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SOME CHARACTERISTICS OF INTUITIONISTIC FUZZY MODAL OPERATORS WITH USING MATRIX REPRESENTATIONS

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Abstract. In this study, we discuss the some properties of intuitionistic fuzzy modal operators with matrix interpretations of the IFMOs.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [13] as an extension of crisp sets by expanding the truth value set to the real unit interval $[0,1]$. Let X be a fixed set. Function $\mu : X \to [0,1]$ is called a fuzzy set over X. The class of the fuzzy sets over X is denoted by $FS(X)$. For $x \in X$, $\mu(x)$ is the membership degree of x and the non-membership degree is $1-\mu(x)$. Intuitionistic Fuzzy Sets (IFSs) have been introduced in [1], as an extension of fuzzy sets.

Definition 1.1. [2] L et $L = [0,1]$ then

$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\} \tag{1.1}$$

is a lattice with

$$(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow \text{"} x_1 \leq y_1 \text{ } and \text{ } x_2 \geq y_2 \text{"} \tag{1.2}$$

For $(x_1, y_1), (x_2, y_2) \in L^*$,the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as following;

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2)), \tag{1.3}$$

$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2)). \tag{1.4}$$

Definition 1.2. [1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \tag{1.5}$$

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where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x), (\nu_A : X \to [0,1])$is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X. \tag{1.6}$$

The hesitation, indeterminacy, or uncertainty degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 1.3. [1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 1.4. [1]Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the above set is callede the complement of A

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\} \tag{1.7}$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\} \tag{1.8}$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\} \tag{1.9}$$

The notion of Intuitionistic Fuzzy Operators(IFO) was discussed in [1, 6]. After than new Intuitionistic Fuzzy Operators were defined by several autors[3, 5, 8, 9, 11, 12] and some properties of these operators were studied.

Definition 1.5. [1] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$.

(1) $\Box A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}$

(2) $\Diamond A = \{< x, 1 - \nu_A(x), \nu_A(x) >: x \in X\}$

Definition 1.6. [6] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, for $\alpha, \beta \in I$

(1) $(A) = \left\{\left\langle x. \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\right\rangle : x \in X\right\}$
(2) $(A) = \left\{\left\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\right\rangle : x \in X\right\}$
(3) $_{\alpha}(A) = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha\rangle : x \in X\}$
(4) $_{\alpha}(A) = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x)\rangle : x \in X\}$
(5) for $\max\{\alpha, \beta\} + \gamma \in I$, $_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma >: x \in X\}$
(6) for $\max\{\alpha, \beta\} + \gamma \in I$, $_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) >: x \in X\}$

Definition 1.7. [6] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta, \alpha + \beta \in I$

(1) $_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta\rangle : x \in X\}$
(2) $_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x)\rangle : x \in X\}$

The operators $_{\alpha,\beta,\gamma},_{\alpha,\beta,\gamma} areanextensionsof_{\alpha,\beta},_{\alpha,\beta}$ (resp.).

In 2007, the author[8] defined a new operator and studied some of its properties. This operator is named $E_{\alpha,\beta}$ and defined as follows:

Definition 1.8. [8]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0, 1]$. We define the following operator:

$$E_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta) >: x \in X\} \quad (1.10)$$

If we choose $\alpha = 1andwrite\alpha insteedof\beta wegettheoperator.Similarly, if\beta = 1ischosenandwriteninsteedof\beta, wegettheoperator_{\alpha}$.

In2007, Atanassov introduced the operator
$boxdot_{\alpha,\beta,\gamma,\delta}$ which is a natural extension of all these operators in [6].

Definition 1.9. [6]Let Xbe a set, A$\in IFS(X)$, $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that

$$\max(\alpha, \beta) + \gamma + \delta 1 \quad (1.11)$$

then the operator $_{\alpha,\beta,\gamma,\delta}$ defined by

$$_{\alpha,\beta,\gamma,\delta}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta >: x \in X\} \quad (1.12)$$

In 2010, the author [8] defined a new operator which is a generalization of $E_{\alpha,\beta}$.

Definition 1.10. [8]Let X be a set and $A \in IFS(X)$, $\alpha, \beta, \omega \in [0, 1]$. We define the following operator:

$$Z^{\omega}_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta) >: x \in X\} \quad (1.13)$$

We have defined a new OTMO on IFS, that is generalization of the some OTMOs. $Z^{\omega,\theta}_{\alpha,\beta}$ defined as follows:

Definition 1.11. [8]Let X be a set and $A \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0, 1]$. We define the following operator:

$$Z^{\omega,\theta}_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta) >: x \in X\} \quad (1.14)$$

The operator $Z^{\omega,\theta}_{\alpha,\beta}$ is a generalization of $Z^{\omega}_{\alpha,\beta}$,
and also,
$E_{\alpha,\beta\,\alpha,\beta},_{\alpha,\beta}$.

