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Numerical Solution for High-Order Linear Complex Differential Equations By Hermite Polynomials

Faruk DÜŞÜNCELİ[1], Ercan ÇELİK[2]

ABSTRACT: In this paper, the numerical solutions of complex differential equations are provided by the Hermite Polynomials and carried on two problems. As a result, the exact solutions and numerical one's have compared by tables and graphs that the method is practical, reliable and functional.

Keywords: Hermite polynomials, linear complex differential equations, numerical solution

Yüksek Mertebeden Lineer Kompleks Diferansiyel Denklemlerin Hermite Polinomları ile Nümerik Çözümleri

ÖZET: Bu makalede lineer kompleks diferansiyel denklemleri hermite polinomları vasıtasıyla nümerik çözümünü sağladık ve iki test problemine uyguladık. Tam çözümler ile nümerik çözümleri tablo ve grafikler ile karşılaştırdık. Sonuç olarak metodumuzun güvenilir, pratik ve kullanışlı olduğunu gördük.

Anahtar Kelimeler: Hermite polinomları, lineer kompleks diferansiyel denklemler, nümerik çözüm

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[1] Faruk DÜŞÜNCELİ (0000-0002-2368-7963), Mardin Artuklu Üniversitesi, Mimarlık Fakültesi, Mimarlık Bölümü, Mardin, Türkiye
[2] Ercan ÇELİK (0000-0002-1402-1457), Atatürk Üniversitesi, Fen Fakültesi, Matematik Bölümü, Erzurum, Türkiye
Sorumlu yazar/Corresponding Author: Faruk DÜŞÜNCELİ, farukdusunceli@artuklu.edu.tr

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INTRODUCTION

Different type of differential equations have been solved with taylor (Sezer and Yalçınbaş, 2009), Bessel (Yüzbaşı et al., 2011), laguerre (Gülsu et al., 2011), hermite (Yüzbaşı et al., 2011), legendre (Tohidi, 2012; Düşünceli and Çelik, 2015) and Fibonacci polynomials (Düşünceli and Çelik, 2017). In this paper, the matrix operates between the Hermite polynomials and their derivatives, we utilized the Hermite method to solve linear complex differential equation.

$$\sum_{n=0}^{m} P_n(z) f^{(n)}(z) = g(z) \tag{1}$$

with the initial conditions

$$f^{(t)}(\alpha) = \vartheta_t \quad t = 0,1, \dots, m-1 \tag{2}$$

We accept $f(z)$ is unknown function,$P_n(z)$ and $g(z)$ are analytical functions in the circular domain which$D = \{z = x + iy, z \in C, |z| \leq r, r \in R^+\}$; $\alpha \in D$, $\vartheta_t$is appropriate complex or real constant.

Suppose that the solution of (1) under the initial conditions (2) is approximated

$$f(z) = \sum_{n=0}^{N} a_n H_n(z),\ z \in D \tag{3}$$

which is the Hermite series of the unknown function $f(z)$, where all of a_n are the Hermite coefficients to be determined.Hermite polynomials defined by

$$H_n(z) = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m}{m!\,(n-2m)!} (2z)^{n-2m}$$

Where$\left[\frac{n}{2}\right] = \frac{n}{2}$ if n is even and $\left[\frac{n}{2}\right] = \frac{n-1}{2}$ if n is odd and we use the collocation points

$$z_{pp} = \frac{r}{N} p e^{\frac{i\theta}{N} p}\ , 0 < \theta \leq 2\pi, r \in R^+, p \in 0,1, \dots, N \tag{4}$$

