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New Characterizations for Pseudo Null and Partially Null Curves in R_2^4

Zühal KÜÇÜKARSLAN YÜZBAŞI[1]

ABSTRACT: In this paper, pseudo-spherical pseudo null and partially null curves are defined by using curvature functions in R_2^4, respectively. Also, some new characterizations for pseudo null and partially null curves are obtained in R_2^4, respectively.

Key words: Pseudo null curve, Partially null curve, Semi-Euclidean space

R_2^4 de Pseudo Null ve Partially Null Eğriler İçin Yeni Karakterizasyonlar

ÖZET: Bu makalede, R_2^4 de eğrilik fonksiyonları kullanılarak sırasıyla pseudo-küresel pseudo null ve partially null eğriler tanımlandı. Ayrıca, sırasıyla R_2^4 de pseudo null ve partially null eğriler için yeni karakterizasyonlar elde edildi.

Anahtar Kelimeler: Pseudo null eğri, Partially null eğri, Semi-Euclidean uzay

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[1] Fırat universitesi, Fen Fakultesi, Matematik, Elazig, Türkiye
Sorumlu yazar/Corresponding Author: Zühal KÜÇÜKARSLAN YÜZBAŞI, zuhal2387@yahoo.com.tr

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INTRODUCTION

A pseudo null or a partially null curves in R_1^4 is defined as a spacelike curves along which the first binormal B_1 is the null vector and the second binormal B_2 is the null vector, respectively, in (Ilarslan , 2002). The Frenet equations and Frenet frame for a pseudo null or a partially null curve such that it lies fully in R_1^4 are obtained (Walrave, 1995). Also, such curves had at most two curvatures in R_1^4.

Recently, M. Petrovic-Torgasev and et al. obtained the Frenet equations of a pseudo null or a partially null curve such that it lies fully in R_2^4. Moreover, they characterized all W-pseudo null and W-partially null curves lying in R_2^4 (Petrovic-Torgasev et al. 2005). In particular, when the Frenet frame along a spacelike or a timelike curve contains a null vectors, such curve is said to be a pseudo null or a partially null curve (Walrave, 1995.).

In this paper, we characterize the pseudo-spherical pseudo null and partially null curves by using the curvature functions in R_2^4. Moreover, we obtain some new characterizations of pseudo null and partially null curves in R_2^4, respectively.

MATERIAL AND METHODS

In this section, we construct the Frenet frames and obtain the Frenet equations of pseudo null and partially null curves, lying fully in R_2^4. Hence, we consider the following two cases.

The semi-Euclidean space R_2^4 is the standard vector space R^4 equipped with an indefinite flat metric $\langle , \rangle$ given by

$$\langle , \rangle = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where $(x_1,..., x_4)$ is rectangular coordinate system of R_2^4. A tangent vector u to R_2^4 is spacelike, if $\langle u,u \rangle > 0$ or $u = 0$ timelike, if $\langle u,u \rangle < 0$ null, if $\langle u,u \rangle = 0$ and $u \neq 0,$ (Synge , 1967). Arbitrary two vectors v and w in R_2^4 are called be orthogonal, if $\langle v, w \rangle = 0$. The norm of a vector u is given by $\|v\| = \sqrt{|\langle u,u \rangle|}$.

Case 1. Pseudo Null Curves

Let $\alpha : I \rightarrow R_2^4$ be a spacelike or a timelike curve in R_2^4, parametrized by the arclenght parameter s, such that respectively hold $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$. Assume that $\langle \alpha''(s), \alpha''(s) \rangle = 0$ and that $\alpha''(s) \neq 0$ for each $s \in I \subset R$. Define the tangent and the principal normal vector fields by $T(s) = \alpha'(s)$, $N(s) = \alpha''(s)$, respectively. By differentiation with respect to s of the relation

$$\langle \alpha'(s), \alpha'(s) \rangle = \pm 1,$$

we obtain

$$\langle \alpha'(s), \alpha''(s) \rangle = 0.$$

Taking the derivative with respect to s of the previous equation, it follows that

$$\langle \alpha'(s), \alpha'''(s) \rangle = 0.$$

Thus, the vector $\alpha'''(s)$ is orthogonal to both of vectors $\alpha'(s)$ and $\alpha''(s)$. Next, assume that $\langle \alpha'''(s), \alpha'''(s) \rangle \neq 0$ for each s. We define the first binormal vector field B_1 by

$$B_1(s) = \frac{\alpha'''(s)}{\|\alpha'''(s)\|}.$$

Then in the space R_2^4 there exists the unique null vector field B_2 such that

$$\langle T, B_2\rangle = \langle B_1, B_2\rangle = \langle B_2, B_2\rangle = 0, \langle N, B_2\rangle = 1,$$

and such that the orientation of the Frenet frame $\{T, N, B_1, B_2\}$ is the same as the orientation of the space R_2^4. We call B_2 the second binormal vector field.

