NUMERICAL SOLUTION OF A NONLOCAL FRACTIONAL BOUNDARY VALUE PROBLEM BY HDG METHOD

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Abstract

This paper is concerned with numerically solving of a nonlocal fractional boundary value problem (NFBVP) by hybridizable discontinuous Galerkin method (HDG). The HDG methods have been successfully applied to ordinary or partial differential equations in an efficient way through a hybridization procedure. These methods reduce the globally coupled unknowns to approximations at the element boundaries. The stability parameter has to be suitably defined to guarantee the existence and uniqueness of the approximate solution. Some numerical examples are given to show the performance of the HDG method for NFBVP.

Keywords: Hybridizable discontinuous Galerkin methods, nonlocal fractional boundary value problem, stability parameter, hybridization.

1. Introduction

Consider a fractional differential equation with nonlocal boundary conditions as follows [1]

$$^{C}D^{\alpha}u(x) + a_0(x)u'(x) + a_1(x)u(x) = f(x), \text{ in } \Omega = (0,1) \tag{1}$$

$$u = g, \text{ on } \partial\Omega = \{0,1\} \tag{2}$$

Where $1 < \alpha < 2$, g is a given function such that $g(0) = 0, \; g(1) = \sum_{i=1}^{m}\beta_i u(\mu_i), \; 0 < \mu_i < 1$,

$a_j \; (j = 0,1)$, and f are given functions such that satisfy the existence and uniqueness of the solutions of the problem. Fractional differential equations arise in modeling many natural phenomena of physics, engineering, electrodynamics of complex medium, aerodynamics, porous media, continuum and statistical mechanics [2-7]. The multipoint boundary conditions arise from many phenomena such as thermodynamics, elasticity, and wave propagation of applied sciences [8] and the references therein. These boundary conditions can be interpreted

as the controllers at the end of the points distribute or add energy with respect to censors settled at intermediate points. Some basic studies and recent researches on the NFBVP s can be found in [9-15].

Benchohra et al. [16] established sufficient conditions for the existence results of NFBVP with Caputo derivative

$$^{C}D^{\alpha}u(x) = f(x,u(x)),\ 0 \le x \le T,\ 1 < \alpha \le 2, \tag{3}$$
$$u(0) = g(u),\ u(T) = u_{T} \tag{4}$$

where $f:[0,T] \times R \to R$ and $g:C(J,R) \to R$ are continuous functions. Bai [17] studied the existence and uniqueness solution of the following nonlocal problem

$$D^{\alpha}u(x) + f(x,u(x)) = 0,\ 0 < x < 1, \tag{5}$$
$$u(0) = 0,\ \beta u(\eta) = u(1), \tag{6}$$

where $1 < \alpha \le 2$, $0 < \beta\eta^{\alpha-1} < 1$, $0 < \eta < 1$ and f is a continuous function on $[0,1] \times [0,\infty]$. El-Sayed and Bin-Thaer [18] obtained the existence of at least one solution for nonlocal multipoint boundary value problem as follows

$$u'' + f(x,^{C}D^{\alpha}u(x)) = 0, \qquad 0 < x < 1,\ 0 < \alpha < 1, \tag{7}$$
$$u(0) = 0,\ u(1) = \sum_{j=1}^{k} a_{j}u(\tau_{j}),\ \tau_{j} \in (a,b) \subset (0,1). \tag{8}$$

Li and Wu [19] proposed a new numerical method based on the reproducing kernel method for the following NFBVP

$$^{C}D^{\alpha}u(x) + a(x)u'(x) + b(x)u(x) = f(x),\ x \in [0,1],\ 1 < \alpha \le 2, \tag{9}$$
$$\beta_{1}u = \mu_{1},\ \beta_{2}u = \mu_{2} \tag{10}$$

where $\beta_{1}u = \mu_{1}$ and $\beta_{2}u = \mu_{2}$ are linear nonlocal boundary conditions, $a(x),b(x) \in W^{1}[0,1]$, $u(x) \in W^{4}[0,1]$, the existence and uniqueness of the solution are provided for a suitable $f(x)$.

Cockburn et al. [16] firstly presented the HDG methods in the framework of second order elliptic problems. These methods involve hybridized mixed, continuous Galerkin, nonconforming, and HDG methods. The HDG methods have an efficient convergence results for the ordinary or partial differential equations and a built-in stabilization mechanism which do not degrade their accuracy [20-23].