MATERIAL AND METHOD

We can write the desired solution $f(z)$ of Equation (3)

$$f(z) = H(z)A \tag{5}$$

where

$$H(z) = [H_0(z) \quad H_1(z) \quad \dots \quad H_N(z)]$$

and

$$A = [a_0 \quad a_1 \quad \dots \quad a_N]^T$$

The Hermite polynomials$H_n(z)$ can be formed in matrix form as

$$H(z) = Z(z)B^T \tag{6}$$

where

$$Z(z) = [1 \quad z \quad z^2 \quad \cdots \quad z^N]$$

and whether N is odd,

$$B = \begin{bmatrix} 0!\frac{(-1)^0}{0!\,0!}2^0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1!\frac{(-1)^0}{0!\,1!}2^1 & 0 & 0 & 0 & \cdots & 0 \\ 2!\frac{(-1)^1}{1!\,0!}2^0 & 0 & 2!\frac{(-1)^0}{0!\,2!}2^2 & 0 & 0 & \cdots & 0 \\ 0 & 3!\frac{(-1)^1}{1!\,1!}2^1 & 0 & 3!\frac{(-1)^0}{0!\,3!}2^3 & 0 & \cdots & 0 \\ 4!\frac{(-1)^2}{2!\,0!}2^0 & 0 & 4!\frac{(-1)^1}{1!\,2!}2^2 & 0 & 4!\frac{(-1)^0}{0!\,4!}2^4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & n!\frac{(-1)^{\left(\frac{n-1}{2}\right)}}{\left(\frac{n-1}{2}\right)!\,1!}2^1 & 0 & n!\frac{(-1)^{\left(\frac{n-3}{2}\right)}}{\left(\frac{n-3}{2}\right)!\,3!}2^3 & 0 & \cdots & n!\frac{(-1)^0}{0!\,n!}2^n \end{bmatrix}_{N+1xN+1}$$

whether N is even,

$$B=\begin{bmatrix} 0!\frac{(-1)^0}{0!\,0!}2^0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1!\frac{(-1)^0}{0!\,1!}2^1 & 0 & 0 & \cdots & 0 & 0 \\ 2!\frac{(-1)^1}{1!\,0!}2^0 & 0 & 2!\frac{(-1)^0}{0!\,2!}2^2 & 0 & \cdots & 0 & 0 \\ 0 & 3!\frac{(-1)^1}{1!\,1!}2^1 & 0 & 3!\frac{(-1)^0}{0!\,3!}2^3 & \cdots & 0 & 0 \\ 4!\frac{(-1)^2}{2!\,0!}2^0 & 0 & 4!\frac{(-1)^1}{1!\,2!}2^2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n!\frac{(-1)^{\left(\frac{n}{2}\right)}}{\left(\frac{n}{2}\right)!\,0!}2^0 & 0 & n!\frac{(-1)^{\left(\frac{n-2}{2}\right)}}{\left(\frac{n-2}{2}\right)!\,2!}2^2 & 0 & \cdots & n!\frac{(-1)^0}{0!\,n!}2^n & 0 \end{bmatrix}_{N+1xN+1}$$

Then, the relation between the matrix $H(z)$ and its derivatives$H'(z), H^{(2)}(z), \dots, H^{(n)}(z)$are

$$\begin{aligned} H'(z) &= H(z)K^T \\ H^{(2)}(z) &= H(z)(K^T)^2 \\ &\vdots \\ H^{(n)}(z) &= H(z)(K^T)^n \end{aligned} \qquad (7)$$

where

$$K=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{N+1xN+1}$$

By using the relations (6) and (7) we obtain the relation

$$f^{(n)}(z) = H^{(n)}(z)K^TA = H(z)(K^T)^nA = Z(z)B^T(K^T)^nA \qquad (8)$$

By changing the collocation points $z = z_{pp}$into the relation (8), we get the following matrix equations

$$f^{(n)}(z_{pp}) = Z(z_{pp})B^T(K^T)^n A,\ p \in 0,1,\dots,N \qquad (9)$$

For $p = 0,1,\dots,N$, we can write the relation (9)