Intuitionistic Fuzzy Operators with matrices were examined in[7] as following;

Let us define that (max)(min)$\{a, b\}$ has property P if and only if (max$\{a, b\}$ has property P and $\wedge$(min$\{a, b\}$ has property P.

Let for brevity $(a_{i,j})$ denote a matrice with elements, denoted also by a and let $M_{3\times3}(\mathbb{R})$ be the set of (3×3)-matrices with elements – real numbers.

Let X be a fixed set. Then Ω and Γ are defining as following;

$$\Omega = \{\Theta \mid \Theta : IFS(X) \to IFS(X) \text{ is an IFMO}\} \quad (1.15)$$

(1.16)

$$\Gamma = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times3}(\mathbb{R}) \;\&\; 0 \le (\max)(\min)\{a_{1,1} + a_{1,2}, a_{2,1} + a_{2,2}\} \le 1$$

$$\& \; 0 \le a_{3,1} + a_{3,2} \le 1\}. \quad (1.17)$$

Definition 1.12. [7]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$.

The mapping $\varphi_A : \Gamma \to \Omega$,

(1.18)
$$\varphi_A((a_{i,j})) = \{\langle x, a_{1,1}\mu_A(x) + a_{2,1}\nu_A(x) + a_{3,1}, a_{1,2}\mu_A(x) + a_{2,2}\nu_A(x) + a_{3,2}\rangle :$$

(1.19)
$$x \in X \ \& \ 0 \leq (\max)(\min)\{a_{1,1} + a_{12}, a_{2,1} + a_{2,2}\} + a_{3,1} + a_{3,2} \leq 1 \ \& \ 0 \leq a_{3,1} + a_{3,2} \leq 1\}.$$

After this we show the second type of IFMOs with matrices as follows. Let $a_{1,1}, a_{2,1}, a_{3,1}, a_{1,2}, a_{2,2}, a_{3,2} \in [0,1]$ satisfy inequalities

$$0 \leq (\max)(\min)\{a_{1,1} + a_{12}, a_{2,1} + a_{2,2}\} + a_{3,1} + a_{3,2} \leq 1 \tag{1.20}$$

and

$$0 \leq a_{3,1} + a_{3,2} \leq 1. \tag{1.21}$$

Then

(1.22)
$$\Theta(A) = \left\{\left\langle x, a_{1,1}\ \mu_A\ (x) + a_{2,1}\ \nu_A(x) + a_{3,1},\ a_{1,2}\ \mu_A\ (x) + a_{2,2}\ \nu_A\ (x) + a_{3,2}\right\rangle : x \in X\right\}$$

$$= \begin{bmatrix} \mu_A\ (x) & \nu_A\ (x) & 1 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \tag{1.23}$$

It is clear that for the present case, sets

$$M_1 = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times 2}(\mathbb{R}) \ \& \ (a_{i,j}) = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix}\} \tag{1.24}$$

and

$$M_1 = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times 2}(\mathbb{R}) \ \& \ (a_{i,j}) = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & 1 \end{bmatrix}\} \tag{1.25}$$

are equal.

For shortly, in this paper, if $\varphi_A((a_{i,j})) = \Theta$, then we will use the notation

$$\Theta = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & 1 \end{bmatrix}. \tag{1.26}$$

Example 1.13. [7]Let X be universe and $A \in IFS(X)$. Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$,

$$\max\ (\alpha\zeta, \beta\varepsilon) + \gamma + \delta\ 1, \tag{1.27}$$

$$\min\ (\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0 \tag{1.28}$$

Then

$$\{\text{circle}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} = [\ \begin{matrix} \alpha & -\zeta & 0 \\ -\varepsilon & \beta & 0 \\ \gamma & \delta & 1 \end{matrix} \quad (1.29)$$

Thanks to this property we can show whether the discussed type of IFMOs satisfy conditions or not.

Theorem 1.14. [7] $(\Gamma, .)$ *is a monoid with multiplication operation over matrices.*

Theorem 1.15. *Let X be a fixed set and $A \in IFS(X)$. If $(a_{i,j}), (b_{i,j}) \in \Gamma$. Then*

$$\varphi_A((a_{i,j})(b_{i,j})) = \varphi_A((b_{i,j})) \circ \varphi_A((a_{i,j})). \tag{1.30}$$

Lemma 1.16. [7]$(\Omega, \circ)$ *is a monoid.*

2. Main Results

In this paper, we will discuss the some properties of intuitionistic fuzzy modal operators can be prove using matrices or not.