Then, let $\langle T,T\rangle = \varepsilon_1 = \pm 1, \langle B_1, B_1\rangle = \varepsilon_2 = \pm 1,$ where by $\varepsilon_1\varepsilon_2 = -1$. By using the conditions as follows

$$\langle T,T\rangle = e_1, \langle B_1, B_1\rangle = e_2, \langle N, B_2\rangle = 1, \langle N, N\rangle = \langle B_2, B_2\rangle = 0, \quad (1)$$

$$\langle T, N\rangle = \langle T, B_1\rangle = \langle T, B_2\rangle = \langle N, B_1\rangle = \langle B_1, B_2\rangle = 0.$$

Since $\langle T', B_2\rangle = \langle N, B_2\rangle = 1$ It follows that $k_1(s) = 1$ for each s. Thus, the first curvature $k_1(s)$ can only take two values: $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases.

The following Frenet equations of a pseudo null curve are given by

$$\begin{aligned} T'(s) &= N(s), \quad (2) \\ N'(s) &= k_2(s)B_1(s) \\ B_1'(s) &= k_3(s)N(s) - \varepsilon_2 k_2(s)B_2(s), \\ B_2'(s) &= -\varepsilon_1 T(s) - \varepsilon_2 k_3(s)B_1(s) \end{aligned}$$

where are only two curvatures $k_2(s)$ and $k_3(s)$ (Petrovic-Torgasev et al. 2005).

Case 2. Partially Null Curves

$\alpha : I \to R_2^4$ be a spacelike or a timelike curve in R_2^4, parametrized by the arclenght parameter s, such that hold $\langle \alpha''(s), \alpha''(s)\rangle < 0$ or $\langle \alpha''(s), \alpha''(s)\rangle > 0$ for each $s \in I \subset R$, respectively. Define the tangent and the principal normal vector fields respectively by $T(s) = \alpha'(s)$, $N(s) = \dfrac{\alpha''(s)}{\|\alpha''(s)\|}$. Then $\{T, N\}$ is the timelike plane of index 1.

Since α is a partially null curve, B_1 is a null vector. Thus there exist the unique null vector field B_2 such that

$$\langle T, B_2\rangle = \langle N, B_2\rangle = \langle B_2, B_2\rangle = 0, \quad \langle B_1, B_2\rangle = 1,$$

and such that the orientation of the Frenet frame $\{T, N, B_1, B_2\}$ is the same as the orientation of the space R_2^4. We call B_2 the second binormal vector field.

Moreover, let $\langle T,T\rangle = \varepsilon_1 = \pm 1$, $\langle N,N\rangle = e_2 = \pm 1$ whereby $\varepsilon_1\varepsilon_2 = -1$. By using the conditions

$$\langle T,T\rangle = \varepsilon_1, \langle N,N\rangle = \varepsilon_2, \langle B_1,B_2\rangle = 1, \langle B_1,B_1\rangle = \langle B_2,B_2\rangle = 0, \tag{3}$$

$$\langle T,N\rangle = \langle T,B_1\rangle = \langle T,B_2\rangle = \langle N,B_1\rangle = \langle N,B_2\rangle = 0.$$

The following Frenet equations of a partially null curve are given by:

$$\begin{aligned} T'(s) &= k_1(s)N(s), \\ N'(s) &= k_1(s)T(s) + k_2(s)B_1(s), \\ B_1'(s) &= k_3(s)B_1(s), \\ B_2'(s) &= -\varepsilon_2 k_2(s)N(s) - k_3(s)B_2(s). \end{aligned} \tag{4}$$

In the result, we prove that $k_3(s) = 0$ for each s, (Petrovic-Torgasev et al. 2005)ç

RESULTS AND DISCUSSION

Pseudo-spherical Pseudo Null Curves

In this section, we characterize pseudo-spherical pseudo null curves by using curvature functions in R_2^4.