The application of the HDG methods for fractional differential equations is limited in the literature. Cockburn and Mustapha [24] used the HDG method for numerically solving subdiffusion fractional model problem for $-1 < \alpha < 0$. Then, Mustapha et al. [25] studied the method for the spatial discretization of time fractional diffusion equation with Caputo derivative of order $0 < \alpha < 1$. Karaaslan et al. [26, 27] obtained the numerical solution of the Bagley-Torvik equation and a class of fractional boundary value problem (FBVP) with $1 < \alpha < 2$ and $0 < \beta < 1$ by HDG method.

The aim of this paper is to derive an efficient numerical solution of (1)-(2) problem by HDG method. To the best of my knowledge, this is the first time an HDG method is applied to the NFBVPs in a systematic way.

The organization of this paper is as follows. In Section 2, the HDG method is presented with its basis components for the NFBVP. The characterization of the HDG approximation is given in Section 3. The convergence results of the method are given on numerical examples in Section 4.

2. The HDG methods

The HDG method is applied to the (1)-(2) problem by defining Caputo fractional derivate with an operator T as follows

$$
{}_0^C D_x^\alpha u = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{u''(\xi)}{(x-\xi)^{\alpha-1}} d\xi := T[u''(x)]. \tag{11}
$$

Using two unknown variables p and q in (1), the HDG formulation is obtained as follows

$$
\begin{aligned}
q(x) \quad &+ u'(x) &&= 0, &&\text{in } \Omega, \\
p(x) \quad &+ q'(x) &&= 0, &&\text{in } \Omega, \\
a_0 T[p(x)] &- a_1 q(x) + a_2 u(x) &&= f(x), &&\text{in } \Omega.
\end{aligned} \tag{12}
$$

Given a positive integer N, Ω is divided into N elements by setting

$$
\Omega_h := \{I_k = (x_{k-1}, x_k) \mid 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1\}.
$$

The set of nodes and interior nodes are defined as $\varepsilon_h := \{x_0, x_1, \ldots, x_N\}$, $\varepsilon_h^0 := \varepsilon_h \setminus \partial\Omega$; and $\partial\Omega_h := \{\partial K : K \in \Omega_h\}$, respectively. For $K \in \Omega_h$, h_K is the length of K, $h := \max_{K \in \Omega_h} \{h_K\}$. Let $P^k(K)$ denote the set of polynomials of degree less than or equal to $k \geq 0$ that is a given polynomial degree on K. The space of piecewise polynomials of degree k on Ω is defined as follows

$$
V_h^k := \{v : \Omega_h \to \mathbb{R} : v|_K \in P^k(K) \text{ for all } K \in \Omega_h\}
$$

and

$$
L_0^2(\varepsilon_h) := \{m \in L^2(\varepsilon_h) : m = 0 \text{ on } \partial\Omega\}
$$

where $L^2(\varepsilon_h)$ is a copy of the $\mathbb{R}^{N+1}$.

The HDG method find a numerical approximation given as $(q_h, p_h, u_h, \hat{u}_h)$ to the $(q, p, u, u_{\varepsilon_h})$ that is the exact solution of (12) where the space is defined as $V_h^k \times V_h^k \times V_h^k \times L^2(\varepsilon_h)$. This approximate solution is defined for $(v, w, z, \mu) \in V_h^k \times V_h^k \times V_h^k \times L_0^2(\varepsilon_h)$ as follows

$$
\begin{aligned}
(q_h, v) &- (u_h, v') &+ &< \hat{q}_h, v.n >= 0, \\
(p_h, w) &- (q_h, w') &+ &< \hat{q}_h, w.n >= 0, \\
a_0(T[p_h], z) &- a_1(q_h, z) + a_2(u_h, z) &= &(f, z), \\
& &< &\hat{q}_h.n, \mu >= 0,
\end{aligned}
\tag{13}
$$

The boundary condition (2) is imposed by requiring that

$$
\hat{u}_h = u_D \quad \text{on } \partial\Omega
\tag{14}
$$

and the numerical trace $\hat{q}_h$ is given as follows

$$
\hat{q}_h = q_h + \tau(u_h - \hat{u}_h)n
\tag{15}
$$

where τ is a stability parameter of the linear system that is a nonnegative function on $\partial\Omega_h$. (13), (14) and (15) construct the framework of the HDG method.