Lemma 2.1. *Let $A \in IFS(X)$ then, for every $\alpha, \beta \in I$, $(E_{\alpha,\beta}(A))^c = E_{\beta,\alpha}(A^c)$.*

Proof. For every $\alpha, \beta \in I$,

$$E_{\alpha,\beta} = \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \beta(1-\alpha) & \alpha(1-\ \beta) & 1 \end{bmatrix}^c \tag{2.1}$$

$$= \begin{bmatrix} 0 & \alpha\beta & 0 \\ \alpha\beta & 0 & 0 \\ \alpha\,(1-\beta) & \beta\,(1-\alpha) & 1 \end{bmatrix} \tag{2.2}$$

$$= E_{\beta,\alpha}(A^c) = \tag{2.3}$$

□

Theorem 2.2. *Let $\alpha, \beta, \omega, \theta \in I$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A^c) = Z^{\theta,\omega}_{\beta,\alpha}(A)^c \tag{2.4}$$

Proof. If we use definition of A^c it is clear that

$$Z^{\theta,\omega}_{\beta,\alpha}(A)^c = \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \beta\omega(1-\alpha) & \alpha\omega(1-\ \beta) & 1 \end{bmatrix}^c \tag{2.5}$$

$$= \begin{bmatrix} 0 & \alpha\beta & 0 \\ \alpha\beta & 0 & 0 \\ \alpha\omega(1-\ \beta) & \beta\omega(1-\alpha) & 1 \end{bmatrix} \tag{2.6}$$

$$= Z^{\omega,\theta}_{\alpha,\beta}(A^c) \tag{2.7}$$

□

Theorem 2.3. *Let $\alpha, \beta, \omega, \theta \in I, \alpha\beta\theta = \omega$ and let $A \in IFS(X)$ then*

$$Z^{\omega}_{\alpha,\beta}(Z^{\theta}_{\beta,\alpha}(A)) = Z^{\omega}_{\beta,\alpha}(Z^{\theta}_{\alpha,\beta}(A)) \tag{2.8}$$

Proof. If $\alpha = \beta$ then the statement is clear. Let $\alpha \neq \beta$ and $\alpha\beta\theta = \omega$.

$$Z^{\omega}_{\alpha,\beta}(Z^{\theta}_{\beta,\alpha}(A)) = \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \beta\omega(1-\alpha) & \alpha\omega(1-\beta) & 1 \end{bmatrix} \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \alpha\theta(1-\beta) & \beta\theta(1-\alpha) & 1 \end{bmatrix} \tag{2.9}$$

$$= \begin{bmatrix} \alpha^2\beta^2 & 0 & 0 \\ 0 & \alpha^2\beta^2 & 0 \\ \alpha^2\beta\theta - \alpha^2\beta^2\theta + \beta\omega - \alpha\beta\omega & \alpha\beta^2\theta - \alpha^2\beta^2\theta + \alpha\omega - \alpha\beta\omega & 1 \end{bmatrix} \tag{2.10}$$

$$= \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \alpha\omega(1-\beta) & \beta\omega(1-\alpha) & 1 \end{bmatrix} \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \beta\theta(1-\alpha) & \alpha\theta(1-\beta) & 1 \end{bmatrix} \tag{2.11}$$

therefore

$$Z^{\omega}_{\alpha,\beta}(Z^{\theta}_{\beta,\alpha}(A)) = Z^{\omega}_{\beta,\alpha}(Z^{\theta}_{\alpha,\beta}(A)) \tag{2.12}$$

□

Theorem 2.4. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega = \theta, \omega.\theta = \alpha.\beta$ and let $A \in IFS(X)$ then*

$$Z^{\beta,\alpha}_{\omega,\theta}(A) = Z^{\omega,\theta}_{\alpha,\beta}(A) \tag{2.13}$$

Proof. If we use $\omega = \theta$ and $\omega.\theta = \alpha.\beta$ then

$$Z^{\omega,\theta}_{\alpha,\beta}(A) = \begin{bmatrix} \alpha\beta & 0 & 0 \\ 0 & \alpha\beta & 0 \\ \beta\omega(1-\alpha) & \alpha\theta(1-\beta) & 1 \end{bmatrix} \tag{2.14}$$

$$= \begin{bmatrix} \omega\theta & 0 & 0 \\ 0 & \omega\theta & 0 \\ \theta\beta(1-\omega) & \omega\alpha(1-\theta) & 1 \end{bmatrix} \tag{2.15}$$

$$= Z^{\beta,\alpha}_{\omega,\theta}(A) \tag{2.16}$$

□

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