The pseudo-sphere of radius r and center p_0 in R_2^4 is given by

$$S_2^3 = \left\{ X \in R_2^4 : \langle x - p_0, x - p_0\rangle = r^2 \right\}$$

(Duggal and Bejancu ,1996.). A pseudo null curve $\alpha(s)$ in R_2^4 is called pseudo-spherical if it lies on a pseudo sphere. A Pseudo null curve $\alpha(s)$ in R_2^4 parameterized by the Frenet curvatures $\{k_1, k_2\}$ and $k_i \neq 0, 1 \leq i \leq 2.$

Teorem 3.1. Let $\alpha(s)$ be a pseudo null curve in R_2^4 parametrized by the pseudo-arc such that $k_i \neq 0$ and $\{a_1, a_2, a_3, a_4\}$ be differentiable functions.

$\alpha(s)$ lies on a pseudo-sphere of radius r if and only if the following condition is satisfied

$$\lambda(s) = r^2,$$

where $\lambda(s) = 2a_2a_4$.

Proof. Assume that $\alpha(s)$ lies on a pseudo-sphere of radius r. That is, there exists a fixed point $p_0 \in R_2^4$ such that

$$\langle \alpha(s) - p_0, \alpha(s) - p_0\rangle = r^2. \tag{5}$$

Set

$$\alpha(s) - p_0 = \alpha_1 T + \alpha_2 N + \alpha_3 B_1 + \alpha_4 B_2.$$

From differentiation (5) and by using (2), we have

$$\langle \alpha(s) - p_0, T \rangle = 0, \tag{6}$$

and $a_1 = 0$ From differentiation of (6), we have

$$\langle T, T \rangle + k_1 \langle \alpha(s) - p_0, N \rangle = 0, \tag{7}$$

$$\langle a(s) \square p_0, N \rangle = \square e_1,$$

and $a_4 = -\varepsilon_1$. From differentiation of (7), we get

$$\langle T, N \rangle + \langle a(s) - p_0, k_1 B_1 \rangle = 0, \tag{8}$$

$$\langle \alpha(s) - p_0, B_1 \rangle = 0,$$

and $a_3 = 0$ From differentiation of (8), we obtain

$$\langle T, B_1 \rangle + \langle a(t) - p_0, k_3 N - e_2 k_2 B_2 \rangle = 0, \tag{9}$$

$$\langle \alpha(t) - p_0, B_2 \rangle = -\frac{\alpha_1 k_3}{\alpha_2 k_2}$$

$$\langle \alpha(t) - p_0, B_2 \rangle = \frac{k_3}{k_2},$$

and $a_2 = \dfrac{k_3}{k_2}$, we have

$$\alpha(s) - p_0 = \alpha_2 N + \alpha_4 B_2,$$

and by (5),

$$2a_2 a_4 = r^2,$$

and so we can write,

$$\lambda(s) = r^2.$$

Conversely, assume that

$$\lambda(s) = r^2 \tag{10}$$

for some positive constant r. Set

$$B(s) = \alpha(s) - \alpha_2 N - \alpha_4 B_2.$$

Then, using Frenet equations in (2) and the definition of $\{a_i\}$, we can obtain

$$B'(s) = T - \left(\frac{k_3}{k_2}\right)' N - \frac{k_3}{k_2}(k_2 B_1) + \varepsilon_1(-\varepsilon_1 T - \varepsilon_2 k_3 B_1),$$

$$B'(s) = -(\frac{k_3}{k_2})' N,$$

by using the (2) we can easily show that $\|B'\| = 0$, and so we get $B(s) = p_0$ for some fixed point $p_0 \in R_2^4$ Thus, we have

$$\alpha(s) - p_0 = a_2 N + a_4 B_2,$$

and using (10), we find

$$\langle \alpha(s) - p_0, \alpha(s) - p_0 \rangle = r^2.$$

Thus $\alpha(s)$ lies on a pseudo-sphere of radius r.

Pseudo-spherical Partially Null Curves

In this section, we characterize pseudo-spherical partially null curves in R_2^4 by using the curvature functions.

The pseudo-sphere of radius r and center p_0 in R_2^4 is given by

$$S_2^3 = \{X \hat{I}\ R_2^4 : \langle x - p_0, x - p_0 \rangle = r^2\}$$

(Duggal and Bejancu ,1996.). A partially null curve $a(s)$ in R_2^4 is called pseudo-spherical if it lies on a pseudo-sphere.

A Partially null curve $\alpha(s)$ in R_2^4 parametrized by the Frenet curvatures $\{k_1, k_2\}$ and $k_i = 0, 1 \leq i \leq 2.$

Teorem 3.2 Let $\alpha(s)$ be a partially null curve in in R_2^4 such that $k_i \neq 0$ and $\{a_1, a_2, a_3, a_4\}$ be differentiable functions.