$$\begin{gathered} f^{(n)}(z_{00}) = Z(z_{00})B^T(K^T)^n A \\ f^{(n)}(z_{11}) = Z(z_{11})B^T(K^T)^n A \\ \vdots \\ f^{(n)}(z_{NN}) = Z(z_{NN})B^T(K^T)^n A \end{gathered}$$

where

$$Z = \begin{bmatrix} Z_{00} \\ Z_{11} \\ \vdots \\ Z_{NN} \end{bmatrix} = \begin{bmatrix} Z_0(z_{00}) & Z_1(z_{00}) & Z_2(z_{00}) & \cdots & Z_N(z_{00}) \\ Z_0(z_{11}) & Z_1(z_{11}) & Z_2(z_{11}) & \cdots & Z_N(z_{11}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_0(z_{NN}) & Z_1(z_{NN}) & Z_2(z_{NN}) & \cdots & Z_N(z_{NN}) \end{bmatrix}$$

Let us modify the collocation points (4) into equation(1),

$$\textstyle\sum_{n=0}^{m} P_n(z_{pp}) f^{(n)}(z_{pp}) A = g(z_{pp}) \qquad (10)$$

We attain the basic matrix equation of the relations (8)–(10),

$$\textstyle\sum_{n=0}^{m}\sum_{p=0}^{N} P_n(z_{pp}) Z(z_{pp}) B^T (K^T)^n \mathrm{A} = \sum_{p=0}^{N} G_p \qquad (11)$$

where

$$P_n = \begin{bmatrix} P_n(z_{00}) & 0 & \cdots & 0 \\ 0 & P_n(z_{11}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_n(z_{NN}) \end{bmatrix} \text{and} G_p = \begin{bmatrix} g(z_{00}) \\ g(z_{11}) \\ \vdots \\ g(z_{NN}) \end{bmatrix}$$

Since the A is unknown and should be determined that the matrix equation (11) could be rewritten in the subsequent form:

$$WA = G\text{or}[W;G] = [w_{pq}; g_p] \quad p,q = 0,1,\dots,N \qquad (12)$$

where,

$W = \sum_{n=0}^{m}\sum_{p=0}^{N} P_n(z_{pp}) Z(z_{pp}) B^T (K^T)^n \text{and} A = [a_0 \quad a_1 \quad \dots \quad a_N]^T$

or real constant.

We write the matrix shape of the initial conditions (2) by the aid of (8),

$$f^{(t)}(\alpha) = H(\alpha)(K^T)^t A = \vartheta_t \quad t = 0,1,\dots.,m-1$$

In another hand the matrix shape of the initial conditions could be reformed as

$$U_t A = \vartheta_t \quad t = 0,1,\dots.,m-1$$

where

$$U_t = H(\alpha)(K^T)^t \quad t = 0,1,\dots.,m-1$$

the augmented form of these equations are

$$[U_t;\vartheta_t] = [u_{t0},u_{t1},\dots,u_{tN};\vartheta_t] \quad t = 0,1,\dots.,m-1 \tag{13}$$

Finally, to find the unknown Hermite coefficientsconnected to the approximate solution of the problem (1) under the initial conditions (2), we require to replace the$m$ rows of (13) by the last m rows of the augmented matrix (12) and hence we have new augmented matrix

$$[\tilde{W};\tilde{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g_0 \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-m\,0} & w_{N-m\,1} & \cdots & w_{N-m\,N} & ; & g_N \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \vartheta_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \vartheta_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ u_{m-1\,0} & u_{m-1\,1} & \cdots & u_{m-1\,N} & ; & \vartheta_{m-1} \end{bmatrix} \tag{14}$$

or the matrix equation

$$\tilde{W}A = \tilde{G} \tag{15}$$

If can rewrite (15) in the form and the Ais uniquely set. The solution is given by the Hermite series are determined. And so on, the mth order linear complex differential equation under the initial conditions has an approximated. We can control the precision of the acquired solutions. Since the Hermite series are numerical solution of (1). So the equation should be approximately efficient for

$$z = z_j \in D, j = 0, 1, 2, \ldots$$

$$E(z_j) = \left|\sum_{n=0}^{m} P_n(z_j) f^{(n)}(z_j) - g(z_j)\right| \cong 0 \tag{16}$$

or

$E(z_j) \leq 10^{-l_j}$ (l_j is any positive integer).

