is a $K > 0$ such that $Q_r \leq K$ for all r. Let

$x_k \to L\left(S_{(R,\theta)}\left(\mathrm{I}\right)\right)$ and

$$N_r = \left|\left\{k \in I_r{}': p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \tag{3.2}$$

By (3.2), given $\varepsilon > 0$, there is a $r_0 \in \mathbb{N}$ such that $\frac{N_r}{H_r} < \varepsilon$ for all $r > r_0$. Now, let $M = \max\left\{N_r : 1 \leq r \leq r_0\right\}$ and let n be any integer satisfying $k_{r-1} < n \leq k_r$, then we can write

$\frac{1}{P_n}\left|\left\{k \leq P_n : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \leq \frac{Mr_0}{P_{k_{r-1}}} + \varepsilon K$. So for a given $\delta > 0$

$$\left\{n \in \mathbb{N}: \frac{1}{P_n}\left|\left\{k \leq P_n : p_k\left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N}: \frac{Mr_0}{P_{k_{r-1}}} + \varepsilon K \geq \delta\right\}.$$

CONCLUSION

There we might also be interested in an analogue of the classical Korovkin Theorem (Korovkin, 1960) which states that for a sequence (T_n) of positive linear operators from $C[a;b]$ into $C[a;b]$ $\lim_n \|T_n - f(x)\|_\infty = 0$; for all $f \in C[a;b]$; if and only if $\lim_n \|T_n(f_i; x) - f_i(x)\|_\infty = 0$ for

$i = 0,1,2$ where $f_0(x) = 1; f_1(x) = x$ and $f_2(x) = x^2$ by using the concept of $(R,\ p_r,\ \theta)^I$-summability. All the results obtained in (Altundağ and Sözbir, 2015) also hold for our new concept.

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