$\alpha(s)$ lies on a pseudo-sphere of radius r if and only if $\varepsilon_1 = -1$, $e_2 = 1$ and the following condition is satisfied

$$\mu(s) = r^2,$$

where $\mu(s) = a_2^2 + 2a_3a_4$.

Proof. Assume that $\alpha(s)$ lies on a pseudo-sphere of radius r. That is, there exists a fixed point $p_0 \in R_2^4$ such that

$$\langle \alpha(s) - p_0, \alpha(s) - p_0 \rangle = r^2. \tag{11}$$

Set

$$\alpha(s) - p_0 = a_1 T + a_2 N + a_3 B_1 + a_4 B_2.$$

From differentiation of (11) and by using Frenet equations in (4), we obtain

$$\langle \alpha(s) - p_0, T \rangle = 0, \qquad (12)$$

and $a_1 = 0$ From differentiation of (12), we find

$$\langle T, T \rangle + k_1 \langle \alpha(s) - p_0, N \rangle = 0, \qquad (13)$$

$$\langle \alpha(s) - p_0, N \rangle = -\frac{\varepsilon_1}{k_1},$$

and $a_2 = -\frac{\varepsilon_1}{k_1}$. From differentiation of (13), we get

$$\langle T, N \rangle + \langle \alpha(s) - p_0, k_1 T + k_2 B_1 \rangle = \left(-\frac{e_1}{k_1}\right)'$$

$$\langle \alpha(s) - p_0, B_1 \rangle = \frac{1}{k_2} \left(-\frac{\varepsilon_1}{k_1}\right)',$$

and $a_4 = \frac{1}{k_2} \left(-\frac{e_1}{k_1}\right)'$.

From differentiation of $a_3 = \langle \alpha(t) - p_0, B_2 \rangle$, we have

$$\langle \alpha(s) - p_0, B_2 \rangle' = \langle T, B_2 \rangle + \langle \alpha(t) - p_0, -\varepsilon_2 k_2 N \rangle,$$

$$\alpha_3' = \frac{\varepsilon_1 \varepsilon_2 k_2}{k_1} = -\frac{k_2}{k_1},$$

$$a_3 = -\frac{k_2}{k_1} ds. \qquad (14)$$

Hence, we get

$$\alpha(s) - p_0 = a_2 N + a_3 B_1 + a_4 B_2,$$

and using (11), we obtain

$$a_2^2 + 2 a_3 a_4 = r^2,$$

and so, we can write

$$\mu(s) = r^2.$$

Conversely, assume that

$$\mu(s) = r^2, \qquad (15)$$

for some positive constant r. We can write

$$B(s)=\alpha(s)-a_2N-a_3B_1-a_4B_2.$$

Then, using Frenet equations in (4) and the definition of $\{a_i\}$, we can obtain

$$B'(s)=T+\left(\frac{\varepsilon_1}{k_1}\right)'N+\left(\frac{\varepsilon_1}{k_1}\right)(k_1T+k_2B_1)+\frac{k_2}{k_1}B_1+\left(\frac{1}{k_2}\left(\frac{\varepsilon_1}{k_1'}\right)\right)'B_2+\frac{1}{k_2}\left(\frac{\varepsilon_1}{k_1'}\right)(-\varepsilon_2k_2N).$$

If we consider $\varepsilon_1=-1$ and $\varepsilon_2=1$ at above equation, then we get

$$B'(s)=\left(\frac{1}{k_2}\right)-\left(\frac{1}{k_1'}\right)\Big)'B_2,$$

by using (4), we can easily show that $\|B'\|=0$, and so we find $B(s)=p_0$ for some fixed point $p_0\in R_2^4$. Hence, we have

$$\alpha(s)-p_0=a_2N+a_3B_1+a_4B_2,$$

and by (15), we can write

$$\langle\alpha(s)-p_0,\alpha(s)-p_0\rangle=r^2.$$

Thus, $\alpha(s)$ lies on a pseudo-sphere of radius r.

REFERENCES

Duggal K. L, Bejancu A,1996. Ligtlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers.

Ilarslan K, 2002. Some Special Curves on Non-Euclidean Manifolds, Doctoral Thesis, Ankara University, Graduate School of Natural and Applied Sciences.

Synge J. L, 1967. Timelike Helices in Flat Space-time, Proc. Roy. Irish Academy A65: 27-42.

Walrave J, 1995. Curves and Surfaces in Minkowski Space, Doctoral thesis, K. U. Leuven, Fac. of Science, Leuven.
Petrovic-Torgasev M, Ilarslan K, Nesovic E, 2005. On Partially Null and Pseudo Null Curves in the Semi-Euclidean Space R_2^4, J. Geom., 84: 106-116.