

An Application of Nabla Operator for the Radial Schrödinger Equation

Reşat YILMAZER¹

ABSTRACT: The aim of this present study is to obtain the discrete fractional solutions of the radial Schrödinger equation by applying the nabla discrete fractional calculus (DFC) operator.

Keywords: Fractional calculus, discrete fractional calculus, nabla operator, radial Schrödinger equation



Radyal Schrödinger Denklemi İçin Nabla Operatörünün Bir Uygulaması

ÖZET: Bu çalışmanın amacı, nabla ayrık kesirli hesap operatörünün uygulanmasıyla radyal Schrödinger denkleminin ayrık kesirli çözümlerini elde etmektir.

Anahtar Kelimeler: Kesirli hesap, ayrık kesirli hesap, nabla operatörü, radyal Schrödinger denklemi

¹ Fırat Üniversitesi, Fen Fakültesi, Matematik, Elazığ, Türkiye
Sorumlu yazar/Corresponding Author: Reşat YILMAZER, rstyilmazer@gmail.com

INTRODUCTION

Two basic concepts of the ordinary calculus are derivative and integral operators and similarly, two basic concepts of the discrete calculus are sum and difference operators in mathematics. In the fractional calculus, orders of derivative and integral operators consist of arbitrary numbers and similarly, orders of sum and difference operators consist of arbitrary numbers in the discrete fractional calculus (DFC). Kuttner defined the difference operator with reel order in 1956 (Kuttner, 1957). In 1974, Diaz and Osler studied on differences of fractional order (Diaz and Osler, 1974). Gray and Zhang developed a new concept for the fractional difference (Gray and Zhang, 1988). Miller and Ross introduced sum and difference operators with fractional order in 1989 (Miller and Ross, 1989). Thus, many scientific studies take part in literature related to fractional calculus and DFC at the present time.

Nabla DFC operator which is the basic subject of our paper has an important position in the DFC theory. Atici and Eloe mentioned the nabla derivative and, new identities of the gamma function were developed (Atici

and Eloe, 2009). And, two definitions were defined for nabla discrete fractional sum operators (Abdeljawad and Atici, 2012). Nabla discrete Sumudu transform of Taylor monomials, fractional sums, and differences were studied and, this transform was used to obtain the solutions of some fractional difference equations with initial value problems (Jarad et al., 2012). Inc and Yilmazer exhibited some particular solutions of the Chebyshev's equations via nabla DFC operator (Inc and Yilmazer, 2016). Sufficient conditions on global existence and uniqueness of solutions of nonlinear fractional nabla difference systems were introduced and, the dependence of solutions on initial conditions and parameters was studied (Jonnalagadda, 2015). A nabla DFC method was used to solve the confluent hypergeometric equation (Inc et al., 2016). A study related to DFC operator was presented for the vibration equations (Ozturk, 2016).

In our present paper, we used nabla DFC operator for the radial Schrödinger equation and, we obtained the solutions both as fractional forms and as hypergeometric forms.

MATERIALS AND METHODS

Definition 2.1. (Yilmazer and Ozturk, 2012) Fractional derivative and fractional integral definitions of Riemann-Liouville are as follows:

$${}_a D_{\tau}^{\omega} \psi(\tau) = [\psi(\tau)]_{\omega} = \frac{1}{\Gamma(k - \omega)} \frac{d^k}{d\tau^k} \int_a^{\tau} \frac{\psi(\rho)}{(\tau - \rho)^{\omega+1-k}} d\rho, \quad (1)$$

$$(k - 1 \leq \omega < k, k \in \mathbb{N}),$$

and,

$${}_a D_{\tau}^{-\omega} \psi(\tau) = [\psi(\tau)]_{-\omega} = \frac{1}{\Gamma(\omega)} \int_a^{\tau} \frac{\psi(\rho)}{(\tau - \rho)^{1-\omega}} d\rho \quad (\tau > a, \omega > 0). \quad (2)$$

Lemma 2.1. When $\psi(z)$ and $\phi(z)$ are analytic and single-valued functions, linearity rule is

$$[a\psi(z) + b\phi(z)]_{\omega} = a\psi_{\omega}(z) + b\phi_{\omega}(z) \quad (\omega \in \mathbb{R}, z \in \mathbb{C}), \quad (3)$$

where a and b are constants (Ozturk and Yilmazer, 2016).