If $max10^{-l_j} = 10^{-l}$ is described, then the truncation limit N is put on until the values $E(z_j)$ at eachof the points z_j becomes smaller than the prescribed 10^{-l}.

RESULTS AND DISCUSSION

In this part, two examples are given to illustrate the accuracy and effectiveness of the proposed way and all of them are complemented on a computer by using programs typed in Matlab. Therefore, we have recorded in tables, the values of the exact solution

$$\begin{aligned} f(x,y) = {} & x(573/9223372036854775808 + (21i)/144115188075855872) \\ & + y(-21/144115188075855872 \\ & + (573i)/9223372036854775808) \\ & + (x + yi)^2(-18014398509481987/36028797018963968 \\ & + (53i)/576460752303423488) \\ & + (x + yi)^3(-159/2305843009213693952 \\ & - (7i)/36028797018963968) \\ & + (x + yi)^4(1485316528861191/36028797018963968 \\ & - (11906580003201i)/36028797018963968) \\ & + (x + yi)^5(1666248956492659/2305843009213693952 \\ & - (41586247975829i)/36028797018963968) \\ & + 144115188075855857/144115188075855872 \\ & - (19i)/2305843009213693952 \end{aligned}$$

For N=10,

$$
\begin{aligned}
f(x,y) = x(&13882071/147573952589676412928\\
&- (6532991i)/18446744073709551616)\\
&+ y(6532991/18446744073709551616\\
&+ (13882071i)/147573952589676412928)\\
&+ (x + yi)^2(-2305843009202906759/4611686018427387904\\
&- (3280233i)/2305843009213693952)\\
&+ (x + yi)^3(-1814333/18446744073709551616\\
&+ (735701i)/2305843009213693952)\\
&+ (x + yi)^4(47986826942415845/1152921504606846976\\
&+ (38497718388899i)/576460752303423488)\\
&+ (x + yi)^5(23264094237931/18446744073709551616\\
&- (34829294771i)/2305843009213693952)\\
&+ (x + yi)^6(-792491809524217/576460752303423488\\
&- (2823403318151i)/288230376151711744)\\
&+ (x + yi)^7(248913431329/4611686018427387904\\
&- (159064944217i)/576460752303423488)\\
&+ (x + yi)^8(13170887529351/576460752303423488\\
&+ (378968311233i)/288230376151711744)\\
&+ (x + yi)^9(-658775106425/9223372036854775808\\
&+ (4512177617i)/1152921504606846976)\\
&+ (x + yi)^{10}(-29038783659/288230376151711744\\
&- (23378868389i)/144115188075855872)\\
&+ 9223372036851657523/9223372036854775808
\end{aligned}
$$

The solutions of the linear complex differential equation for N = 3,5 and 10 are obtained. The absolute errors shown in Tables 1,2 and in Figures 1,2.

Table 1 Comparison of real parts of the exact solution and numerical one's for some values