3. Characterization of the HDG approximation

This section are first give the local solvers that form the basis of the HDG method. The first local solver is defined on $K \in \Omega_h$ as the mapping $\mu \in L^2(\partial K) \mapsto (Q\mu, P\mu, U\mu) \in [P^k(K)]^3$ where

$$
\begin{aligned}
(Q\mu, v)_K &- (U\mu, v')_K & &= -\langle \mu, vn \rangle_{\partial K}, \\
(-w', Q\mu)_K &+ (P\mu, w)_K &+ \langle w, \hat{Q}\mu.n \rangle_{\partial K} &= 0, \\
a_0(T[P\mu], z)_K &- a_1(Q\mu, z)_K + a_2(U\mu, z)_K &= 0,
\end{aligned}
\tag{16}
$$

for all $v, w, z \in P^k(K)$ where $\hat{Q}\mu = Q\mu + \tau(U\mu - \mu)n$, on ∂K.

The second local solver is defined on the element $K \in \Omega_h$ as the mapping $f \in L^2(K) \mapsto (Qf, Pf, Uf) \in [P^k(K)]^3$ where

$$
\begin{aligned}
(Qf, v)_K &- (Uf, v')_K & &= 0, \\
(-w', Qf)_K &+ (Pf, w)_K &+ \langle w, \hat{Q}f.n \rangle_{\partial K} &= 0, \\
a_0(T[Pf], z)_K &- a_1(Qf, z)_K + a_2(Uf, z)_K &= (f, z)_K
\end{aligned}
\tag{17}
$$

for all $v, w, z \in P^k(K)$ where $\hat{Q}f = Qf + \tau(Uf - 0)n$, on ∂K.

The function u_D, as well as any other function defined only on $\partial\Omega$ is extended to ε_h by zero.

Also,

$$\lambda_h := \begin{cases} \hat{u}_h & \text{on } \partial\Omega_h \setminus \partial\Omega, \\ 0 & \text{on } \partial\Omega, \end{cases}$$

and $\hat{u}_h = \lambda_h + u_D$ where $\lambda_h \in L_0^2(\varepsilon_h)$.

Theorem 3.1 The approximate solution $(q_h, p_h, u_h, \lambda_h) \in [V_h^k]^3 \times L_0^2(\varepsilon_h)$ obtained by the HDG method (13) can be written with respect to the local solvers as

$$\begin{aligned} q_h &= Q\lambda_h + Qu_D + Qf, \\ p_h &= P\lambda_h + Pu_D + Pf, \\ u_h &= U\lambda_h + Uu_D + Uf, \end{aligned} \tag{18}$$

where λ_h satisfies

$$a_h(\lambda_h, m) = b_h(m), \quad \text{for all } m \in L_0^2(\varepsilon_h), \tag{19}$$

where

$$\begin{aligned} a_h(\lambda_h, m) &= - <\hat{Q}\lambda_h mn>_{\partial\Omega_h}, \\ b_h(m) &= <\hat{Q}u_D + \hat{Q}fmn>_{\partial\Omega_h}. \end{aligned} \tag{20}$$

Proof. It is clearly seen as a reference to [20] and [28] rewriting (19) with respect to the last equation called as conservativity condition in (13).

Remark 2.2 The logarithm of the ratio of the two consecutive errors for the two HDG approximations with consecutive meshes are given as convergence order of the approximate solution and

$$\left\| \varphi - \varphi_{h0} \le Ch^\eta \right\|$$

where η is convergence order, C is a constant number independent of h. Also,

$$\left\| u - \hat{u}_h \right\|_\infty = \max_{e \in \varepsilon_h} |(u - \hat{u}_h)(e)|$$

and the stability parameter $\tau = 1$ is taken on $\partial\Omega_h$ in each examples.

4. Numerical examples

In this section, two numerical examples are given to test the convergence order and accuracy of the HDG approximation for NFBVPs using MATLAB 2014a.

Example 4.1 Consider the following nonlocal fractional boundary value problem [19, 29]

$$^C D^\alpha u(x) + \delta u(x) = f(x), \quad \Omega = (0,1), 1 \le \alpha < 2$$
$$u(0) = 0, \quad u(1) = \xi u(\eta), \quad \partial\Omega = \{0,1\}$$

where $\alpha = \dfrac{3}{2}$, $\delta = \dfrac{e^{-3\pi}}{\sqrt{\pi}}$, $\xi = -\dfrac{125}{196}$, $\eta = \dfrac{2}{5}$, $u(1) = -\dfrac{1}{40}$,

$f(x) = \dfrac{e^{-3\pi}}{40\sqrt{\pi}}(x^2(40x^2 - 74x + 33) + 4e^{3\pi}\sqrt{x}(128x^2 - 148x + 33))$ and the exact solution is

given as $u(x) = x^2(x^2 - \dfrac{37}{20}x + \dfrac{33}{40})$. The error value and convergence order for L^2-norm are

presented in Table 1. For $k = 2$ and $k = 3$, u_h and q_h converge with order $k+1$. For $k = 2$,
p_h and $\hat{u}_h$ converge with order $k-1$ whereas the HDG method captures the exact solution
(up to the machine accuracy) for p_h and $\hat{u}_h$ in $k = 3$. However, the exact solution is captured
for u_h and q_h in $k = 4$.