Lemma 2.2. If $\psi(z)$ is an analytic and single-valued function, index rule is

$$[\psi_\nu(z)]_\omega = \psi_{\nu+\omega}(z) = [\psi_\omega(z)]_\nu \quad (\nu, \omega \in \mathbb{R}, z \in \mathbb{C}), \tag{4}$$

where $\left| \frac{\Gamma(\nu+\omega+1)}{\Gamma(\nu+1)\Gamma(\omega+1)} \right| < \infty$ (Yilmazer and Ozturk, 2013).

Lemma 2.3. When $\psi(z)$ and $\phi(z)$ are analytic and single-valued functions, generalized Leibniz rule is

$$[\psi(z)\phi(z)]_\omega = \sum_{k=0}^{\infty} \frac{\Gamma(\omega+1)}{\Gamma(\omega-k+1)\Gamma(k+1)} \psi_{\omega-k}(z)\phi_k(z) \quad (\omega \in \mathbb{R}, z \in \mathbb{C}), \tag{5}$$

where $\left| \frac{\Gamma(\omega+1)}{\Gamma(\omega-k+1)\Gamma(k+1)} \right| < \infty$ (Yilmazer and Ozturk, 2012).

Property 2.1. In the fractional calculus, the following properties are available:

$$(e^{az})_\omega = a^\omega e^{az} \quad (\omega \in \mathbb{R}, z \in \mathbb{C}), \tag{6}$$

$$(e^{-az})_\omega = e^{-i\pi\omega} a^\omega e^{-az} \quad (\omega \in \mathbb{R}, z \in \mathbb{C}), \tag{7}$$

$$(z^a)_\omega = e^{-i\pi\omega} \frac{\Gamma(\omega-a)}{\Gamma(-a)} z^{a-\omega} \quad \left(\omega \in \mathbb{R}, z \in \mathbb{C}, \left| \frac{\Gamma(\omega-a)}{\Gamma(-a)} \right| < \infty \right), \tag{8}$$

$$\Gamma(\omega-k) = (-1)^k \frac{\Gamma(\omega)\Gamma(1-\omega)}{\Gamma(k+1-\omega)} \quad (\omega \in \mathbb{R}, k \in \mathbb{Z}_0^+), \tag{9}$$

where λ ($\lambda \neq 0$) is a constant (Ozturk and Yilmazer, 2016).

Definition 2.2. The rising factorial power $\omega^{\bar{k}}$ is given by

$$\tau^{\bar{k}} = \tau(\tau+1)(\tau+2) \dots (\tau+k-1) \quad (k \in \mathbb{N}, \tau^{\bar{0}} = 1). \tag{10}$$

And, “ τ to the ω rising” is also defined by

$$\tau^{\bar{\omega}} = \frac{\Gamma(\tau + \omega)}{\Gamma(\tau)} \quad (\omega \in \mathbb{R}, \tau \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}, 0^{\bar{\omega}} = 0). \quad (11)$$

Thus, the following equality is available:

$$\nabla(\tau^{\bar{\omega}}) = \omega \tau^{\bar{\omega}-1}, \quad (12)$$

where $\nabla\psi(\tau) = \psi(\tau) - \psi(\tau - 1)$ (Inc et al., 2016).

Definition 2.3. Let $\omega \in \mathbb{R}^+$ such that $0 < k - 1 \leq \omega < k$ ($k \in \mathbb{Z}$).

The ω -th order fractional sum of ψ is defined as follows:

$$\nabla_a^{-\omega}\psi(\tau) = \sum_{\rho=a}^{\tau} \frac{[\tau - \phi(\rho)]^{\bar{\omega}-1}}{\Gamma(\omega)} \psi(\rho) \quad (a \in \mathbb{R}), \quad (13)$$

where $\tau \in \mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$, $\phi(\tau) = \tau - 1$, is backward jump operator of the time scale calculus.

The ω -th order fractional difference of ψ is given as follows:

$$\nabla_a^{\omega}\psi(\tau) = \nabla^k \nabla_a^{-(k-\omega)}\psi(\tau) = \nabla^k \sum_{\rho=a}^{\tau} \frac{[\tau - \phi(\rho)]^{k-\omega-1}}{\Gamma(k-\omega)} \psi(\rho), \quad (14)$$

where $\psi: \mathbb{N}_a \rightarrow \mathbb{R}$ (Atici and Acar, 2013).