z_j	Exact solution(Real)	N=3	N=5	N=10
-.9(1+i)	.8908207824	1	.6216215573	.8908892455
-.8(1+i)	.9317999001	1	.6966493057	.9318501705
-.7(1+i)	.9600062080	1	.7647768674	.9600393451
-.6(1+i)	.9784066646	1	.8252866747	.9895949000
-.5(1+i)	.9895848832	1	.8775540316	.9895949000
-.4(1+i)	.9957335935	1	.9210479812	.9957378882
-.3(1+i)	.9986500259	1	.9553321731	.9986514289
-.2(1+i)	.9997333347	1	.9800657301	.9997336172
-.1(1+i)	.9999833333	1	.9950041154	.9999833512
0(1+i)	1	1	1	1
.1(1+i)	.9999833333	1	.9950041298	.9999833511
.2(1+i)	.9997333347	1	.9800661925	.9997336139
.3(1+i)	.9986500259	1	.9553356851	.9986514034
.4(1+i)	.9957335935	1	.9210627805	.9957377771
.5(1+i)	.9895848832	1	.8775991953	.9895945485
.6(1+i)	.978406646	1	.8253990566	.9784252442
.7(1+i)	.960006208	1	.7650197690	.9600372394
.8(1+i)	.931799900	1	.6971228821	.9318457566
.9(1+i)	.890820782	1	.6224749574	.8908805837

Table 2 Comparison of imaginer parts of the exact solution and numerical one's for some values

z_j	Exact solution(Im.)	N=3	N=5	N=10
-.9(1+i)	-.8040981746	-.81	-.8101521967	-.8043179038
-.8(1+i)	-.6370882357	-.64	- .6400242974	-.6372208337
-.7(1+i)	- .4886930379	-.49	- .4899727890	- .4887680105
-.6(1+i)	- .3594816533	- .36	- .3599629365	- .3595206898
-.5(1+i)	- .2498263975	- .25	- .2499713353	- .2498445938
-.4(1+i)	- .1599544898	- .16	- .1599838390	- .1599617215
-.3(1+i)	- .0899919000	- .09	- .0899934881	- .0899941321
-.2(1+i)	- .0399992888	- .04	- .0399984374	- .0399997218
-.1(1+i)	- .0099999888	- .01	- .0099998850	- .0100000001
0(1+i)	0	0	0	0
.1(1+i)	-.0099999888	-.01	-.009999850	-.0100000000
.2(1+i)	-.0399992888	-.04	-.0399973324	-.0399997251
.3(1+i)	-.0899919000	-.09	-.0899850972	-.0899941575
.4(1+i)	-.1599544898	-.16	-.1599484800	-.1599618328
.5(1+i)	-.2498263975	-.25	-.2498634277	-.2498449507
.6(1+i)	-.3594816533	-.36	-.3596944279	-.3595216345
.7(1+i)	-.4886930379	-.49	-.4893924366	-.4887702076
.8(1+i)	-.6370882357	-.64	-.6388928054	-.6372254971
.9(1+i)	-.8040981746	-.81	-.8081132113	-.8043271515