Table 1. Convergence order and error result for Example 1.

k	N	$\|u - u_h\|_{0,\Omega}$ Error	Order	$\|q - q_h\|_{0,\Omega}$ Error	Order	$\|p - p_h\|_{0,\Omega}$ Error	Order	$\|u - \hat{u}_h\|_{L^2(\varepsilon_h)}$ Error	Order
2	8	1.78e-04	2.99	1.56e-04	2.99	3.87e-07	1.15	3.67e-08	0.87
	16	2.24e-05	2.99	1.95e-05	3.00	1.90e-07	1.03	1.86e-08	0.99
	32	2.80e-06	3.00	2.44e-06	3.00	9.45e-08	1.01	9.32e-09	0.99
	64	3.51e-07	3.00	3.05e-07	3.00	4.73e-08	1.00	4.67e-09	1.00
	128	4.39e-08	3.00	3.85e-08	2.99	2.36e-08	1.00	2.34e-09	1.00
3	8	1.16e-06	4.00	1.31e-06	3.99	4.08e-11		3.86e-12	
	16	7.27e-08	4.00	8.20e-08	3.99	2.92e-11		9.33e-13	
	32	4.54e-09	4.00	5.14e-09	4.00	2.69e-10		5.42e-12	
	64	2.86e-10	3.99	3.45e-10	3.90	2.15e-09		4.95e-11	
4	4	1.93e-13		6.55e-13		3.61e-12		2.95e-13	
	8	4.89e-12		2.08e-11		1.75e-10		7.61e-12	
	16	7.26e-11		3.30e-10		5.70e-09		1.16e-10	

Example 4.2 Consider a fractional differential equation with three-point boundary conditions

$$^C D^\alpha u(x) + u'(x) + 2u(x) = f(x), \quad \Omega = (0,1),$$
$$u(0) = 0, \quad u(1) = \frac{1}{8} + 2u\left(\frac{1}{2}\right) + \frac{31}{49}u\left(\frac{7}{8}\right)$$

where the fractional order $\alpha = 1.3$, $f(x) = 2x^2 + 2x + \dfrac{\Gamma(3)}{\Gamma(1.7)}x^{0.7}$, and the exact solution is

given as $u(x) = x^2$. Table 2 shows that for $k = 1$, the convergence order of the u_h, q_h, and the
numerical trace $\hat{u}_h$ is 1 while p_h slowly convergences to 1 unlike the others. For $k = 2$, the
numerical solution u_h, q_h, p_h, and the numerical trace $\hat{u}_h$ capture the exact solution (up to
the machine accuracy).

Table 2. Convergence order and error result for Example 2.

k	N	$\|u - u_h\|_{0,\Omega}$ Error	Order	$\|q - q_h\|_{0,\Omega}$ Error	Order	$\|p - p_h\|_{0,\Omega}$ Error	Order	$\|u - \hat{u}_h\|_{L^2(\varepsilon_h)}$ Error	Order
1	8	1.79e-03	1.54	4.36e-03	1.04	2.71e-02	0.48	1.90e-03	0.99
	16	7.78e-04	1.20	2.26e-03	0.95	1.83e-02	0.57	9.81e-04	0.95
	32	3.76e-04	1.05	1.17e-03	0.95	1.16e-02	0.65	5.02e-04	0.97
	64	1.87e-04	1.01	5.95e-04	0.97	7.15e-03	0.70	2.54e-04	0.98
	128	9.36e-05	1.00	3.00e-04	0.99	4.31e-03	0.73	1.27e-04	0.99
2	4	4.50e-15		1.21e-14		1.38e-13		4.41e-15	
	8	7.56e-15		2.59e-14		3.30e-13		9.21e-15	
	16	8.70e-15		4.22e-14		5.07e-13		1.53e-14	

4. Conclusion

In this paper, the numerical solution of a NFBVP is investigated using HDG method. The HDG method captures the exact solution (up to the machine accuracy) of these problems by increasing the degree of the polynomial basis functions. The convergence order and error show the effective performance of the HDG method on the problem under consideration.

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