Definition 2.4. (Ozturk, 2016) The shift operator E is

$$E^k\psi(\tau) = \psi(\tau + k) \quad (k \in \mathbb{N}). \quad (15)$$

Theorem 2.1. When $\omega, \nu > 0$ and a, b are scalars, then,

$$\nabla^{-\omega} \nabla^{-\nu} \psi(\tau) = \nabla^{-(\omega+\nu)} \psi(\tau) = \nabla^{-\nu} \nabla^{-\omega} \psi(\tau), \quad (16)$$

$$\nabla^\omega [a\psi(\tau) + b\phi(\tau)] = a\nabla^\omega \psi(\tau) + b\nabla^\omega \phi(\tau), \quad (17)$$

$$\nabla \nabla^{-\omega} \psi(\tau) = \nabla^{-(\omega-1)} \psi(\tau), \quad (18)$$

$$\nabla^{-\omega} \nabla \psi(\tau) = \nabla^{1-\omega} \psi(\tau) - \binom{\tau + \omega - 2}{\tau - 1} \psi(0), \quad (19)$$

where $\psi, \phi: \mathbb{N}_0 \rightarrow \mathbb{R}$ (Inc and Yilmazer, 2016).

Lemma 2.4. (Inc et al., 2016) For $\forall \tau \in \mathbb{N}_a$, , power rule is given by

$$\nabla_a^{-\omega} (\tau - a + 1)^{\bar{\nu}} = \frac{\Gamma(\nu + 1)}{\Gamma(\omega + \nu + 1)} (\tau - a + 1)^{\overline{\omega+\nu}} \quad (\omega, \nu \in \mathbb{R}, \omega > 0). \quad (20)$$

Lemma 2.5. The following equality is available:

$$\nabla_{a+1}^{-\omega} \nabla \psi(\tau) = \nabla \nabla_a^{-\omega} \psi(\tau) - \frac{(\tau - a + 1)^{\overline{\omega-1}}}{\Gamma(\omega)} \psi(a) \quad (\omega > 0), \quad (21)$$

where $\psi: \mathbb{N}_a \rightarrow \mathbb{R}$ (Atici and Acar, 2013).

Lemma 2.6. In DFC, Leibniz rule is as follows:

$$\nabla_0^\omega (\psi\phi)(\tau) = \sum_{k=0}^{\tau} \binom{\omega}{k} [\nabla_0^{\omega-k} \psi(\tau - k)] [\nabla^k \phi(\tau)], \quad (\omega > 0, \tau \in \mathbb{Z}^+), \quad (22)$$

where $\psi, \phi: \mathbb{N}_0 \rightarrow \mathbb{R}$ (Ozturk, 2016).

MAIN RESULTS

In the β - dimensional space, the radial equation of the fractional Schrödinger equation is

$$F''(x) + \frac{\beta - 1}{x} F'(x) + \left[\frac{2m}{\hbar^2} \left(E + e^2 \frac{\delta_a}{x^{a-2}} \right) - \frac{\ell(\ell - \beta - 2)}{x^2} \right] F(x) = 0, \quad (23)$$

where $1 \leq \beta \leq 3$, $0 \leq x \leq \infty$, and δ_a is defined by

$$\delta_a = \frac{\Gamma\left(\frac{a}{2}\right)}{2\pi^{a/2}(a-2)b_0} \quad (a > 2).$$

For “Eq. 23.”, we use the following equalities:

$$F(x) = x^\ell e^{-\delta x} \psi(x) \quad (\delta^2 = -2mE/\hbar^2),$$

and,

$$z = 2\delta x, \quad c = \frac{me^2\delta_a}{\hbar^2}.$$

And then, we have

$$z \frac{d^2\psi}{dz^2} + (\mu - z) \frac{d\psi}{dz} + \left(v z^{3-a} - \frac{\mu}{2} \right) \psi(z) = 0 \quad (z \in \mathbb{C}, z \neq 0), \quad (24)$$

where $\mu = 2\ell + \beta - 1$, $v = \frac{c}{2^{3-a}\delta^{4-a}}$ (Yilmazer and Ozturk, 2013).

Hereafter, we can apply the nabla DFC operator to “Eq. 24.” by means of the following theorem.

Theorem 3.1. Let $a=2$ in “Eq. 24.”. Thus, we write

$$z \frac{d^2\psi}{dz^2} + (\mu - z) \frac{d\psi}{dz} + \left(v z - \frac{\mu}{2} \right) \psi(z) = 0. \quad (25)$$

The discrete fractional solutions of “Eq. 25.” are given by

$$\psi^I = AZ^{1-\mu}e^{(1-\sqrt{1-4v})z/2} \left[z^{-(\xi+2-\mu)} e^{\sqrt{1-4v}z} \right]_{-(1+E^{-1}\xi)}, \tag{26}$$

$$\psi^{II} = BZ^{1-\mu}e^{(1+\sqrt{1-4v})z/2} \left[z^{-(\xi+2-\mu)} e^{-\sqrt{1-4v}z} \right]_{-(1+E^{-1}\xi)}, \tag{27}$$

where $\psi \in \{\psi: 0 \neq |\psi_\omega| < \infty; \omega \in \mathbb{R}\}$, A, B, μ and v are constants.