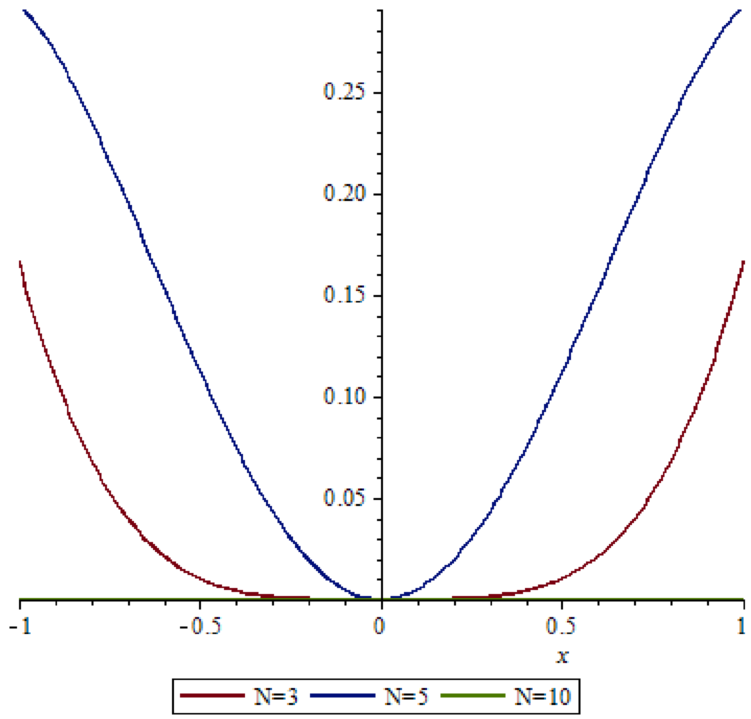

Figure 1. The real parts of the absolute errors functions

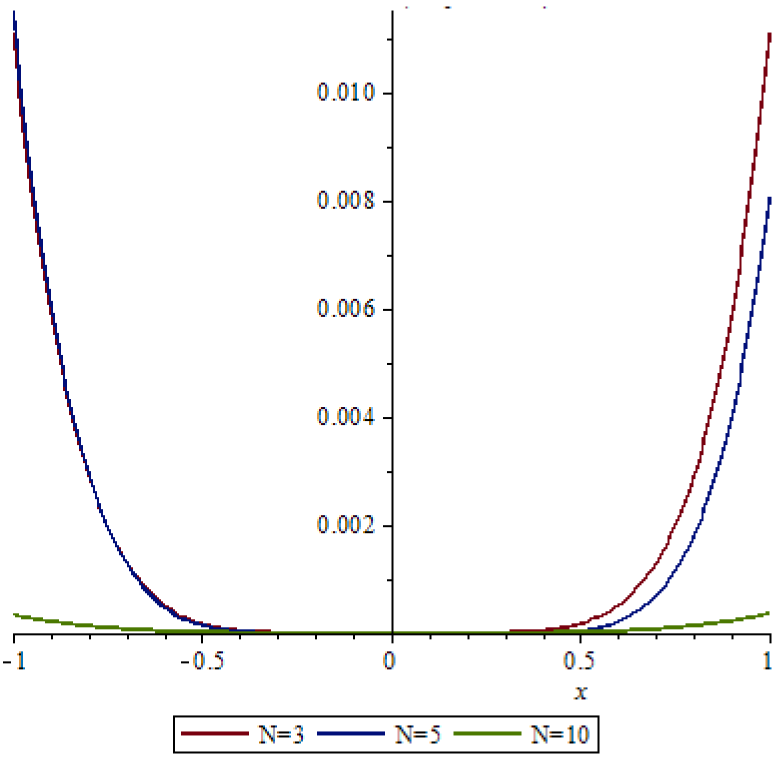

Figure 2. The imaginer parts of the absolute errors functions

Example 2: Finally, consider the linear complex differential equation

$$f''(z) + 2f'(z) + zf(z) = z^3 + e^z(z+3) + 6z + 2$$

with initial conditions $f(0) = 3$, $f'(0) = 1$. The exact solution is $f(z) = z^2 + e^z(z+3) + 6z + 2$. The next step of our method, we obtain the numerical solution for N = 3,5,10. The values of the numerical solution in the issue of N = 3,5,10 for both parts of real and imaginary together with the exact solution and absolute errors are supported in figures 3,4,5,6 as follows.