Proof. At first, we choose $\psi = z^\alpha \varphi$ ($\varphi = \varphi(z)$) for “Eq. 25.”, and then, we have

$$z\varphi_2 + (2\alpha + \mu - z)\varphi_1 + \left[\alpha(\alpha + \mu - 1)z^{-1} - \left(\alpha + \frac{\mu}{2} \right) + vz \right] \varphi = 0. \tag{28}$$

If we get $\alpha(\alpha + \mu - 1) = 0$ in “Eq. 28.”, so $\alpha = 0$ or $\alpha = 1 - \mu$. When $\alpha = 0$, we obtain “Eq. 25.” from “Eq. 28.”. So, we use $\alpha = 1 - \mu$ for “Eq. 28.”, and

$$z\varphi_2 + (2 - \mu - z)\varphi_1 + \left(\frac{\mu}{2} - 1 + vz \right) \varphi = 0. \tag{29}$$

Now, we get $\varphi = e^{\eta z} f$ ($f = f(z)$) for “Eq. 28.”, and so,

$$zf_2 + [2 - \mu + (2\eta - 1)z]f_1 + \left[\eta(2 - \mu) + \frac{\mu}{2} - 1 + (\eta^2 - \eta + v)z \right] f = 0. \tag{30}$$

If $\eta^2 - \eta + v = 0$ in “Eq. 30.”, thus, $\eta = \frac{1 \pm \sqrt{1-4v}}{2}$ and, we obtain

$$zf_2 + [2 - \mu - \sqrt{1-4v}z]f_1 + \left[\sqrt{1-4v} \left(\frac{\mu}{2} - 1 \right) \right] f = 0. \tag{31}$$

where $\eta = \frac{1-\sqrt{1-4v}}{2}$. Hereafter, we can use the nabla (∇^ω) DFC operator to “Eq. 31.”, and we have

$$zf_{2+\omega} + [\omega E + 2 - \mu - \sqrt{1-4vz}]f_{1+\omega} - \sqrt{1-4v} \left(\omega E - \frac{\mu}{2} + 1 \right) f_{\omega} = 0. \quad (32)$$

For $\omega E - \frac{\mu}{2} + 1 = 0$ in “Eq. 32.”, $\omega = E^{-1}\xi$ ($\xi = \frac{\mu}{2} - 1$), and “Eq. 32.” is be written as

$$zf_{2+E^{-1}\xi} + [\xi + 2 - \mu - \sqrt{1-4vz}]f_{1+E^{-1}\xi} = 0. \quad (33)$$

Now, we suppose that $f_{1+E^{-1}\xi} = g = g(z)$ ($f = g_{-(1+E^{-1}\xi)}$) for “Eq. 33.”. Therefore,

$$g_1 + [(\xi + 2 - \mu)z^{-1} - \sqrt{1-4v}]g = 0, \quad (34)$$

and,

$$g = Az^{-(\xi+2-\mu)}e^{\sqrt{1-4v}z}, \quad (35)$$

where A is a constant. After all, we obtain “Eq. 26.” by means of backwards processes.

By means of the similar steps, we have “Eq. 26.” for $\eta = \frac{1+\sqrt{1-4v}}{2}$ in “Eq. 31.”.

Theorem 3.2. Let ${}_2\varphi_0$ be Gauss hypergeometric function. “Eq. 26.” is written as the following hypergeometric form:

$$\psi(z) = z^{-(\xi+1)}e^{(1+\sqrt{1-4v})z/2} {}_2\varphi_0 \left[1 - E^{-1}\xi, \xi + 2 - \mu; \frac{1}{\sqrt{1-4vz}} \right], \quad (36)$$

where $\left| [z^{-(\xi+2-\mu)}]_k \right| < \infty$ ($k \in \mathbb{Z}^+ \cup \{0\}$) and $\left| \frac{1}{\sqrt{1-4vz}} \right| < 1$.

Proof. At first, if “Eq. 5.” is applied to “Eq. 26.”, we have

$$\begin{aligned} \psi(z) &= Az^{1-\mu}e^{(1-\sqrt{1-4v})z/2} \\ &\times \sum_{k=0}^{\infty} \frac{\Gamma(-E^{-1}\xi)}{\Gamma(-E^{-1}\xi - k)k!} [z^{-(\xi+2-\mu)}]_k (e^{\sqrt{1-4v}z})_{-(1+E^{-1}\xi+k)}. \end{aligned} \quad (37)$$

By means of “Eq. 6.”, “Eq. 8.” and “Eq. 9.”, “Eq. 37.” is obtained as follows:

$$\psi(z) = A(\sqrt{1-4v})^{-(1+E^{-1}\xi)} z^{-(\xi+1)} e^{(1+\sqrt{1-4v})z/2} \times \sum_{k=0}^{\infty} (1-E^{-1}\xi)_k (\xi+2-\mu)_k \frac{1}{k!} \left(\frac{1}{\sqrt{1-4vz}}\right)^k. \tag{38}$$

At the end of, we have “Eq. 36.” for $1/A = (\sqrt{1-4v})^{-(1+E^{-1}\xi)}$ in “Eq. 38.”

Theorem 3.3. Let ${}_2\varphi_0$ be Gauss hypergeometric function. “Eq. 27.” is written as the following hypergeometric form:

$$\psi(z) = z^{-(\xi+1)} e^{(1+\sqrt{1-4v})z/2} {}_2\varphi_0 \left[1 - E^{-1}\xi, \xi + 2 - \mu; \frac{-1}{\sqrt{1-4vz}} \right], \tag{39}$$

where $\left| [z^{-(\xi+2-\mu)}]_k \right| < \infty$ ($k \in \mathbb{Z}^+ \cup \{0\}$) and $\left| \frac{-1}{\sqrt{1-4vz}} \right| < 1$.

CONCLUSION

Our present paper is related to an application of nabla DFC operator for the radial Schrödinger equation. In this context, we obtained the successful results for

the fractional calculus studies under favour of discrete fractional solutions and hypergeometric forms. And, we will use this operator (*or, similar operators*) for the different equations in our future studies.

REFERENCES

Abdeljawad T, Atici FM, 2012. On the definitions of nabla fractional operators. *Abstr. Appl. Anal.*, 2012: 13p.

Atici FM, Acar N, 2013. Exponential functions of discrete fractional calculus. *Appl. Anal. Discrete Math.*, 7: 343-353.

Atici FM, Eloe PW, 2009. Discrete fractional calculus with the nabla operator. *Electron. J. Qual. Theory Differ. Equ.*, 3: 1-12.

Diaz JB, Osler TJ, 1974. Differences of fractional order. *Amer. Math. Soc.*, 28: 185-202.

Gray HL, Zhang N, 1988. On a new definition of the fractional difference. *Math. Comp.*, 50: 513-529.

Inc M, Yilmazer R, 2016. On some particular solutions of the Chebyshev’s equation by means of ∇^α discrete fractional calculus operator. *Progr. Fract. Differ. Appl.*, 2 (2): 123-129.

Jarad F, Kaymacalan B, Tas K, 2012. A new transform method in nabla discrete fractional calculus. *Adv. Difference Equ.*, 190, doi: 10.1186/1687-1847-2012-190.

Jonnalagadda JM, 2015. Analysis of nonlinear fractional nabla difference equations. *Int. J. Anal. Appl.*, 7 (1): 79-95.

Kuttner B, 1957. On differences of fractional order. *Proc. London Math. Soc.*, 3: 453-466.

- Miller KS, Ross B, 1989. Fractional difference calculus. Ellis Horwood Ser. Math. Appl., 139-152.
- Ozturk O, 2016. A study on nabla discrete fractional operator in mass-spring-damper system. New Trends Math. Sci., 4 (4): 137-144.
- Ozturk O, Yilmazer R, 2016. Solutions of the radial component of the fractional Schrödinger equation using N -fractional calculus operator. Differ. Equ. Dyn. Syst., doi: 10.1007/s12591-016-0308-8.
- Yilmazer R, Inc M, Tchier F, Baleanu D, 2016. Particular solutions of the confluent hypergeometric differential equation by using the nabla fractional calculus operator. Entropy, 18 (2): 49.
- Yilmazer R, Ozturk O, 2012. N -fractional calculus operator N^n method applied to a Gegenbauer differential equation. Cankaya Univ. J. Sci. Eng., 9 (1): 37-48.
- Yilmazer R, Ozturk O, 2013. Explicit solutions of singular differential equation by means of fractional calculus operators. Abstr. Appl. Anal., 2013: 6p.