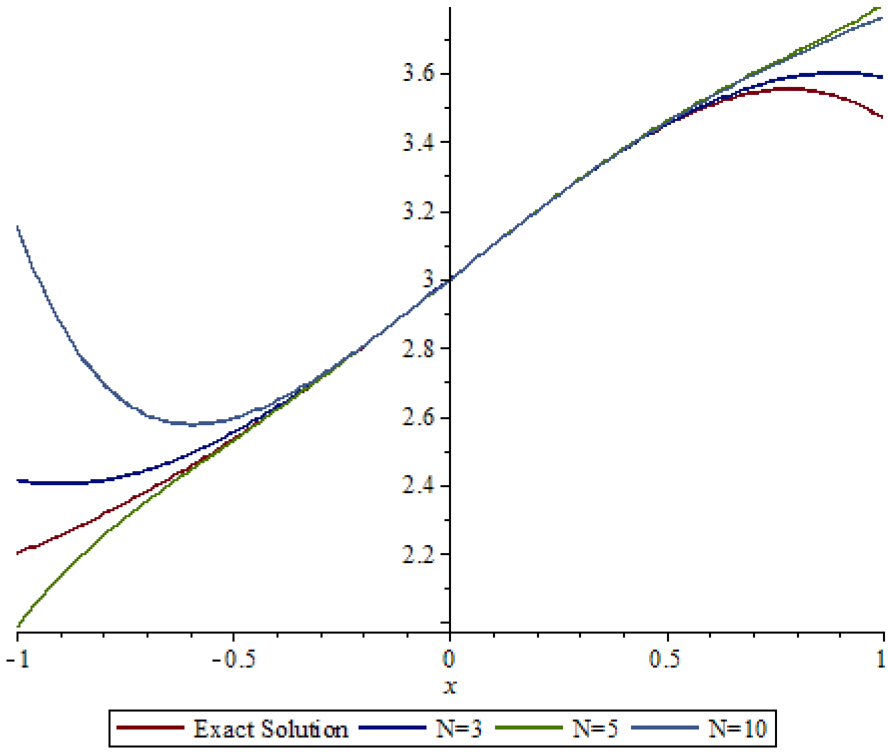

Figure 3. The real parts of the exact solution and numerical one's

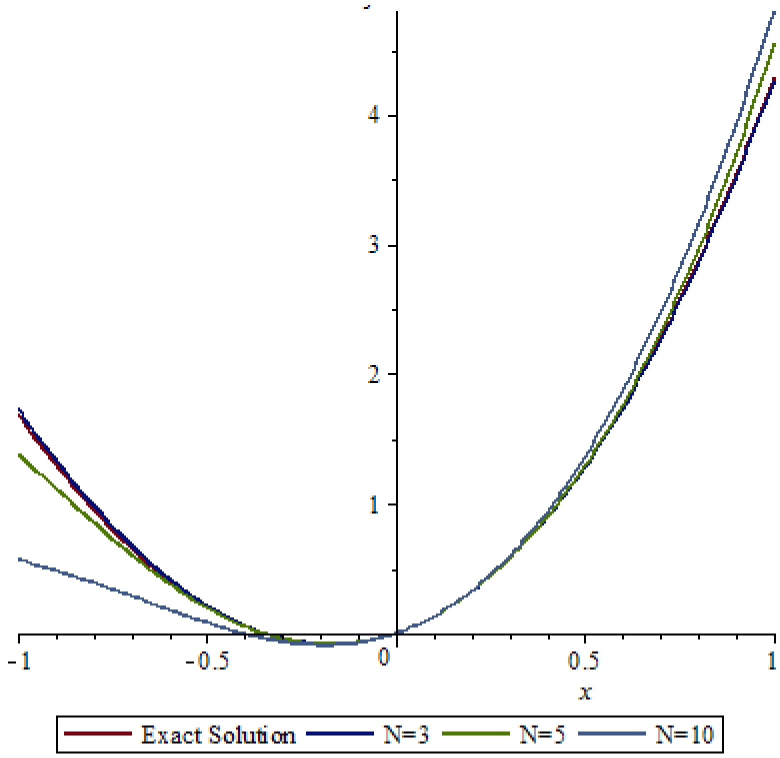

Figure 4. The imaginer parts of the exact solution and numerical one's

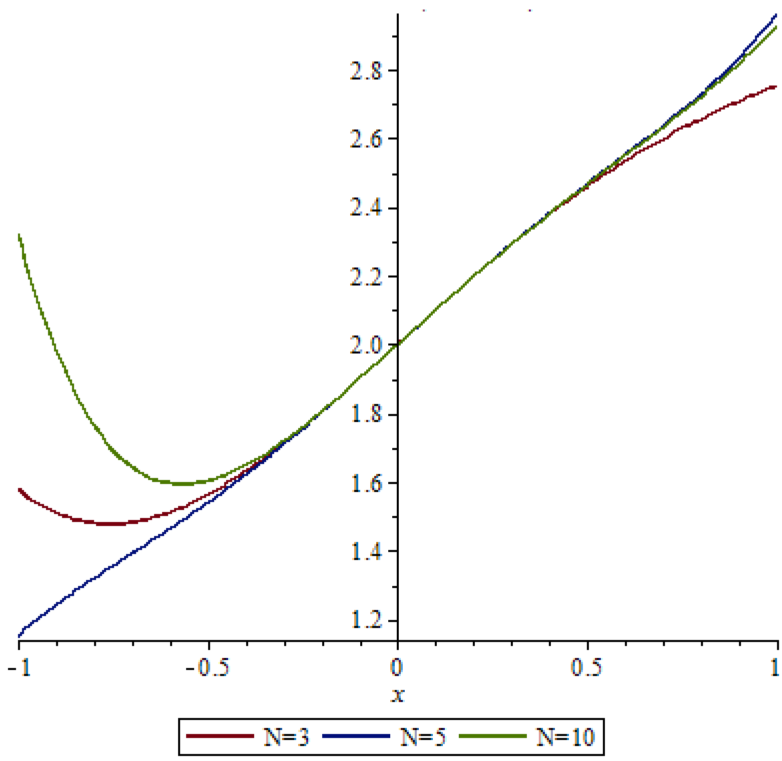

Figure 5. The real parts of the absolute errors functions

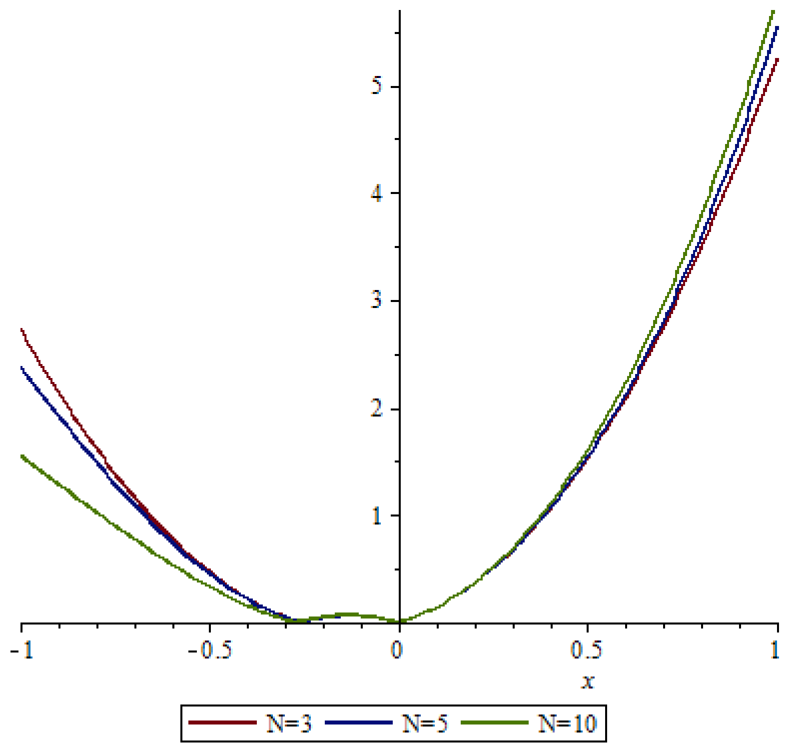

Figure 6. The imaginer parts of the absolute errors functions

CONCLUSIONS

This way is established on reckoning the coefficients in the Hermite series extension of the solution of a linear complex differential equations providing the circular domain is identified by the functions and by the functions Pn(z) and Gn(z). This way is righteous. It may be concluded that the method is an efficient way to discover numerical solutions for linear complex differential equations. On the other hand, the results are quite reliable and good-agreement with the exact solutions.

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